Introduction

In the present book I give a detailed exposition of my and my collaborators' work on boundedness, continuity, and differentiability properties of solutions to elliptic equations in general domains. By "general" I mean domains which are not a priori restricted by assumptions like "piecewise smoothness", "Lipschitz graph", etc.

The maximum modulus principle for harmonic functions is the earliest result of such nature, known from the very beginning of the theory of partial differential equations. The question of continuity of a solution up to the boundary proved to be delicate and it was completely solved by Wiener in 1924 [165]. He gave his famous criterion for the so-called regularity of a boundary point.

A point O on the boundary $\partial\Omega$ of a domain $\Omega \subset \mathbb{R}^n$ is called regular if solutions of the Dirichlet problem for the Laplace equation in Ω with Dirichlet data continuous at O, are continuous at this point. (I do not want to explain here in which sense the solution is understood — this is not quite trivial and is also due to Wiener [166], 1924.)

Before Wiener's result only some special facts concerning regularity were known. For example, since (by Riemann's conformal mapping theorem) an arbitrary Jordan domain in $\mathbb{R}^2$ is conformally homeomorphic to the unit disc, it follows that any point of its boundary is regular.

As for the n-dimensional case, it was known for years that a boundary point O is regular provided the complement of Ω near O is so thick that it contains an open cone with O as the vertex (Poincaré [134], 1890, Zaremba [167], 1909). Lebesgue noticed that the vertex of a sufficiently thin cusp in $\mathbb{R}^3$ is irregular [74], 1913. Thus it became clear that, in order to characterize the regularity, one should find proper geometric or quasi-geometric terms describing the massiveness of $\mathbb{R}^n \setminus \Omega$ near O.

To this end Wiener introduced the harmonic capacity $\operatorname{cap}(K)$ of a compact set K in $\mathbb{R}^n$, which corresponds to the electrostatic capacity of a body when $n = 3$. Up to a constant factor, the harmonic capacity in the case $n > 2$ is equal to

$$\inf\left\{ \int_{\mathbb{R}^n} |\operatorname{grad} u|^2\, dx \;:\; u \in C_0^\infty(\mathbb{R}^n),\ u > 1 \text{ on } K \right\}.$$

For $n = 2$ this definition of capacity needs to be modified.

The notion of capacity can be extended in a standard way to arbitrary Borel sets (see, for instance, [26]).

The notion of capacity enabled Wiener to state and prove the following result.

Theorem 0.0.1. *The point O of the boundary of the domain $\Omega \subset \mathbb{R}^n$, $n > 2$, is regular if and only if*

$$\sum_{k \geq 1} 2^{(n-2)k} \operatorname{cap}(\overline{B_{2^{-k}}} \setminus \Omega) = \infty. \tag{0.0.1}$$

It is straightforward that (0.0.1) can be rewritten in integral form as

$$\int_0 \frac{\operatorname{cap}(\overline{B_\sigma} \setminus \Omega)}{\operatorname{cap}(\overline{B_\sigma})} \frac{d\sigma}{\sigma} = \infty, \tag{0.0.2}$$

where $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$.

Wiener's theorem was the first necessary and sufficient condition characterizing the dependence of properties of solutions on geometric properties of the boundary. The theorem strongly influenced potential theory, partial differential equations, and probability theory. Especially striking was the impact of the notion of the Wiener capacity, which provided an adequate language to answer many important questions. During the years many attempts have been made to extend the range of Wiener's result to different classes of linear equations of the second order, although some of them were successful only in the sufficiency part.

For uniformly elliptic operators with measurable bounded coefficients in divergence form

$$u \longmapsto \sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i})_{x_j}, \tag{0.0.3}$$

Littman, Stampacchia and Weinberger [77], 1963, discovered that the regularity of a boundary point is equivalent to the Wiener condition (0.0.1).

