Preface

0. The book in brief. Sir Michael Atiyah said in his Fields Lecture [13]: "What about 21st century? ... the 21st century might be the era of quantum mathematics or, if you like of infinite-dimensional mathematics."

The aim of this book is to start a systematic development of the non-commutative harmonic analysis on infinite-dimensional (non-locally compact) groups. To do this we generalize the notion of *regular, quasi-regular and induced representations* for infinite-dimensional groups and study when they are irreducible. We also start to develop the *orbit method* in the case of infinite-dimensional "*nilpotent*" groups.

Since almost all constructions in the harmonic analysis on a *locally compact group* G are based on existence (and uniqueness) of the G-invariant measure (*Haar measure*) on the group G, it is rather natural to try to construct something similar for non-locally compact groups. Since the initial group G is not locally compact, there exists neither a Haar (G-invariant) measure (Weil, [188]), nor a G-quasi-invariant measure (Xia Dao-Xing, [191]) on it. The most direct approach to construct an analog of the Haar measure is as follows. Try to construct some larger *topological group* $\widetilde{G}$ containing the initial group G as a *dense subgroup* (i.e., $\widetilde{G}$ is a completion of G) and a G*-right-quasi-invariant measure* μ on $\widetilde{G}$.

Problem 0.0.1. *Thus, the starting point is to construct, for an infinite-dimensional group G, a triple $(\widetilde{G}, G, \mu)$ with the aforementioned properties:*

$$G \longmapsto (\widetilde{G}, G, \mu). \tag{1}$$

Having such a measure we can try to construct *regular*, *quasi-regular* and, with some additional efforts, *induced representations* for the infinite-dimensional group G and study their irreducibility. The main problem we study in the book is as follows:

Problem 0.0.2. *Find irreducibility criteria for the regular, quasi-regular, and induced representations of infinite-dimensional groups.*

In particular, *Ismagilov's conjecture* 0.0.7 explains, in terms of the corresponding measures, when regular representations are *irreducible*. All these representations (except for the induced ones) are Koopman representations (see (4)), so we try to find criteria of irreducibility of Koopman representations (Conjecture 0.0.8).

We generalize the Mackey construction of induced representations for the infinite-dimensional groups and start the development of the *orbit method for the infinite-dimensional "nilpotent" group $B_0^{\mathbb{Z}}$.*

We study the *von Neumann algebra* $\mathfrak{A}^{R,\mu}(G) = \left(T_t^{R,\mu} \mid t \in G\right)''$ generated by the right $T^{R,\mu}$ (or left $T^{L,\mu}$, see (6)) regular representations of the *infinite-dimensional nilpotent groups* $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$ (see Definition 0.0.3 below). Here M' is the *commutant of the von Neumann algebra M* (see (2)). First, we give a condition on the measure μ for the right von Neumann algebra $\mathfrak{A}^{R,\mu}(G)$ to be the *commutant* of the left algebra $\mathfrak{A}^{L,\mu}(G) = \left(T_s^{L,\mu} \mid t \in G\right)''$. This is an analogue of the well-known *Dixmier commutation theorem* for locally compact groups. Second, we determine when the von Neumann algebra M generated by the right (or left) regular representations is a *factor*, i.e., when $M \bigcap M'$ is trivial, that is, consists of only scalar operators. Finally, we show that under some natural conditions on the measure μ the corresponding *factors are of type* III_1. In the case when the ground field is a finite field $\mathbb{F}_p$, we find new irreducibility conditions of the Koopman representation.

We show how the von Neumann infinite tensor product of Hilbert spaces is involved in unitary representations of infinite-dimensional groups. More precisely we define a Hilbert space H_i as an inductive limit $H_i = \varinjlim_{n,i} H_n$ of Hilbert spaces H_n when the sequence of embedding i is fixed, see Section 2.4. We try to define a C^*-group algebra for an infinite-dimensional group G.

Definition 0.0.3. *We call an infinite-dimensional group G "nilpotent" (resp. "solvable") or residually nilpotent (resp. residually solvable) if $\bigcap_{n\in\mathbb{N}} G_n = \{e\}$ (resp. $\bigcap_n G^{(n)} = \{e\}$), where $G_n = \{G_{n-1}, G\}$, $G^{(n)} = \{G^{(n-1)}, G^{(n-1)}\}$, $G_1 = G^{(1)} = G$ and $\{a, b\} = aba^{-1}b^{-1}$.*

1. Representation theory of finite-dimensional (locally compact) groups. The main problem in the representation theory for a locally compact group G is to find the set of all unitary irreducible representations of G up to unitary equivalence and decompose reducible representations into a direct sum or direct integral of irreducible representations. This set is called the unitary dual of G and is denoted by $\widehat{G}$. For many locally compact groups this problem has been solved, but in general it remains open, for example, for the group $\mathrm{SO}(p, q)$. For compact groups this problem is simpler. It is sufficient to consider the *right* or the *left regular representation* of the initial group. These representations are reducible since they commute with each other and are equivalent. The decomposition of, e.g., the right regular representation contains all the irreducible representations for any compact group.

For non-compact, locally compact groups, the regular representation is no longer sufficient to describe $\widehat{G}$. One must go further and consider, for example, the simplest generalization of the regular representation, the so-called *quasi-regular* representation. The decomposition of these representations may give new irreducible representations.

The next step is to introduce *induced representations*. For connected and simply connected nilpotent Lie group, the induced representations are sufficient for obtaining all irreducible representations. Moreover, A. Kirillov ([72], 1962) using his *orbit method* showed that there exists a one-to-one correspondence between the set $\widehat{G}$, the unitary dual of the group G, and the set of all orbits of the co-adjoint action of the group G on the dual space $\mathfrak{g}^*$ to its Lie algebra $\mathfrak{g}$.

The book [75] is a short review of the classical part of representation theory. The main chapters of representation theory are discussed: representations of finite and compact groups, finite- and infinite-dimensional representations of Lie groups.

The structure of the theory is carefully exposed, so that the reader can easily see the essence of the theory without being overwhelmed by details. The final chapter is devoted to the method of orbits for different types of groups.

The survey [126] introduces the readers to the subjects of harmonic analysis on homogenuous spaces and group theoretical methods, and prepares them for the study of more specialised literature.

All the representations considered in the book, except the induced ones, are *Koopman representations*, i.e., are associated with some G-spaces and quasi-invariant measures (see (4)). To study the properties of these representations, in particular, their irreducibility, we need some conjectures to describe the commutant of the von Neumann algebras generated by these representations. The Schur–Weyl duality and the Dixmier commutation theorem below give us a very good hint for such a conjecture (see Conjecture 0.0.5) in a general context.

2. Schur–Weyl duality. The *Schur–Weyl duality* [155, 156, 187] is a typical situation in representation theory involving two kinds of symmetry that determine each other.

Quoting from [193]: "If V is a finite-dimensional complex vector space, then the symmetric group S_n naturally acts on the tensor power $V^{\otimes n}$ by permuting the factors. This action of S_n commutes with the action of $\mathrm{GL}(V)$, so all permutations $\sigma : V^{\otimes n} \to V^{\otimes n}$ are morphisms of $\mathrm{GL}(V)$-representations. This defines a morphism $\mathbb{C}[S_n] \to \mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes n})$, and a natural question to ask is whether this map is surjective.

Part of Schur–Weyl duality asserts that the answer is yes. The *double commutant theorem* plays an important role in the proof and also highlights an important corollary, namely that $V^{\otimes n}$ admits a canonical decomposition

$$V^{\otimes n} = \bigoplus_{\lambda} V_\lambda \otimes S_\lambda,$$

where λ runs over partitions, V_λ are some irreducible representations of $\mathrm{GL}(V)$, and S_λ are the *Specht modules*, which describe all irreducible representations of S_n. This gives a fundamental relationship between the representation theories of the general linear and symmetric groups; in particular, the assignment $V \mapsto V_\lambda$ can be upgraded to a functor called a *Schur functor*, generalizing the construction of the exterior and symmetric products."

