Introduction

This set of notes is based upon the graduate course with the same name that I taught at ETH Zurich in Spring Semester 2016. During the semester Yannick Krifka typed up lecture notes, and I have since edited them to form this volume. These notes follow the course quite closely and most chapters represent a two-hour class. The only substantial omission is the class in which I surveyed recent results on boundaries of Coxeter groups, since this material would quickly become out of date.

The assumed background for most of these notes is basic group theory, including group actions, along with a first course in algebraic topology, and some familiarity with Riemannian geometry, particularly the geometry of the hyperbolic plane. There are also some examples or remarks directed at those with additional background in, for example, symmetric spaces or Lie theory.

My main sources when preparing the lectures and these notes were reference books, particularly the works of Davis [16] and Humphreys [23] on Coxeter groups, and Brown [10] and Ronan [32] on buildings. My intention is that this volume will serve as a bridge to these and other more substantial treatments of this beautiful theory, much of which is due to Tits. In this work I have tried to emphasise examples, pictures and intuition, but this comes at the cost of rigour and completeness, and references should be checked before relying upon any results presented here.

Inevitably, the approach taken here reflects my background as a geometric group theorist. Coxeter groups and buildings are important to many areas of mathematics, and this is part of their attraction. My focus is on the geometry and topology of infinite Coxeter groups and nonspherical buildings. I hope that these notes will assist other geometric group theorists to learn about these topics, and for those from other areas will introduce new examples and points of view.

Part I considers Coxeter groups. The first examples, to which I refer throughout these notes, are discrete groups of isometries generated by reflections. I then establish some basic algebraic and combinatorial properties of Coxeter groups, before describing various geometric realisations for Coxeter groups. The final chapter of Part I concerns the Davis complex and its key properties, as established by Davis and Moussong [16]. The reader should be aware that I am not using "geometric realisation" as a precise mathematical term. It does sometimes refer to what I call the Tits representation, but in my view this is too restrictive. I am instead using "geometric realisation" whenever a Coxeter group W acts on some space X with the generating involutions behaving reasonably like geometric reflections. The space X could belong to many different categories, and the action of W may or may not be by isometries. The “best” geometric realisation depends on the group W and what one would like to know about it.

Part II then considers buildings. Tits formulated the general definition of a Coxeter group in order to formulate the definition of a building, and some understanding of Coxeter groups and their geometric realisations is required before reading Part II. The term “building” is perhaps best understood as a cluster of definitions, with enough overlap or equivalence to merit the same word being used. After considering the two main definitions of buildings, I discuss a key technical tool for studying them — retractions — and give the main constructions of buildings. Some of the difficulty with buildings is that they can be approached from many different points of view, with experts often implicitly switching perspectives. However, buildings are such interesting spaces, and they are such powerful tools for understanding the groups which act on them, precisely because of the interplay between their algebraic, combinatorial and geometric structures.