In 1963 [86] I proved the estimate for the continuity modulus of a harmonic function equal to zero on the boundary near O:

$$\sup_{\Omega\cap B_r} |u| \le c(n) \sup_{\Omega\cap B_R} |u| \exp\Big(-\frac{n-2}{2(n-1)} \int_r^R \frac{\operatorname{cap}(\overline{B_\rho} \setminus \Omega)}{\rho^{n-1}} d\rho\Big), \tag{0.0.4}$$

where R is sufficiently small, $r < R$, and $c(n)$ is a constant depending only on n.

This result directly implies the sufficiency of the Wiener test. Inequality (0.0.4) was extended to the equation

$$\sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i})_{x_j} = 0 \tag{0.0.5}$$

with measurable bounded coefficients in Maz′ya [88], 1967.

An exposition of various results related to (0.0.4) is given in Chapter 1 of the present book.

Chapter 2 follows the paper [54], 1988, by Kerimov, Maz′ya and Novruzov. It concerns the so-called Zaremba problem in the half-cylinder $C = \{x = (x', x_n) : x_n > 0, x' \in \omega\}$, where $\omega \subset \mathbb{R}^{n-1}$ is an open set with bounded closure and smooth boundary:

$$\begin{aligned} \Delta v &= 0 && \text{in } C, \\ \frac{\partial v}{\partial \nu} &= 0 && \text{on } \partial C \setminus F, \\ v &= \varphi && \text{on } F. \end{aligned} \tag{0.0.6}$$

Here F is a closed subset of ∂C with limit points at infinity.

The point at infinity of the domain C is called regular if for every function $\varphi(x)$ continuous on F and possessing a limit Φ as $x_n \to \infty$, $x \in F$, the solution $v(x)$ of the problem (0.0.6) tends to Φ as $x_n \to \infty$, $x \in \overline{C} \setminus F$. The main result of Chapter 2 is the following theorem. A necessary and sufficient condition for the point at infinity to be regular is that

$$\sum_{j=1}^{\infty} j \cdot \mathrm{cap}(\{x \in F : j \le x_n \le j+1\}) = \infty.$$

An extension of this criterion to the equation (0.0.5) is due to my former PhD student J. Björn [16], 1997. Her result is presented in detail in Chapter 3.

Now I turn to the topic of Chapter 4. In [89], 1970, I considered the question of regularity for a certain class of quasilinear operators more general than the p-Laplacian

$$\Delta_p u = \mathrm{div}\big(|\,\mathrm{grad}\, u|^{p-2}\,\mathrm{grad}\, u\big) \ \text{ in } \ \Omega, \tag{0.0.7}$$

where $p > 1$. I found that the condition

$$\int_0 \Big(\frac{p\text{-cap}(\overline{B_\sigma} \setminus \Omega)}{p\text{-cap}(\overline{B_\sigma})}\Big)^{\frac{1}{p-1}} \frac{d\sigma}{\sigma} = \infty \tag{0.0.8}$$

is sufficient for the regularity of a boundary point $O \in \partial\Omega$. Here $1 < p \le n$ and the p-capacity is a modification of the Wiener capacity generated by the p-Laplacian. In [89], an estimate of the modulus of continuity similar to (0.0.4) as well as two-sided estimates for the p-capacitary potential were obtained. The results of [89] are presented in Chapter 4.

The next chapter contains a construction of a special positive homogeneous solution to the Dirichlet problem for $\Delta_p u = 0$ which shows explicitly the loss of continuity of solutions near an irregular point. Here the presentation follows the paper [65], 1972, by myself and my PhD student I. Krol.

So far I spoke only about the regularity of a boundary point for second-order elliptic equations. However, the topic can be extended to include other equations and systems. The first result in this direction was stated by V. Maz′ya [91], 1977 (complete proofs are given in [92], 1979). I showed that for $n = 4, 5, 6, 7$ the Wiener type condition

$$\int_0 \frac{\mathrm{cap}_2(\overline{B_\sigma} \setminus \Omega)}{\mathrm{cap}_2(\overline{B_\sigma})} \frac{d\sigma}{\sigma} = \infty \tag{0.0.9}$$

guarantees the regularity of O with respect to the operator Δ^2. The difference between the conditions (0.0.2) and (0.0.9) is that the harmonic capacity is replaced by the biharmonic one cap_2, introduced by Maz′ya [85], 1963.

