

Introduction

The Erlangen program is a perspective on geometry through invariants of the automorphism group of a space. The original reference to this program is a paper by Felix Klein which is usually presented as the exclusive historical document in this matter. Even though Klein's viewpoint was generally accepted by the mathematical community, its re-interpretation in the light of modern geometries, and especially of modern theories of physics, is central today. There are no books on the modern developments of this program. Our book is one modest step towards this goal.

The history of the Erlangen program is intricate. Klein wrote this program, but Sophus Lie made a very substantial contribution, in promoting and popularizing the ideas it contains. The work of Lie on group actions and his emphasis on their importance were certainly more decisive than Klein's contribution. This is why Lie's name comes first in the title of the present volume. Another major figure in this story is Poincaré, and his role in highlighting the importance of group actions is also critical.

Thus, groups and group actions are at the center of our discussion. But their importance in mathematics had already been crucial before the Erlangen program was formulated.

From its early beginning in questions related to solutions of algebraic equations, group theory is merged with geometry and topology. In fact, group actions existed and were important before mathematicians gave them a name, even though the formalization of the notion of a group and its systematic use in the language of geometry took place in the 19th century. If we consider group theory and transformation groups as an abstraction of the notion of symmetry, then we can say that the presence and importance of this notion in the sciences and in the arts was realized in ancient times.

Today, the notion of group is omnipresent in mathematics and, in fact, if we want to name one single concept which runs through the broad field of mathematics, it is the notion of group. Among groups, Lie groups play a central role. Besides their mathematical beauty, Lie groups have many applications both inside and outside mathematics. They are a combination of algebra, geometry and topology.

Besides groups, our subject includes geometry.

Unlike the word "group" which, in mathematics has a definite significance, the word "geometry" is not frozen. It has several meanings, and all of them (even the most recent ones) can be encompassed by the modern interpretation of Klein's idea. In the first version of Klein's Erlangen program, the main geometries that are emphasized are projective geometry and the three constant curvature geometries (Euclidean, hyperbolic and spherical), which are considered there, like affine geometry, as part of projective geometry. This is due to the fact that the transformation groups of all these geometries can be viewed as restrictions to subgroups of the transformation group of projective geometry. After these first examples of group actions in geometry, the stress shifted to Lie transformation groups, and it gradually included many new notions, like Riemannian manifolds, and more generally spaces equipped

with affine connections. There is a wealth of geometries which can be described by transformation groups in the spirit of the Erlangen program. Several of these geometries were studied by Klein and Lie; among them we can mention Minkowski geometry, complex geometry, contact geometry and symplectic geometry. In modern geometry, besides the transformations of classical geometry which take the form of motions, isometries, etc., new notions of transformations and maps between spaces arose. Today, there is a wealth of new geometries that can be described by transformation groups in the spirit of the Erlangen program, including modern algebraic geometry where, according to Grothendieck's approach, the notion of morphism is more important than the notion of space.¹ As a concrete example of this fact, one can compare the Grothendieck–Riemann–Roch theorem with the Hirzebruch–Riemann–Roch. The former, which concerns morphisms, is much stronger than the latter, which concerns spaces.

Besides Lie and Klein, several other mathematicians must be mentioned in this venture. Lie created Lie theory, but others' contributions are also immense. About two decades before Klein wrote his Erlangen program, Riemann had introduced new geometries, namely, in his inaugural lecture, *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (On the hypotheses which lie at the bases of geometry) (1854). These geometries, in which groups intervene at the level of infinitesimal transformations, are encompassed by the program. Poincaré, all across his work, highlighted the importance of groups. In his article on the Future of mathematics², he wrote: "Among the words that exerted the most beneficial influence, I will point out the words group and invariant. They made us foresee the very essence of mathematical reasoning. They showed us that in numerous cases the ancient mathematicians considered groups without knowing it, and how, after thinking that they were far away from each other, they suddenly ended up close together without understanding why." Poincaré stressed several times the importance of the ideas of Lie in the theory of group transformations. In his analysis of his own works,³ Poincaré declares: "Like Lie, I believe that the notion, more or less unconscious, of a continuous group is the unique logical basis of our geometry." Killing, É. Cartan, Weyl, Chevalley and many others refined the structures of Lie theory and they developed its global aspects and applications to homogeneous spaces. The generalization of the Erlangen program to these new spaces uses the notions of connections and gauge groups, which were

¹See A. Grothendieck, Proceedings of the International Congress of Mathematicians, 14–21 August 1958, Edinburgh, ed. J.A. Todd, Cambridge University Press, p. 103–118. In that talk, Grothendieck sketched his theory of cohomology of schemes.