Let $\dim V = m$; then $\mathrm{GL}(V) = \mathrm{GL}(m, \mathbb{C})$. The abstract form of the Schur–Weyl duality asserts that two algebras of operators on the tensor space generated by the actions of $\mathrm{GL}(m, \mathbb{C})$ and S_n are the full mutual centralizers in the algebra of endomorphisms $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^m \otimes \mathbb{C}^m \otimes \cdots \otimes \mathbb{C}^m)$.

Denote by α and β the corresponding homomorphisms of S_n and $\mathrm{GL}(m, \mathbb{C})$ into the group of all automorphisms $\mathrm{Aut}(X)$ where $X = \mathbb{C}^m \otimes \mathbb{C}^m \otimes \cdots \otimes \mathbb{C}^m$:

$$\alpha : S_n \to \mathrm{Aut}(X), \quad \beta : \mathrm{GL}(m, \mathbb{C}) \to \mathrm{Aut}(X).$$

Let M' be the commutant of the subset M in the von Neumann algebra $B(H)$ of all bounded operators in a Hilbert space H:

$$M' = \{B \in B(H) \mid [B,a] = 0 \ \ \forall a \in M\}, \text{ where } [B,a] = Ba - aB. \qquad (2)$$

Set $M_1 = (\alpha(S_n))''$ and $M_2 = (\beta(\mathrm{GL}(m,\mathbb{C})))''$. Then the Schur–Weyl duality states that $M_1' = M_2$, hence $M_2' = M_1$.

In [143] the infinite-dimensional versions of Schur–Weyl duality for representation of $S(\infty) = \varinjlim_n S_n$ is established. Sometime this group is denoted by S_∞. In [133] the analogue of Schur-Weyl duality for the unitary group of an arbitrary II_1-factor is obtained.

In [179] the authors extend the classical Schur–Weyl duality between representations of the groups $\mathrm{SL}(m,\mathbb{C})$ and S_n to the case of $\mathrm{SL}(m,\mathbb{C})$ and the infinite symmetric group S_∞. In [148] the authors extend Weyl's results to the classical infinite-dimensional locally finite algebras $\mathfrak{gl}_\infty$, $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, $\mathfrak{so}_\infty$.

3. The Dixmier commutation theorem, locally compact groups. Let G be a locally compact group and let h be the right invariant Haar measure on G, i.e., $h^{R_t} = h$ for all $t \in G$. Consider the left L and the right R action of the group G on itself defined as follows:

$$R_t(x) = xt^{-1}, \quad L_s(x) = sx, \quad x,t,s \in G.$$

The right and the left regular representations of the group G are defined in the Hilbert space $L^2(G,h)$ by

$$(\rho_t f)(x) = f(xt), \ \ (\lambda_s f)(x) = \big(dh(s^{-1}x)/dh(x)\big)^{1/2} f(s^{-1}x), \quad f \in L^2(G,h),$$

where $dh(s^{-1}x)/dh(x)$ is the Radon–Nikodým derivative.

Theorem 0.0.4 (Dixmier's commutation theorem [35]). *The commutant of the von Neumann algebra generated by the right regular representation is generated by the left regular representation. More precisely, let $\rho, \lambda : G \to U(L^2(G,h))$ be the right and the left regular representations of the group G, and let $\mathfrak{A}^\rho = (\rho_t \mid t \in G)''$ and $\mathfrak{A}^\lambda = (\lambda_s \mid s \in G)''$ be the corresponding von Neumann algebras. Then*

$$(\mathfrak{A}^\rho)' = \mathfrak{A}^\lambda \quad \textit{and} \quad (\mathfrak{A}^\lambda)' = \mathfrak{A}^\rho. \qquad (3)$$

4. *G*-action and irreducibility of the Koopman representation. In the two previous examples we have two commuting actions of the groups G_1 and G_2 on the same space X. Let $Z_G(H)$ be the *centralizer* of the subgroup H in the group G:

$$Z_G(H) = \{g \in G \mid \{g,a\} = e \ \ \forall a \in H\},$$

where $\{g,a\} = gag^{-1}a^{-1}$. In the first example, we have two commuting actions α and β of the groups $G_1 = S_n$ and $G_2 = \mathrm{GL}(m,\mathbb{C})$ on the space X such that

$Z_{\mathrm{Aut}(X)}(\alpha(G_1)) \supseteq \beta(G_2)$. In the second example, we have two commuting actions R and L of the same group G in the space $X = G$. In this case we have $\{R(G), L(G)\} = e$ or $Z_{\mathrm{Aut}(G)}(R(G)) \supseteq L(G)$. In the general case, if we have only one group G acting via α on the space X, the second group should be the *centralizer* of the group $\alpha(G)$ in the group $\mathrm{Aut}(X)$, i.e., it is natural to consider $G_2 = Z_{\mathrm{Aut}(X)}(\alpha(G))$.

In a general case, let us fix a Borelian action $\alpha : G \to \mathrm{Aut}(X)$ of a group G (not necessarily locally compact) on a Borelian space X with a G-quasi-invariant measure μ on X, where $\mathrm{Aut}(X)$ is the group of all measurable bijections of the space X. In this case one can naturally define the unitary representation $\pi := \pi^{\alpha,\mu,X}$ of the group G on the space $L^2(X, \mu)$ by the formula:

$$(\pi_t^{\alpha,\mu,X} f)(x) = (d\mu(\alpha_t^{-1}(x))/d\mu(x))^{1/2} f(\alpha_t^{-1}(x)), \quad f \in L^2(X,\mu),\ t \in G. \tag{4}$$

In the case of invariant measure μ this representations is called the *Koopman representation*, see [78]. Consider the centralizer $Z_{\mathrm{Aut}(X)}(\alpha(G))$ of the subgroup $\alpha(G) = \{\alpha_t \mid t \in G\}$ in the group $\mathrm{Aut}(X)$ and its subgroup G_2 defined as follows:

$$G_2 := Z^{\mu}_{\mathrm{Aut}(X)}(\alpha(G)) := \left\{g \in Z_{\mathrm{Aut}(X)}(\alpha(G)) \mid \mu^g \sim \mu\right\}.$$

Define the representation T of the group G_2 by:

$$(T_g f)(x) = (d\mu(g^{-1}x)/d\mu(x))^{1/2} f(g^{-1}x). \tag{5}$$

Consider the two von Neumann algebras

$$\mathfrak{A}^{\pi}(G) = (\pi_t \mid t \in G)'', \quad \mathfrak{A}^{T}(G_2) = (T_g \mid g \in G_2)''.$$

It would be interesting to find conditions under which the following conjecture is true.

Conjecture 0.0.5. *The commutant of the von Neumann algebra generated by the representation π of the group G coincides with the von Neumann algebra generated by the representation T of the subgroup G_2 in the centralizer $Z_{\mathrm{Aut}(X)}(\alpha(G))$:*

$$(\mathfrak{A}^{\pi}(G))' = \mathfrak{A}^{T}(G_2).$$

The book [57] by Helgason is devoted to harmonic analysis on homogeneous spaces. In particular, it gives description of the commutant of the right quasi-regular representations associated with the homogeneous spaces $X = H\backslash G$, where G is locally compact group and H its closed subgroup.

For regular and quasi-regular representations of the groups $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$ the Conjecture 0.0.5 holds, but in general it fails. In Chapter 1, Subsection 1.3.9 we give the example of the group $O(3)$ acting on the homegeneous space $O(2) \setminus O(3) \simeq S^2$ for which Conjecture 0.0.5 fails, Example 1.3.18. Nevertheless, almost all results of the book concerning properties of different types of representations are particular cases of Conjecture 0.0.5. Let us consider three particular cases of Conjecture 0.0.5.

