Preface

Hay que saber buscar aunque uno no sepa qué es lo que busca.
ROBERTO BOLAÑO

— Les maths, m'sieur, ça fonctionne toujours à côté de ses grolles.
JEAN AMILA

In 1748 Leonhard Euler published *Introductio in analysin infinitorum* where, among several fundamental results, he established the relationship $e^{i\pi} = -1$ and gave explicitly the continued fraction expansions of e and e^2. He also made a conjecture concerning the nature of quotients of logarithms of rational numbers, which can be formulated as follows:

For any two positive rational numbers r, s with r different from 1, the number ${}^{\log s}/_{\log r}$ is either rational (in which case there are non-zero integers a, b such that $r^a = s^b$) or transcendental.

Recall that a complex number is called algebraic if it is a root of a non-zero polynomial with integer coefficients and a complex number which is not algebraic is called transcendental. Euler's conjecture implies, for example, that $2^{\sqrt{2}}$ is irrational (if it were rational, then $\log 2^{\sqrt{2}}$ divided by $\log 2$, which is equal to $\sqrt{2}$, would be rational or transcendental). It can be reformulated as follows:

If a is a positive rational number different from 1 and β an irrational real algebraic number, then a^β is irrational.

In 1900, David Hilbert proposed a list of twenty-three open problems and presented ten of them in Paris at the second International Congress of Mathematicians. His seventh problem expands the arithmetical nature of the numbers under consideration in Euler's conjecture and asks whether (observe that $e^\pi = (-1)^{-i}$)

the expression α^β for an algebraic base α different from 0 and 1 and an irrational algebraic exponent β, e.g. the number $2^{\sqrt{2}}$ or e^π, always represents a transcendental or at least an irrational number.

Here and below, unless otherwise specified, by algebraic number we mean complex algebraic number. Hilbert believed that the Riemann Hypothesis would be settled long before his seventh problem. This was not the case: the seventh problem was eventually solved in 1934, independently and simultaneously, by Aleksandr Gelfond and Theodor Schneider, by different methods. They established that, for any non-zero algebraic numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ with $\log \alpha_1$ and $\log \alpha_2$ linearly independent over the rationals (here and below, log denotes the principal determination of the logarithm function), we have

$$\Lambda_2 := \beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 \neq 0.$$

Since the formulation is different, let us add some explanation. Under the hypotheses of Hilbert's seventh problem, the complex numbers $\log \alpha$ and $\log \alpha^\beta$ are linearly independent over the rationals and, assuming furthermore that α^β is algebraic, we derive from the Gelfond–Schneider theorem that β, equal to the quotient of the logarithm of α^β by the logarithm of α, cannot be algebraic, a contradiction.

Subsequently, Gelfond derived a lower bound for $|\Lambda_2|$ and, a few years later, he realized that an extension of his result to linear forms in an arbitrarily large number of logarithms of algebraic numbers would enable one to solve many challenging problems in Diophantine approximation and in the theory of Diophantine equations.

This program was realized by Alan Baker in a series of four papers published between 1966 and 1968 in the journal *Mathematika*. He made the long awaited breakthrough, by showing that, if $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers such that $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the rationals, and if $\beta_1, \ldots, \beta_n$ are non-zero algebraic numbers, then

$$\Lambda_n := \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n \neq 0.$$

In addition, he derived a lower bound for $|\Lambda_n|$, thereby giving the expected extension of the Gelfond–Schneider theorem. In his work, Baker generated a large class of transcendental numbers not previously identified and showed how the underlying theory can be used to answer a wide range of Diophantine problems, including the effective resolution of many classical Diophantine equations. He was awarded a Fields Medal in 1970 at the International Congress of Mathematicians in Nice.