²H. Poincaré, *L'Avenir des mathématiques*, Revue générale des sciences pures et appliquées 19 (1908) p. 930–939. [Parmi les mots qui ont exercé la plus heureuse influence, je signalerai ceux de groupe et d'invariant. Ils nous ont fait apercevoir l'essence de bien des raisonnements mathématiques ; ils nous ont montré dans combien de cas les anciens mathématiciens considéraient des groupes sans le savoir, et comment, se croyant bien éloignés les uns des autres, ils se trouvaient tout à coup rapprochés sans comprendre pourquoi.]

³*Analyse de ses travaux scientifiques, par Henri Poincaré*. Acta Mathematica, 38 (1921), p. 3–135. [Comme Lie, je crois que la notion plus ou moins inconsciente de groupe continu est la seule base logique de notre géométrie]; p. 127. There are many similar quotes in Poincaré's works.

closely linked to new developments in physics, in particular, in electromagnetism, phenomena related to light, and Einstein's theory of general relativity.

Today, instead of the word "geometry" we often use the expression "geometric structure", and there is a wealth of geometric structures which can be described by transformation groups in the spirit of the Erlangen program. We mention in particular the notion of (G, X) structure introduced by Charles Ehresmann in the 1930s, which is of paramount importance. Here X is a homogeneous space and G a Lie group acting transitively on X . A (G, X) structure on a manifold M is then an atlas whose charts are in X and whose coordinate changes are restrictions of elements of G acting on X . Ehresmann formulated the notions of developing map and of holonomy transformations, which are basic objects in the study of these structures and their moduli spaces. (G, X) structures have several variants and they have been developed and adapted to various settings by Haefliger, Kuiper, Benzécri, Thurston, Goldman and others to cover new structures, including foliations and singular spaces. The most spectacular advancement in this domain is certainly Thurston's vision of the eight geometries in dimension three, his formulation of the geometrization conjecture and the work around it, which culminated in the proof of the Poincaré conjecture by Perelman.

We talked about mathematics, but the Erlangen program also encompasses physics. In fact, geometry is closely related to physics, and symmetry is essential in modern physics. Klein himself investigated the role of groups in physics, when he stressed the concept of geometric invariants in his description of Einstein's theories of special and general relativity, in particular by showing the importance of the Lorentz group, and also in his work on the conservation laws of energy and momentum in general relativity. Another milestone that led to conceptual clarifications and made it possible to systematically exploit the notion of symmetry in physics was E. Noether's work that related symmetries of physical systems to conserved quantities.

In conclusion, the central questions that are behind the present volume are:

- What is geometry?
- What is the relation between geometry and physics?
- How are groups used in physics, especially in contemporary physics?

Let us now describe briefly the content of this volume.

Chapters 1 and 2, written by Lizhen Ji, are introductions to the lives and works on Lie and Klein. Even though Klein was a major mathematician, surprisingly enough, there is no systematic English biography of him. The author's aim is to fill this gap to a certain extent. Besides providing convenient short biographies of Lie and Klein, the author wishes to convince the reader of the importance of their works, especially those which are in close relation with the Erlangen program, and also to show how close the two men were in their ideas and characters. They both learned from each other and they had a profound influence on each other. This closeness, their ambitiousness, the competition among them and their disputes for priority of some discoveries were altogether the reasons that made them split after years of collaboration and friendship. The conflict between them is interesting and not so well known. The author describes this conflict, also mentions the difficulties that these two men encountered

in their professional lives and in their relations with other mathematicians. Both of them experienced nervous breakdowns.⁴ The chapter on Lie also contains an outline of his important theories as well as statements of some of his most significant theorems. In particular, the author puts forward in modern language and comments on three fundamental theorems of Lie. Concerning Klein, it is more difficult to pick out individual theorems, because Klein is known for having transmitted ideas rather than specific results. The author explains how Klein greatly influenced people and the world around him through his lectures and conversations, his books, the journals he edited, and he also recalls his crucial influence in shaping up the university of Göttingen to be the world's most important mathematics center. In these surveys, the author also mentions several mathematicians who were closely related in some way or another to Lie and Klein, among them Hilbert, Hausdorff, Engel, Plücker, Sylow, Schwarz and Poincaré. The chapter on Lie also reviews other aspects of Lie's work besides Lie groups.