5. The Dixmier commutation theorem, infinite-dimensional groups. To define a regular representation for an infinite-dimensional group G, we find some larger topological group $\widetilde{G}$ and a measure μ on $\widetilde{G}$ such that G is a dense subgroup in $\widetilde{G}$, and $\mu^{R_t} \sim \mu$ for all $t \in G$, (or $\mu^{L_t} \sim \mu$ for all $t \in G$). The right and left representations $T^{R,\mu}, T^{L,\mu} : G \to U(L^2(\widetilde{G}, \mu))$ are naturally defined on the Hilbert space $L^2(\widetilde{G}, \mu)$:

$$\begin{aligned}
\left(T_t^{R,\mu} f\right)(x) &= \left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2} f(xt), \\
\left(T_s^{L,\mu} f\right)(x) &= \left(\frac{d\mu(s^{-1}x)}{d\mu(x)}\right)^{1/2} f\left(s^{-1}x\right).
\end{aligned} \tag{6}$$

Consider the two von Neumann algebras $\mathfrak{A}^{T^{R,\mu}}$ and $\mathfrak{A}^{T^{L,\mu}}$ generated respectively by the right $T^{R,\mu}$ and left $T^{L,\mu}$ regular representations:

$$\mathfrak{A}^{T^{R,\mu}} = \left(T_t^{R,\mu} \mid t \in G\right)'', \quad \mathfrak{A}^{T^{L,\mu}} = \left(T_s^{L,\mu} \mid s \in G\right)''.$$

Conjecture 0.0.6. *The von Neuman algebras $\mathfrak{A}^{T^{R,\mu}}$ and $\mathfrak{A}^{T^{L,\mu}}$ are the commutants of each other or, in other words, they are the mutual centralizers: $\left(\mathfrak{A}^{T^{R,\mu}}\right)' = \mathfrak{A}^{T^{L,\mu}}$.*

We prove Conjecture 0.0.6 only in two particular cases of the groups $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$, see Section 1.1.2 formulas (1.4) and (1.8) for notations.

6. The Ismagilov conjecture. The right regular representation of an infinite-dimensional group can be irreducible if no left actions are admissible for the measure μ, i.e., if the von Neumann algebra $\mathfrak{A}^{T^{L,\mu}}$ generated by the left regular representation $T^{L,\mu}$ is trivial. More precisely:

Conjecture 0.0.7 (Ismagilov, 1985). *The right regular representation*

$$T^{R,\mu} : G \longrightarrow U(L^2(\widetilde{G}, \mu))$$

is irreducible if and only if

1) $\mu^{L_t} \perp \mu$ *for all* $t \in G\backslash\{e\}$, *where* $\perp$ *stands for* singular,
2) *the measure* μ *is* G*-ergodic.*

It is clear that these two conditions are necessary conditions for irreducibility. We recall that two probability measures μ and ν on X are said to be *orthogonal* or *singular* if for some subspace $X_0 \subset X$ the relation $\mu(X_0) = \nu(X\backslash X_0) = 1$ holds.

Conjecture 0.0.7 was verified by the author for some particular cases of the group $B_0^{\mathbb{N}}(\mathbb{R})$ over the real numbers $\mathbb{R}$ and Gaussian product measures on its completion (see Chapter 2, [82, 84]). In the general case, the problem remains open.

In the case of the group $B_0^{\mathbb{N}}(\mathbb{F}_p)$ over a finite field $\mathbb{F}_p$ we need some additional conditions to ensure irreducibility (see Chapter 7, or [106])

The following conjecture can be considered as a natural generalization of the Ismagilov conjecture. It is a particular case of Conjecture 0.0.5 and was first discussed for infinite-dimensional groups by the author in [6, 8, 97, 98, 100, 102].

Conjecture 0.0.8 (Kosyak, [98, 100]). *The Koopman representation* $\pi^{\alpha,\mu,X}: G \to U(L^2(X,\mu))$ *defined by* (4) *is irreducible if and only if*

1) $\mu^g \perp \mu$, *for all* $g \in Z_{\mathrm{Aut}(X)}(\alpha(G))\backslash\{e\}$,
2) *the measure* μ *is* G*-ergodic.*

In Subsection 1.3.9 we show that Conjectures 0.0.5 and 0.0.8 in general fail, Examples 1.3.18 and 1.3.19. It would be interesting to find the conditions when they are true. We recall the following definition

Definition 0.0.9. *A measure* μ *is* G*-ergodic if* $f(\alpha_t(x)) = f(x)$ μ-a.e. *for all* $t \in G$ *implies* $f(x) = \mathrm{const}\,\mu$-a.e. *for all functions* $f \in L^1(X,\mu)$, *where* a.e. *means almost everywhere.*

7. From dynamical systems to representation theory, a single step.

$$\text{Representation theory} \qquad \Longleftarrow \qquad \text{Dynamical systems}$$

$$\begin{array}{cc} \text{a group } G & \\ \Downarrow & (G,\alpha,\mu,X) \\ \widehat{G} = \mathrm{IrrUniRep}(G)/\sim & \\ \alpha: G \longrightarrow \mathrm{Aut}(X) & \text{measurable action,} \\ (X,\mu): \mu^{\alpha_t} \sim \mu \quad \forall t \in G, & \\ \pi^{\alpha,\mu,X}: G \longrightarrow U(L^2(X,\mu)). & \end{array}$$

8. Quasi-regular representations for infinite-dimensional groups. Let us consider the special case of a *G-space* (a space on which a group G acts), namely, the homogeneous space $\widetilde{X} = \widetilde{H} \setminus \widetilde{G}$, where H is a subgroup of G and μ is a quasi-invariant measure on $\widetilde{X}$ (if it exists) with respect to the right action of the group G on $\widetilde{H} \setminus \widetilde{G}$. In this case we call the corresponding representation $\pi^{\alpha,\mu,\widetilde{H}\setminus\widetilde{G}}$ an *analogue of the quasi-regular or geometric representation* of the group G (Kosyak [98, 100], 2002). We note that in this case we should also take some *completion* of the initial homogeneous space $H \setminus G$.

9. Induced representations for infinite-dimensional groups. Induced representations $\mathrm{Ind}_H^G S$ were introduced and studied by F.G. Frobenius [40] for finite groups and developed by G.W. Mackey [112, 113] for locally compact groups. In [160] the

Mackey irreducibility criterion for induced representations for finite groups is explained.

We generalize the Mackey construction for infinite-dimensional groups. To do this, we construct some G-quasi-invariant measures on an appropriate completion $\widetilde{X} = \widetilde{H}\backslash\widetilde{G}$ of the initial space $X = H\backslash G$ (since there is no Haar measure on G) and extend the representation S of the subgroup H to the representation $\widetilde{S}$ of the corresponding completion $\widetilde{H}$. The induced representation defined in such a way allows us to develop an analogue of the orbit method for the infinite-dimensional "nilpotent" group $B_0^{\mathbb{Z}} = \varinjlim_n G_{2n-1}$ of doubly infinite matrices. In particular, we find irreducibility criteria for induced representations corresponding to the so-called *generic orbits*.

10. Von Neumann algebras. A powerful tool in the theory of von Neumann algebras is the *Tomita–Takesaki modular theory*. In particular, this theory is used for the description of the commutant of a von Neumann algebra in terms of the *canonical conjugation operator* J and the *modular operator* Δ (see Subsection 1.5.2). Note that the Dixmier commutation theorem is a particular case of this theory, see Example 1.5.13. We prove the analogue of the Dixmier commutation theorem for the infinite-dimensional groups $B_0^{\mathbb{Z}}$ in Chapter 5 by constructing the corresponding *modular conjugation operator* and the *modular operator*. These tools allow us to prove that the corresponding von Neumann algebra is a type III_1 factor.