The restriction to dimensions $n < 8$ is dictated by the method of proof, which relies on the property of weighted positivity of the biharmonic operator:

$$\int_{\mathbb{R}^n} u(x)\Delta^2 u(x) \frac{dx}{|x|^{n-4}} \geq 0,$$

which unfortunately fails for $n \geq 8$.

In Chapter 6, following my paper [99], 2002, I deal with strongly elliptic differential operators of an arbitrary even order $2m$ with constant real coefficients and introduce a notion of the regularity of a boundary point with respect to the Dirichlet problem. It is shown that a capacitary Wiener-type criterion is necessary and sufficient for regularity if $n = 2m$. In the case $n > 2m$, the same result is obtained for a subclass of strongly elliptic operators which possess the property of weighted positivity.

Boundary behaviour of solutions to the polyharmonic equation is considered in Chapter 7. First, conditions of weighted positivity of $(-\Delta)^m$ with zero Dirichlet data are studied which, together with results in Chapter 2, give a Wiener-type criterion for the space dimensions $n = 2m,\ 2m+1,\ 2m+2$ with $m > 2$ and $n = 4, 5, 6, 7$ with $m = 2$. Second, certain pointwise estimates for the polyharmonic Green function and solutions of the polyharmonic equation are derived for the same n and m. Here I mostly follow my paper [98].

Chapter 8 contains the results of my former PhD student S. Eilertsen [37], 2000, who studied the regularity of a boundary point for certain fractional powers of the Laplacian. The main result of this chapter is the sufficiency of the Wiener type regularity test.

Chapter 9 addresses results by G. Luo and V. Maz′ya [79], 2010. We consider the three-dimensional Lamé system and establish its weighted positive definiteness for a certain range of elasticity constants. By modifying the general theory developed in Chapter 6, it is shown, under the assumption of weighted positivity, that the divergence of the classical Wiener integral for a boundary point guarantees the continuity of solutions to the Lamé system at this point.

The results of the last three chapters are related to the questions of boundedness and continuity of derivatives of solutions to the Dirichlet problem for the polyharmonic equation, and were obtained together with S. Mayboroda in [81], 2009, [83], 2014, and [84], 2017.

Chapter 10 is devoted to the Dirichlet problem for the equation $\Delta^2 u = f$. The biharmonic operator is considered separately in Chapter 10, since it is simpler and is studied in more detail in comparison with the case of the polyharmonic operator. One of the main results of the chapter is the boundedness of the gradient ∇u and in particular of $\nabla_x G(x, y)$, where G is the Green function, in arbitrary bounded domains. By introducing a generalized notion of capacity we obtain separately necessary and sufficient conditions for the first-order regularity of a boundary point O, i.e., $\nabla u(x) \to 0$ as $x \to O$. For a compact set $K \subset \mathbb{R}^3 \setminus \{0\}$, the capacity is defined as the infimum of

$\|u\|^2_{L^2(\mathbb{R}^3)}$, where $u \in H^2(\mathbb{R}^3)$ is required to be of the form $u(x) = b_0 + |x|^{-1}\langle b, x\rangle$ in a neighbourhood of K.

A more involved treatment of the operator Δ^m, $m > 2$, is presented in the next chapters. Boundedness of $[m - \frac{n}{2} + \frac{1}{2}]$ derivatives for solutions to the polyharmonic equation of order $2m$, $2 \leq n \leq 2m + 1$, without any restrictions on the geometry of the underlying domain is established in Chapter 11.

It remains to mention that results in Chapter 11 are used in the concluding Chapter 12, where Wiener-type conditions are found for the continuity of higher-order derivatives at a boundary point.

Each chapter of the book ends with short historical comments on the included material. The notations used in different chapters are essentially independent.

This volume is addressed to students and researchers working in the theory of partial differential equations and potential theory.