It then became clear that further progress, refinements, and extensions of the theory would have important consequences. This area was at that time flourishing and developing very rapidly, both from a theoretical point of view (with improvements obtained by Baker and Feldman, among others, on the lower bounds for $|\Lambda_n|$) and regarding its applications. A spectacular achievement was the proof by Robert Tijdeman in 1976 that the Catalan equation $x^m - y^n = 1$, in the integer unknowns x, y, m, n all greater than 1, has only finitely many solutions (Preda Mihăilescu established in 2002 that $3^2 - 2^3 = 1$ is the only solution to this equation).

The aim of the present monograph is to serve as an introductory text to Baker's theory of linear forms in the logarithms of algebraic numbers, with a special emphasis on a large variety of its applications, mainly to Diophantine questions. We wish to help students and researchers to learn what is hidden inside the blackbox "Baker's theory of linear forms in logarithms" (in complex or in p-adic logarithms) and how this theory applies to many Diophantine problems.

Chapter 1 gives the reader a concise historical introduction to the theory. In Chapter 2, we gather several explicit lower bounds for $|\Lambda_n|$ and its p-adic analogue, which were established by Waldschmidt, Matveev, Laurent, Mignotte and Nesterenko, Yu, and Bugeaud and Laurent, and which will be used in the subsequent chapters. In all but one of these estimates, $\beta_1, \ldots, \beta_n$ are integers, a special case sufficient for most of the applications. The lower bounds are then expressed in terms of the maximum B of their absolute values and take the form

$$\log |\Lambda_n| > -c(n, D) (\log 2A_1) \ldots (\log 2A_n) (\log 2B),$$

where $c(n, D)$ is an explicit real number depending only on n and the degree D of the algebraic number field generated by $\alpha_1, \ldots, \alpha_n$ and A_j is the maximum of the absolute values of the coefficients of the minimal defining polynomial of α_j over the rational integers, for $j = 1, \ldots, n$. The crucial achievements of Baker are the logarithmic dependence on B and the fact that an admissible value for $c(n, D)$ can be explicitly computed.

We consider in Chapter 3 Diophantine problems for which the reduction to linear forms in complex logarithms is almost straightforward. These problems include explicit lower bounds for the distance between powers of 2 and powers of 3, effective irrationality measures for n-th roots of rational numbers, lower bounds for the greatest prime factor of $n(n + 1)$, where n is a positive integer, perfect powers in linear recurrence sequences of integers, etc.

Chapter 4 is devoted to applications to classical families of Diophantine equations. In the works of Thue and Siegel, it was established that unit equations, Thue equations, and super- and hyperelliptic equations have only finitely many integer solutions, but the proofs were ineffective, in the sense that they did not yield upper bounds for the absolute values of the solutions and, consequently, were of very little help for the complete resolution of the equations. The theory of linear forms in logarithms induced dramatic changes in the field of Diophantine equations and we explain how it can be applied to establish, in an effective way, that unit equations, Thue equations, super- and hyperelliptic equations, the Catalan equation, etc., have only finitely many integer solutions. This chapter also contains a complete proof, following Bilu and Bugeaud [72], of an effective improvement of Liouville's inequality (which states that an algebraic number of degree d cannot be approximated by rational numbers at an order greater than d) derived ultimately from an estimate for linear forms in two complex logarithms proved in Chapter 11.

When the algebraic numbers $\alpha_1, \ldots, \alpha_n$ occurring in the linear form Λ_n are all rational numbers very close to 1, the lower bounds for $|\Lambda_n|$ can be considerably improved. Several applications of this refinement are listed in Chapter 5. They include effective irrationality measures for n-th roots of rational numbers close to 1 and striking results on the Thue equation $ax^n - by^n = c$.

Chapter 6 presents various applications of the theory of linear forms in p-adic logarithms, in particular towards Waring's problem and, again, to perfect powers in linear recurrence sequences of integers. It also includes extensions of results established in Chapter 4: unit equations, Thue equations, super- and hyperelliptic equations have only finitely many solutions in the rational numbers, whose denominators are divisible by prime numbers from a given, finite set, and, moreover, the size of these solutions can be effectively bounded.