Factors. We study the von Neumann algebras $\mathfrak{A}^{R,\mu}(G) = (T_t^{R,\mu} \mid t \in G)''$ and $\mathfrak{A}^{L,\mu}(G) = (T_s^{L,\mu} \mid s \in G)''$, generated by the right and the left regular representations of the infinite-dimensional "nilpotent" groups $G = B_0^{\mathbb{N}}$ and $G = B_0^{\mathbb{Z}}$. First, we give a condition on the measure μ for the right von Neumann algebra $\mathfrak{A}^{R,\mu}(G)$ to be the commutant of the left one $\mathfrak{A}^{L,\mu}(G)$. This is an analogue of the *Dixmier commutation theorem* 0.0.4 for locally compact groups. Second, we determine when the von Neumann algebra M generated by the right (or left) regular representation is a *factor*, i.e., when $M \bigcap M'$ is trivial.

Type III *factors.* In both cases (for the groups $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$) we prove that the von Neumann algebra $\mathfrak{A}^{R,\mu}(G)$ is the *type* III_1 *hyperfinite factor* provided some natural conditions on the measure μ hold. We would like to stress that the first non-type I factor (namely, type II_1 factor) was obtained by von Neumann [135] as the von Neumann algebra generated by the regular representation of a discrete ICC group (i.e., a group for which all conjugacy classes are infinite, except the trivial one).

11. What can you find only in this book? Why it can be useful?

1) We generalize systematically the notions of the *regular, quasi-regular, and induced representations for infinite-dimensional groups*.
2) The essential part of this program is to deal with the lack of a *Haar measure* on the initial group by introducing a suitable *G-quasi-invariant measure* on an appropriate completion of the infinite-dimensional groups G or on the completions of the homogeneous spaces $H\backslash G$. This measure is not unique, there are a lot of non-equivalent measures.

3) The central idea for verifying the irreducibility is the *Ismagilov conjecture*. Together with the Dixmier commutation theorem and the Schur–Weyl duality, this conjecture allows us to formulate and verify a reasonable irreducibility hypothesis for all the considered cases, i.e, for the regular, quasi-regular, induced and more general, *the Koopman representations*.
4) We show that the set of all quasi-invariant measures are very important for developing the harmonic analysis on infinite-dimensional groups. They are useful not only to construct the representations themselves, but serve as an essential ingredient in the description of the dual of the group G. For example, two irreducible regular representations corresponding to non-equivalent measures are non-equivalent. For induced representations of the infinite-dimensional "nilpotent" group $B_0^{\mathbb{Z}}$ corresponding to a point on a generic orbit and a measure on the completion of a homogeneous space, it is reasonable to expect that two such representations are equivalent if and only if the mentioned points are on the same G-orbit in $\mathfrak{g}^*$ of the coadjoint action and the measures are equivalent. Thus, the measures become parameters of the description of the dual $\widehat{G}$. This is a completely infinite-dimensional phenomenon.
5) We have discovered a family of Hilbert–Lie group $\mathrm{GL}_2(a)$, $a \in \mathfrak{A}_{\mathrm{GL}}$, having the property that every unitary continuous representation of $\mathrm{GL}_0(2\infty, \mathbb{R})$ can be extended by continuity to some Hilbert–Lie group $\mathrm{GL}_2(a)$ depending on the representation. This family play en important role in the definition of the induced representation for infinite-dimensional groups G and in the description of the dual $\widehat{G}$.

12. A brief history of the representation theory of infinite-dimensional groups. The representation theory of infinite-dimensional groups is a very broad area. We mention here only the work of some authors connected with our approach. The representation theory of infinite-dimensional unitary groups began with I.E. Segal's 1957 paper [157] in which he studies unitary representations of the full group $U(H)$, called *physical representations*. In order to study the current commutation relations of quantum field theory, H. Araki and E. Woods [11] and H. Araki [10] introduced the notion of current groups and *factorisable representations* of such groups. In the work of I.M. Gel'fand, A.M. Vershik and M.I. Graev, [44], 1973, the representations of *current groups*, i.e., groups $C(X, U)$ of continuous mappings $X \to U$, where X is a finite-dimensional Riemannian manifold and U is a finite-dimensional Lie group, were studied (see also [146]). The first examples of regular representations for infinite-dimensional groups (in the case of current groups) were given by S. Albeverio, R. Høegh-Krohn and D. Testard, [3], 1981, and R.S. Ismagilov [62], 1981. The work of I.M. Gel'fand played a decisive role in the representation theory of groups in general, and that of infinite-dimensional groups, in particular, see [46, 47, 48].

Regular representations of infinite-dimensional groups, in the case of current groups, were studied in [2, 3, 4, 62] (see also [5]). An analogue of the regular representation for an arbitrary infinite-dimensional group G, using a G-quasi-invariant measure on some completion $\widetilde{G}$ of such a group, is defined in [82, 85] in 1990. For $S(\infty)$ and inductive limits of classical compact groups there are analogs of regular representations of another type (with a well-developed harmonic analysis), [21, 70].

For $X = S^1$ and U a compact or non-compact connected Lie group, a Wiener measure on the loop group $\widetilde{G} = C(X, U)$ was constructed and its quasi-invariance was proved in [2, 3, 4, 5, 120, 121, 122].

Using his orbit method developed in [72], A.A. Kirillov described in [73] all unitary irreducible representations of the group $U_\infty(H)$ (see (7)), the *completion in the strong operator topology of the group* $U(\infty) = \varinjlim_n U(n)$. The group $U_\infty(H)$ consists of all unitary operators of the form $1 + a$, where a is compact. If L is a finite-dimensional complex Hilbert space and U is the unitary group of L, a classical theorem of Hermann Weyl asserts that the irreducible unitary representations of U are realized in subspaces of the tensor algebra over L that are defined by suitable symmetry conditions. Kirillov announces here a series of results extending this theorem to the case in which L is infinite-dimensional.

This approach was generalized by G.I. Olshanskii for the inductive limits of other classical groups $K(\infty) = \varinjlim_n K(n)$, where K is U, O or Sp. In [142] the complete classification of the so-called "tame" representations of the group $K(\infty)$ was obtained.

The aim of the book [167] by S. Stratila and D. Voiculescu is the in-depth study of the factor representations of the group $U(\infty)$, see more details in the review by Ola Bratteli MR0458188.

Quoting from another review by Ola Bratteli MR0442153: "If ρ is a continuous unitary finite factor representation of $U(\infty)$ (shortened c.u.f.f.r.) and Tr is the unique normalized trace on the corresponding factor, define the character χ_ρ of ρ as $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$ for $g \in U(\infty)$. Let $\mathfrak{B}$ be the collection of bilateral sequences $(c_n)_{n\in\mathbb{Z}}$ such that (1) $\det((c_{m_i+j-i})_{1\le i,j\le N}) \ge 0$ for all $m_1, \dots, m_2 \in \mathbb{Z}$ and $N \subset \mathbb{N}$, and (2) $\sum_{n\in\mathbb{Z}} c_n = 1$. In [186] D, Voiculescu proves that there is a one-to-one correspondence between the characters χ_ρ of c.u.f.f.r. of $U(\infty)$ and the elements in $\mathfrak{B}$ given by $\chi_\rho(g) = \prod_{j=1}^N p(z_j)$ if $g \in U(N) \subset U(\infty)$, where $z_1, \dots, z_n$ are the eigenvalues of g and $p(z) = \sum_{n\in\mathbb{Z}} c_n z^n$. The explicit characterization of characters allows him to deduce that any c.u.f.f.r. of $U(\infty)$ has a unique extension by continuity to a c.u.f.f.r. of the group $U_1(\infty)$."