Primitive divisors of terms of binary recurrence sequences are discussed in Chapter 7. We partially prove a deep result of Bilu, Hanrot, and Voutier [77] on the primitive divisors of Lucas and Lehmer numbers and discuss some of its applications to Diophantine equations. Then, following Stewart [400], we confirm a conjecture of Erdős and show that, for every integer $n \geq 3$, the greatest prime factor of $2^n - 1$ exceeds some positive real number times $n\sqrt{\log n / \log\log n}$.

In Chapter 8, we follow Stewart and Yu [405] to establish partial results towards the abc-conjecture, which claims that, for every positive real number ε, there exists a positive real number $\kappa(\varepsilon)$, depending only on ε, such that, for all coprime, positive integers a, b, and c with $a + b = c$, we have

$$c < \kappa(\varepsilon) \Big(\prod_{p|abc} p \Big)^{1+\varepsilon},$$

the product being taken over the distinct prime factors of abc. Specifically, we show how to combine complex and p-adic estimates to prove the existence of an effectively computable positive real number κ such that, for all positive coprime integers a, b, and c with $a + b = c$, we have

$$\log c < \kappa \Big(\prod_{p|abc} p \Big)^{1/3} \Big(\log\Big(\prod_{p|abc} p \Big)\Big)^{3}.$$

There are only a few known applications of the theory of simultaneous linear forms in logarithms, developed by Loxton in 1986. Two of them are presented in Chapter 9. A first gives us an upper bound for the number of perfect powers in the interval $[N, N + \sqrt{N}]$, for every sufficiently large integer N. A second shows that, under a suitable assumption, a system of two Pellian equations has at most one solution.

Given a finite set of multiplicatively dependent algebraic numbers, we establish in Chapter 10 that these numbers satisfy a multiplicative dependence relation with small exponents. A key ingredient for the proof is a lower bound for the Weil height of a non-zero algebraic number which is not a root of unity.

Full proofs of estimates for linear forms in two complex logarithms, which, in particular, imply lower estimates for the difference between integral powers of real algebraic numbers, are given in Chapter 11. Analogous estimates for linear forms in two p-adic logarithms, that is, upper estimates for the p-adic valuation of the difference between integral powers of algebraic numbers are given in Chapter 12. An estimate for linear forms in an arbitrary number of complex logarithms is derived in Chapter 4 from the estimate for linear forms in two complex logarithms established in Chapter 11. While the former estimate is not as strong and general as the estimates stated in Chapter 2, it is sufficiently precise for many applications.

We collect open problems in Chapter 13. The thirteen chapters are complemented by six appendices, which, mostly without proofs, gather classical results on approximation by rational numbers, the theory of heights, algebraic number theory, and p-adic analysis.

We have tried, admittedly without too much success, to curb our taste for extensive bibliographies. No effort has been made towards exhaustivity, including in the list of bibliographic references, and the topics covered in this textbook reflect somehow the personal taste of the author.

Inevitably, there is some overlap between this monograph and the monograph [376] of Shorey and Tijdeman, which, although over thirty years old, remains an invaluable reference for anyone interested in Diophantine equations. In particular, the content of Chapter 4 (except Section 4.1) is treated in [376] in much greater generality. There is also some overlap with Sprindžuk's book [386] and the monograph of Evertse and Győry [182]. Regarding the theory of linear forms in logarithms, Chapters 2 and 11 can be seen as an introduction to the book of Waldschmidt [432]. As far as we are aware, the content of Chapters 5, 7, 8, 9, and 12 and several other parts of the present monograph have never appeared in books.

To keep this book reasonably short and accessible to graduate and post-graduate students, the results are not proved in their greatest generality and proofs of the best known lower bounds for linear forms in an arbitrary number of complex (*resp.,* p-adic) logarithms are not given.

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This book was written while I was director of the 'Institut de Recherche Mathématique Avancée'.