Freely quoting K.H. Neeb [127, 128]: "One of the most drastic differences between the representation theory of finite-dimensional Lie groups and infinite-dimensional ones is that an infinite-dimensional Lie group G may carry many different group topologies and any such topology leads to a different class of continuous unitary representations... For an infinite-dimensional Hilbert space H, there is a large variety of unitary groups. First of all, there is the full unitary group $U(H)$, endowed with the norm topology ... However, the much coarser (or weaker) strong operator topology also turns it into another topological group $U(H)_s$. The third variant of a unitary group is the subgroup $U_\infty(H)$ of all unitary operators g for which $g - 1$ is compact. This is a Banach–Lie group. If H is separable ... and $(e_n)_{n\in N}$ is an orthonormal basis, then we obtain natural embeddings $U(n) \mapsto U(n+1)$ and the group $U(\infty, \mathbb{C}) = \varinjlim_n U(n, \mathbb{C})$ ... Introducing also the Banach–Lie groups $U_p(H)$, consisting of unitary operators g, for which $g - 1$ is of *Schatten class* $p \in [1, \infty]$, i.e., $\mathrm{Tr}(|U - 1|^p) < \infty$, we thus obtain an infinite family of groups with continuous inclusions:

$$U(\infty,\mathbb{C}) \hookrightarrow U_1(H) \hookrightarrow \cdots \hookrightarrow U_p(H) \hookrightarrow \cdots \hookrightarrow U_\infty(H) \hookrightarrow U(H) \to U(H)_s." \tag{7}$$

Quoting G. Segal [159]: "We construct projective unitary representations of (a) $\mathrm{Map}(S^1;G)$, the group of smooth maps from the circle into a compact Lie group G, and (b) the group of diffeomorphisms of the circle. We show that a class of representations of $\mathrm{Map}(S^1;T)$, where T is a maximal torus of G, can be extended to representations of $\mathrm{Map}(S^1;G)$."

In [149] D. Pickrell considered the infinite-dimensional group $G = U_\infty(H)$. He showed that there are analogues of the Peter–Weyl theorem and Frobenius reciprocity for $U_\infty(H)$. N.I. Nessonov [131, 132] gave the complete classification of all admissible representations of the group $\mathrm{GL}(\infty)$ and the infinite-dimensional orthogonal and symplectic group. The book [141] by G.I. Olshanskii deals with the representation theory of the automorphism groups of infinite-dimensional Riemannian symmetric spaces. The book [64] by R.S. Ismagilov is devoted to the representations of two classes of infinite-dimensional Lie groups: groups of currents, groups of diffeomorphisms and some of their semidirect products.

The book [129] by Yu.A. Neretin is devoted to the representation theory of the following infinite-dimensional groups: groups of diffeomorphisms of manifolds, groups associated to Virasoro or Kac–Moody algebras, infinite groups of permutations S_∞, groups of operators in Hilbert spaces, groups of currents, and finally, groups of automorphisms of measure spaces.

The book of L. Guieu and C. Roger [53] studies the Virasoro group (the central extension of the group of diffeomorphisms of the circle) and the Virasoro algebra. These objects play an important role in various branches of mathematics and in theoretical physics, for example, in the study of integrable systems, characteristic classes, quantization, dynamical systems, string theory, and conformal field theory. The book of S. Albeverio and coauthors [5] is devoted to representation theory of gauge groups and related topics. Let $S_\infty = \bigcup_{n\geq 1} S_n$ be the group of finite permutations of the natural numbers. All indecomposable central positive definite functions on S_∞, which are related to factor representations of type II_1 were given by E. Thoma [175].

Later A.M. Vershik and S.V. Kerov obtained the same result by a different method in [183] and gave a realization of the representations of type II_1 in [184].

In [136, 137] N. Obata constructed and classified an uncountable family $U^{\theta,\chi}$ of irreducible representations of the group S_∞. This family consists of induced representations. In [69] the generalized regular representations $\{T_z : z \in \mathbb{C}\}$ of the group $S_\infty \times S_\infty$ were studied. These representations are deformations of the biregular representation of S_∞ in $l^2(S_\infty)$. A two-parameter family of generalized regular representations $T_{z,z'}$ of the group S_∞ was also mentioned in [69]. In [20] the corresponding spectral measure $P_{z,z'}$ was investigated. The correlation functions are of a determinantal form similar to those studied in *random matrix theory*.

In [19] the asymptotics of the *Plancherel measures* M_n for the symmetric groups S_n is studied. It is shown that M_n converge to the delta measure supported on a certain subset Ω of $\mathbb{R}^2$ closely connected to *Wigner's semicircle law* for the distribution of eigenvalues of random matrices. In particular, this gives a positive answer to a conjecture of J. Baik, P.A. Deift, and K. Johansson [14].

Conjecture 0.0.7 was formulated by R.S. Ismagilov in his referee report on the author's PhD thesis [81] for the group $G = B_0^{\mathbb{N}}$ and any Gaussian product measure on the group $\widetilde{G} = B^{\mathbb{N}}$, and was proved for this case by the author in [82, 84].

The first result in this direction was proved by N. Nessonov in [130]. For the complex infinite-dimensional Borel group $\mathrm{Bor}_0^{c,\mathbb{N}}$ and the tensor product of the *standard Gaussian measure* on its completion $\mathrm{Bor}^{c,\mathbb{N}}$ the irreducibility of the corresponding regular representation was proved there. Here $\mathrm{Bor}_0^{c,\mathbb{N}}$ (respectively $\mathrm{Bor}^{c,\mathbb{N}}$) is the group of matrices of the form $x = \exp t + s$, where t is a diagonal matrix with a finite number of non-zero real elements (resp. arbitrary real elements) and s is a finite (resp. arbitrary) complex strictly upper-triangular matrix.

In the case when the measure on the group $B^{\mathbb{N}}$ is a product of arbitrary one-dimensional measures, Conjecture 0.0.7 was proved in [92] under some technical assumptions on the measures. In [85] Conjecture 0.0.7 was studied for the groups of the diffeomorphisms of the interval and of the circle. For the group of the diffeomorphisms of the interval the Shavgulidze measure [163] i.e., the image of the classical Wiener measure with respect to a suitable bijection, was used. For the group of diffeomorphisms of the circle the Malliavin measure [121] was used.

In [37] the Koopman and quasi-regular representations corresponding to the action of an arbitrary weakly branch group G on the boundary of a rooted tree T is studied. One of the main results is that in the case of a quasi-invariant Bernoulli measure on the boundary of T, the corresponding Koopman representation of G is irreducible (under some general conditions). It is shown also that quasi-regular representations of G corresponding to different orbits on the boundary of T are pairwise distinct.

13. Segal–Shale–Weil representation. Because of its importance, we want to mention this representation separately, see details in Subsection 1.3.7. The consideration of the *symplectic group* $\mathrm{Sp}(n, \mathbb{R})$ as a group of automorphisms of the commutation relations (i.e., Heisenberg group) leads to the definition of the *Segal–Shale–Weil representation* of the *metaplectic group* $\mathrm{Mp}(n)$ in $L^2(\mathbb{R}^n)$. Infinite-dimensional versions of the corresponding groups and representations were considered e.g., in [157, 158, 161, 67, 22, 17]. Weil representations were introduced in a context of a study [161] dealing with bosons (particles obeying the Bose–Einstein statistics) and electron spin, authored by David Shale and building on work by I.E. Segal [158].

From MR575900 by Pierre de la Harpe: "The group $U(\mathfrak{H})_2$ of those unitary operators on a complex Hilbert space which are Hilbert–Schmidt perturbations of the identity. The author (R.P. Boyer, [22]) shows that Kirillovs method of orbits works for the group $U(\mathfrak{H})_2$–though not trivially, for example, because of the lack of any Haar measure. Norm-continuous representations of $U(\mathfrak{H})_2$ are thus well understood: they are shown to be of type I, and to split into irreducible components characterized by discrete data and isomorphic to standard models. They coincide with those given by Kirillov [73]. The last section contains also results about strongly continuous representations. The author explains how his work relates to and extends results due to Segal, Kirillov, Stratila and Voiculescu."

In [25] A.L. Carey, showed that "the group $U_2(H)$ may be imbedded in the group of Bogoliubov automorphisms of the CAR algebra over H in such a way as to be *weakly inner* in any gauge-invariant quasifree representation. Consequently each such quasifree representation determines a projective representation of $U_2(H)$." From MR0463359 by A.U. Klimyk: "Let G be a semi-simple Lie group and $\widehat{G}$ its unitary dual. Apart from special cases $\widehat{G}$ is not known. There exist isolated points in $\widehat{G}$ which are not members of discrete or "mockdiscrete" series. The authors (M. Kashiwara and M. Vergne, [67]) construct series of such representations for the two-sheeted covering group Mp(n) of the *symplectic group* and for $U(p,q)$. In order to do this they study the decomposition of the tensor products of the harmonic representations into irreducible components. In this way new unitary irreducible representations of Mp(n) and $U(p,q)$ with highest weight vectors are obtained. To construct the intertwining operator from the tensor product of the harmonic representations into a space of vectorvalued holomorphic functions on the associated Hermitian symmetric space G/K, the authors describe the representations of the group $\mathrm{GL}(n,\mathbb{C})\times\mathrm{O}(k,\mathbb{C})$ (resp. $\mathrm{GL}(p,\mathbb{C})\times\mathrm{GL}(q,\mathbb{C})\times\mathrm{GL}(k,\mathbb{C})$) in the space of pluriharmonic polynomials on the space $\mathrm{M}(n,k,\mathbb{C})$ of $n\times k$ complex matrices (resp. $\mathrm{M}(p,k,\mathbb{C})\times\mathrm{M}(q,k,\mathbb{C})$)." From the abstract to [17]: "We produce a connection between the Weil 2-cocycles defining the local and adèlic metaplectic groups defined over a global field, i.e., the double covers of the attendant local and adèlic symplectic groups, and local and adèlic Maslov indices of the type considered by Souriau and Leray."

14. Irreducibility. We study the irreducibility of the Koopman representations (4) when the group G and space X are infinite-dimensional and the measure μ is G-quasi-invariant. The proof of the *irreducibility* is based on the following facts:

(i) the *ergodicity of the measure* μ with respect to the right action of the group G on $\widetilde{G}$ or X,
(ii) the operators of multiplication by the independent variables can be approximated by the generators of one-parameter subgroups of the group G,
(iii) the von Neumann algebra $L^\infty(X,\mu)$ is *maximal abelian*.

Ismagilov's conjecture was proved by the author for certain infinite-dimensional groups and certain quasi-invariant measures. Whether the Ismagilov conjecture and other conjectures mentioned in the book are true in the general case is an *open question*.

The main conclusion is the following: for an infinite-dimensional group even regular representations may be irreducible (in contrast to a locally compact group). Regular representations may be non-equivalent, if the corresponding measures are non-equivalent! We also obtain irreducibility criteria of the induced representations of the group $B_0^{\mathbb{Z}}$ corresponding to generic orbits. Some cases are similar to the locally compact case, others are completely different (Theorem 7.4.3) and use the same irreducibility conditions as in the Ismagilov Conjecture 0.0.7.

15. Examples. We call an infinite matrix **finite** if only a finite number of its non-diagonal elements are non-zero. The examples we are going to study in this book are as follows (see Subsection 1.1.2): the group of *finite* upper triangular real matrices of infinite order with units on the diagonal $B_0^{\mathbb{N}} = \varinjlim_n B(n,\mathbb{R})$ (infinite in one direction), group of *finite* upper triangular real matrices of infinite order with units on the diagonal $B_0^{\mathbb{Z}} = \varinjlim_n B(2n-1,\mathbb{R})$ (infinite in both directions), both are "*nilpotent*" groups; the group of infinite upper triangular matrices with non-zero elements on the diagonal $\mathrm{Bor}_0^{\mathbb{N}}$ (a "*solvable*" group) (see Definition 0.0.3), the group $B_0^{\mathbb{N}}(\mathbb{F}_p) = \varinjlim_n B(n,\mathbb{F}_p)$ over the finite field $\mathbb{F}_p$, the inductive limit of the general linear groups (*reductive* groups) $\mathrm{GL}_0(2\infty,\mathbb{R}) = \varinjlim_n \mathrm{GL}(2n-1,\mathbb{R})$. Almost everywhere we use Gaussian product measures on the corresponding spaces. Only in the Section 3.5 we study arbitrary product measures.

16. Open problems. To develop the harmonic analysis for infinite-dimensional groups it would be useful to solve the following problems:

(I) For an arbitrary infinite-dimensional group G, find a triple $(\widetilde{G}, G, \mu)$, see (1).
(II) To verify the Ismagilov conjecture and its generalization for infinite-dimensional groups over different fields $\mathbf{k}$, Conjecture 0.0.7 and Conjecture 0.0.8.
(III) To verify the Dixmier commutation theorem for infinite-dimensional groups, Conjecture 0.0.6.
(IV) To describe the commutant of the von Neumann algebra $\mathfrak{A}^\pi(G)$ generated by a representation π of the group G, when the representation is reducible, Conjecture 0.0.5.
(V) Find the unitary dual $\widehat{G}$ for infinite-dimensional nilpotent groups $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$.
(VI) Find the *Plancherel measure* on the dual $\widehat{G}$ to the groups $B_0^{\mathbb{N}}$ and $B_0^{\mathbb{Z}}$.
(VII) Construct explicitly the "induced representations" for arbitrary infinite-dimensional groups and establish corresponding irreducibility criteria.

17. The contents of the book. In Chapter 1, we fix the notation and introduce some notions used in the book to make it self-contained. In Chapter 2, we prove the Ismagilov conjecture for the regular representations T^{R,μ_b} of the infinite-dimensional "nilpotent" group $B_0^{\mathbb{N}}$. The corresponding measures are infinite tensor products of one-dimensional arbitrary Gaussian centered measures. We prove also that two irreducible regular representations corresponding to different measures are equivalent if and only if the corresponding measures are equivalent.

In fact, we construct the representation T^{R,μ_b} on the space $L^2(B^{\mathbb{N}}, \mu_b)$ of the inductive limit $G = \varinjlim_n G_n$ as the inductive limit $\varinjlim_n T^{R,\mu_{b,n}}$ of the representations $T^{R,\mu_{b,n}}$ in $\mathcal{H}_n = L^2(G_n, \mu_{b,n})$ equivalent with the regular representations ρ_n of $G_n = B(n,\mathbb{R})$ in $H_n = L^2(G_n, h_n)$, where h_n is the Haar measure on G_n. Since $H_{n+1} = H_{(n+1)} \otimes H_n$ for some Hilbert space $H_{(n+1)}$, the $\varinjlim_n H_n$ can be treated as $H_e = \bigotimes_{n,e} H_{(n)}$, *the von Neumann infinite tensor product of Hilbert spaces* $H_{(n)}$, corresponding to a *stabilizing sequence* $e = (e^{(n)})_n$ depending on the inclusions i, see details in Section 2.4. This means that we define the object $\varinjlim_{n,i} H_n$ in the category of Hilbert spaces, but this object depends on the embedding i. The space $\mathcal{H}_l = \bigotimes_{n,l} \mathcal{H}_{(n)}$ is unitarily equivalent with the space $L^2(B^{\mathbb{N}}, \mu_b)$. Equivalence of two spaces H_e and $H_{e'}$ is the same as the equivalence of two measures μ_b and $\mu_{b'}$. We establish the connection between inclusion i, the stabilizing sequence e, and the measure μ_b.

In Chapter 3, we prove Conjecture 0.0.8 for the quasi-regular representations of the infinite-dimensional "nilpotent" group $G = B_0^{\mathbb{N}}$ and for the "solvable" group $G = \mathrm{Bor}_0^{\mathbb{N}}$ (Section 3.6). The corresponding measures are infinite tensor products of one-dimensional arbitrary non-centered Gaussian measures on some G-spaces of the form $\widetilde{H}\backslash\widetilde{G}$, where $\widetilde{H}$ is a subgroup in $\widetilde{G}$. We also prove that two irreducible quasi-regular representations corresponding to different measures and different G-spaces are equivalent if and only if the corresponding spaces coincide and the corresponding measures are equivalent.

In Chapter 4, we prove the generalized Ismagilov conjecture (Conjecture 0.0.8) for the quasi-regular representations of the infinite-dimensional "nilpotent" group $G = B_0^{\mathbb{N}}$. The corresponding measures are defined on some G-spaces $\widetilde{H}\backslash\widetilde{G}$, where $\widetilde{H}$ is a subgroup of $\widetilde{G}$. They are infinite tensor products of m-dimensional arbitrary centered Gaussian measures on $\mathbb{R}^m$. Since the initial measure $\mu_B = \bigotimes_{n=m+1}^{\infty} \mu_{B_n}$ depends on the infinite set of arbitrary positive operators B_n on the space $\mathbb{R}^m$, the level of the technical problems give rise to more elaborated technique, e.g., the Sylvester identity, the Hadamard–Fischer inequality etc. We have even introduced and studied the *generalized characteristic polynomial* for $n \times n$ matrices to settle the problem.

In Chapter 5 we prove the *Dixmier commutation theorem* for the regular representation of the infinite-dimensional "nilpotent" group $G = B_0^{\mathbb{N}}$. Namely, we prove that the commutant of the von Neumann algebra generated by the right regular representation of the group G coincides with the von Neumann algebra generated by the left regular representation of G. The corresponding measure is an infinite tensor product of one-dimensional centered Gaussian measures on the completion $\widetilde{G} = B^{\mathbb{N}}$ of the group G. The most important observation here is that there are measures $\mu_b = \bigotimes_{k<n} \mu_{b_{kn}}$ on $\widetilde{G}$ such that $\mu_b(x^{-1}) \sim \mu_b(x)$. We give sufficient conditions (close to necessary ones) for the eigenvalues of the covariance operator B of the measure μ_b to have this property. This property allows us to construct the *operator of modular conjugation* and the *modular operator* used in Tomita–Takesaki theory, which are essential for the future study of the properties of the von Neumann algebras generated by regular representations (see Chapter 6).

In Chapter 6, we determine when the von Neumann algebra $\mathfrak{A}^{R,\mu}(G)$ generated by the right (and left) regular representations of the infinite-dimensional "nilpotent" group G is a factor. In Section 6.1, we investigate the case of the group $G = B_0^{\mathbb{N}}$, in Section 6.2 the case of the group $G = B_0^{\mathbb{Z}}$. Moreover, we determine the type of the corresponding factors. We show that the von Neumann algebra $\mathfrak{A}^{R,\mu}(G)$ is the type III_1 hyperfinite factor. The case of the group $G = B_0^{\mathbb{N}}$ is considered in Section 6.3, the case of the group $G = B_0^{\mathbb{Z}}$ in Section 6.4.

The induced representations $\mathrm{Ind}_H^G S$ were introduced and studied by F.G. Frobenius [40] for finite groups and developed by G.W. Mackey [112, 113] for locally compact groups. In Chapter 7, we generalize the Mackey construction to infinite-dimensional groups. To do this, we construct some G-quasi-invariant measures on an appropriate completion $\widetilde{X} = \widetilde{H}\backslash\widetilde{G}$ of the initial space $X = H\backslash G$ (since there is no Haar measure on G) and extend the representation S of the subgroup H to the representation $\widetilde{S}$ of the corresponding completion $\widetilde{H}$. Kirillov's orbit method [72] describes all irreducible unitary representations of the finite-dimensional nilpotent group G_n in terms of induced representations associated with orbits of the coadjoint action of the group G_n on the dual space $\mathfrak{g}_n^*$ of the Lie algebra $\mathfrak{g}_n$. The induced representations defined in this way allow us to start developing the orbit method for the infinite-dimensional "nilpotent" group $B_0^{\mathbb{Z}} = \varinjlim_n G_{2n-1}$ of doubly infinite matrices.

To find an appropriate completion $\widetilde{X}$ of the space X and extend the representation S of the subgroup H to its completion $\widetilde{H}$ it is necessary to use a family of *Hilbert–Lie groups* $B_2(a)$, $a \in \mathfrak{A}$, introduced by the author in [80]. These groups $B_2(a)$ are completions of the group $B_0^{\mathbb{Z}}$ in an appropriate Hilbert topology. This family has the property that any continuous representation U of the group $B_0^{\mathbb{Z}}$ can be extended by continuity to some representation $U_2(a)$ of an appropriate Hilbert–Lie group $B_2(a)$. See Subsection 7.3.3 for detail.

In Chapter 8 we summarize what we know about $\widehat{G}$ for the three groups $B_0^{\mathbb{N}}$, $B_0^{\mathbb{Z}}$, and $\mathrm{GL}_0(2\infty, \mathbb{R})$. For the group $G = \varinjlim_n G_n$, where $G_n = B(n, \mathbb{R})$, we have $\widehat{G} \supset \bigcup_n \widehat{G_n}$ since there are natural projections $G_{n+1} \to G_n$. The sets $\widehat{G_n}$ are known due to the Kirillov's orbit method. We show that $\widehat{G}\backslash\bigcup_n \widehat{G_n} \neq \emptyset$. Namely, $\widehat{G}\backslash\bigcup_n \widehat{G_n}$ contains irreducible "elementary", "regular", "quasi-regular", and "non-local" induced representations of the group G. A precise description of $\widehat{G} \setminus \bigcup_n \widehat{G_n}$ is not known. On the other hand, for all inductive limits $G = \varinjlim_n G_n$ of matrix groups G_n we have $\widehat{G} = \bigcup_{a\in\mathfrak{A}_G} \widehat{G_2(a)}$, where $G_2(a)$ is a Hilbert–Lie group corresponding to the initial group G and some weight $a \in \mathfrak{A}_G$ (see the details below). So it remains to find $\widehat{G_2(a)}$, $a \in \mathfrak{A}_G$; this problem is still open.

In Chapter 9, we study the quasi-regular representations of the infinite-dimensional "nilpotent" group $B_0^{\mathbb{N}}(\mathbb{F}_p)$ over the finite field $\mathbb{F}_p$. The corresponding G-space X is similar to the space used in Chapter 2. The corresponding measures on X are infinite tensor products of arbitrary measures on $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. We obtain irreducibility criteria for quasi-regular representations, but some new additional conditions on the corresponding measure (if we compare them with the Ismagilov conjecture) must be imposed to have irreducibility in this case. The reason is that in the case of a compact field some additional operators appear in the commutant.

In Chapter 10, we find irreducibility criteria for Koopman representations of the group $G = \mathrm{GL}_0(2\infty, \mathbb{R}) = \varinjlim_n \mathrm{GL}(2n-1, \mathbb{R})$, the inductive limit of the general linear groups (see [105]). Thus, we prove Conjecture 0.0.8 in this case. The corresponding measures are infinite tensor products of arbitrary one-dimensional Gaussian non-centered measures. The corresponding G-space X_m is a subspace of the space $\mathrm{Mat}(2\infty, \mathbb{R})$ of doubly infinite real matrices. The space X_m is a collection of m rows.

In Chapter 11, we give examples of regular representations for non-matrix groups. Namely, we consider the group of the diffeomorphisms of the interval, of the circle, the group of local diffeomorhisms of the real line, and the group G^X of smooth mappings of a Riemannian manifold into a compact Lie group G, for the simplest example $X = [0, 1]$.

In Chapter 12, we show how to solve the problem of finding a triple $(\widetilde{G}, G, \mu)$, Problem 0.0.1, for an arbitrary infinite-dimensional group G. For this we consider a Gaussian measure μ_B in a Hilbert space H and its subspace of admissible shifts H_0. The properties of the triple (H, H_0, μ_B) make it a good model for a general triple $(\widetilde{G}, G, \mu)$. In addition we discuss the analogue of the C^*-group algebra for infinite-dimensional groups.

18. Which readership we have in mind? The book is addressed to graduate students with a good background in measure theory and representation theory of locally compact groups. More precisely, a familiarity with the following material is required if one wishes to understand the book:

1) *measure theory* (Gaussian measures in Hilbert spaces, measures on infinite products of spaces, equivalence, singularity, and ergodicity of measures, Hellinger integral), see [32, 108, 166];
2) *representation theory of locally compact groups* (Haar measure, regular, quasi-regular, induced representation, irreducibility, equivalence of unitary representations) see [35, 74, 188],
3) the *orbit method* developed by A. Kirillov [75] for finite-dimensional nilpotent groups, but we explain everything necessary here. In Chapter 7 we generalize the orbit method for infinite-dimensional "nilpotent" groups,
4) the *finite field* $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, where p is a prime, is used only in Chapter 9.
5) Chapter 6, dealing with the *von Neumann algebras*, requires more: algebras of operators on a Hilbert space, factors, type of factors, especially A. Connes' *classification of type* III_1 *factors*, see [29, 172, 173, 174].

All the notions used in the book are defined in Introduction and preliminaries, Chapter 1. But to understand the sections dealing with the von Neumann algebras and with other subjects, some experience and patience are required. This is a rather new field, there are no general theorems, so the proofs are usually complicated. But if the reader makes some additional efforts, he will be richly rewarded! There are plenty of new phenomena that do not arise for locally compact groups. One example: the regular representation of an infinite-dimensional group can be irreducible, which never happens for a locally compact group, except for the trivial one! Another example: the regular representation can be irreducible, being the inductive limit of the regular (hence reducible) representations of locally compact groups. The non-equivalent measures parametrize the description of the dual $\widehat{G}$.

19. The contents of the book in one table. The main conjecture:
Ismagilov's conjecture. *The right regular representation of the infinite-dimensional group G is irreducible if and only if*

(i) $\mu^{L_s} \perp \mu$, for all $s \in G\backslash e$,
(ii) μ is G-right ergodic.

We can present almost all objects and problems we treat in the book in the following table:

		G loc.comp.	$\dim G=\infty$ non loc.comp.
1	Haar measure	$\exists!$ measure h on group G: $h^{R_t}=h\;\forall t\in G$	$(\widetilde{G},G,\mu):\mu^{R_t}\sim\mu^{L_t}\sim\mu\;\forall t\in G$ G is dense subgroup in $\widetilde{G}$ Problem 0.0.1
2	regular representation	$\rho,\lambda:G\to U(L^2(G,dh))$, $(\rho_t f)(x)=f(xt)$, $(\lambda_s f)(x)=\left(\frac{dh(s^{-1}x)}{dh(x)}\right)^{1/2}f\left(s^{-1}x\right)$	$T^{R,\mu},T^{L,\mu}:G\to U(L^2(\widetilde{G},\mu))$ $\left(T_t^{R,\mu}f\right)(x)=\left(\frac{d\mu(xt)}{d\mu(x)}\right)^{1/2}f(xt)$
3	reducibility	$[\rho_t,\lambda_s]=0\;\;\forall t,s\in G$	$\left[T_t^{R,\mu},T_s^{L,\mu}\right]=0\;\;\forall t,s\in G$
4	Ismagilov's conjecture (1985)		Conjecture 0.0.7
5	quasi-regular rep.	$T:G\to U(L^2(X,\mu))$, $X=H\backslash G$	$T:G\to U(L^2(\widetilde{X},\mu))$, $\widetilde{X}=\widetilde{H}\backslash\widetilde{G}$
6	Hilbert–Lie groups $\mathrm{GL}_2(a)$, $a\in\mathfrak{A}_{\mathrm{GL}}$,	$\bigcap_{a\in\mathfrak{A}_{\mathrm{GL}}}\mathrm{GL}_2(a)=\mathrm{GL}_0(2\infty,\mathbb{R})$	$\forall\,U:\mathrm{GL}_0(2\infty,\mathbb{R})\to U(H)\;\exists a\in\mathfrak{A}_{\mathrm{GL}}:U_2(a):\mathrm{GL}_2(a)\to U(H)$
7	induced representation	$H\subset G$, $S:H\to U(V)$, $\mathrm{Ind}_H^G S:G\to U(L^2(X,V,\mu))$	$H\subset G$, $S:H\to U(V)$, $\mathrm{Ind}_H^G S:G\to U(L^2(\widetilde{X},V,\mu))$, $\widetilde{S}:\widetilde{H}\to U(V)$
8	Koopman's representation	$\alpha:G\to\mathrm{Aut}(X,\mu)$, $\mu^{\alpha_t}\sim\mu$, $\pi:G\to U(L^2(X,\mu))$	Conjecture 0.0.8
9	von Neumann algebras, factors	$\mathfrak{A}^\rho(G)=(\rho_t\mid t\in G)''$, $\mathfrak{A}^\lambda(G)=(\lambda_s\mid s\in G)''$, $\mathfrak{A}^\rho(G)$ is type II_1 factor iff G is discret ICC group	$\mathfrak{A}^{R,\mu}(G)=\left(T_t^{R,\mu}\mid t\in G\right)''$, $\mathfrak{A}^{L,\mu}(G)=\left(T_t^{L,\mu}\mid t\in G\right)''$. When $\mathfrak{A}^{R,\mu}(G)$ is factor?
10	commutation theorem	$\mathfrak{A}^\rho(G)'=\mathfrak{A}^\lambda(G)$, $M=\mathfrak{A}^\rho(G)$	When $\mathfrak{A}^{R,\mu}(G)'=\mathfrak{A}^{L,\mu}(G)$? $M=\mathfrak{A}^{R,\mu}(G)$
11	Tomita–Takesaki theory	$h(x^{-1})\sim h(x)$, $\Delta(x)=dh(x^{-1})/dh(x)$	When $\mu(x^{-1})\sim\mu(x)$? $\Delta_\mu(x)=d\mu(x^{-1})/d\mu(x)$
12	canonical conjugation operator	$(Jf)(x)=\Delta^{1/2}(x)\overline{f(x^{-1})}$ $J\rho_t J=\lambda_t\;\forall t\in G$, $JMJ=M'$,	$(J_\mu f)(x)=\Delta_\mu^{1/2}(x)\overline{f(x^{-1})}$, $J_\mu T_t^{R,\mu}J_\mu=T_t^{L,\mu}\;\forall t\in G$, when $J_\mu MJ_\mu=M'$?
13	dual for $G=\varinjlim_n G_n$? $G=B_0^{\mathbb{N}}$, $B_0^{\mathbb{Z}}$, $\mathrm{GL}_0(2\infty,\mathbb{R})$		$\widehat{G}=\bigcup_{a\in\mathfrak{A}_G}\widehat{G_2(a)}$ where $G_2(a)$ is a corr. Hilbert–Lie group, $\widehat{G_2(a)}$?
14	$C^*(G)$-group algebra	$C^*_{red}(G)=(\lambda_t\mid t\in G)''$	$C^*_{red,\mu}(G)=\left(T_t^{R,\mu}\mid t\in G\right)''$
15	inductive limit representation		$G=\varinjlim_n G_n$, $T_n:G_n\to U(H_n)$ $i_{n+1}^n:H_n\to H_{n+1}$ define repres.? $T=\varinjlim_n T_n$, Hilb. space? $\varinjlim_{n,i}H_n$

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In his report on my PhD thesis Rais Ismagilov conjectured that the right regular representation of an infinite-dimensional nilpotent group constructed in the thesis can be irreducible if no left actions are admissible for the measure in question. This remark helped me find my way in mathematics. Despite the fact that this remark has never been published by him, I called it the Ismagilov conjecture, in order to honor his deep observation and his contribution to the subject. I am extremely grateful to Rais Ismagilov.

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