Chapter 3, by Jeremy Gray, is a historical commentary on the Erlangen program. The author starts by a short summary of the program manifesto and on the circumstances of its writing, mentioning the influence of several mathematicians, and the importance of the ideas that originate from projective geometry (specially those of von Staudt). He then brings up the question of the impact of this program on the views of several mathematicians, comparing the opinions of Birkhoff and Bennet and of Hawkins.

In Chapter 4, Hubert Goenner presents a critical discussion of the general impact and of the limitations of the Erlangen program in physics. He starts by recalling that the influence of the Erlangen program in physics was greatly motivated by the geometrization of special relativity by H. Minkowski, in which the Lorentz group appears as one of the main objects of interest, but he stresses the fact that the notion of *field* defined on a geometry – and not the notion of geometry itself – is then the central element. He comments on the relation of Lie transformations with theories of conservation laws and the relations of the Erlangen program with symplectic geometry, analytical mechanics, statistical physics, quantum field theories, general relativity, Yang–Mills theory and supergravity. The paper has a special section where the author discusses supersymmetry. In a final section, the author mentions several generalizations of the notion of Lie algebra.

In Chapter 5, Norbert A'Campo and Athanase Papadopoulos comment on the two famous papers of Klein, *Über die sogenannte Nicht-Euklidische Geometrie* (On the so-called non-Euclidean geometries), *I* and *II*. The two papers were written respectively one year and a few months before the Erlangen program, and they contain in essence the main ideas of this program. We recall that the 19th century saw the birth of non-Euclidean geometry by Lobachevsky, Bolyai and Gauss, and at the same time, the development of projective geometry by Poncelet, Plücker, von Staudt and others, and also of conformal geometry by Liouville and others. Groups made the first link between all these geometries, and also between geometry and algebra. Klein, in the

⁴Klein's nervous breakdown was probably due to overwork and exhaustion, caused in part by his rude competition with Poincaré on Fuchsian functions, whereas Lie's nervous breakdown was the consequence of a chronic illness, pernicious anemia, related to a lack in vitamin B12, which at that time was incurable.

papers cited above, gives models of the three constant-curvature geometries (hyperbolic, Euclidean and spherical) in the setting of projective geometry. He defines the distance functions in each of these geometries by fixing a conic (the “conic at infinity”) and taking a constant multiple of the logarithm of the cross ratio of four points: the given two points and the two intersection points of the line joining them with the conic at infinity. The hyperbolic and spherical geometries are obtained by using real and complex conics respectively, and Euclidean geometry by using a degenerate one. The authors in Chapter 5 comment on these two important papers of Klein and they display relations with works of other mathematicians, including Cayley, Beltrami, Poincaré and the founders of projective geometry.

Klein’s interaction with Lie in their formative years partly motivated Lie to develop Lie’s version of Galois theory of differential equations and hence of Lie transformation theory.⁵ In fact, a major motivation for Lie for the introduction of Lie groups was to understand differential equations. This subject is treated in Chapter 6 of this volume. The author, Alexandre Vinogradov, starts by observing that Lie initiated his work by transporting the Galois theory of the solvability of algebraic equations to the setting of differential equations. He explains that the major contribution of Lie in this setting is the idea that symmetries of differential equations are the basic elements in the search for their solutions. One may recall here that Galois approached the problem of solvability of polynomial equations through a study of the symmetries of their roots. This is based on the simple observation that the coefficients of a polynomial may be expressed in terms of the symmetric functions of their roots, and that a permutation of the roots does not change the coefficients of the polynomial. In the case of differential equations, one can naively define the symmetry group to be the group of diffeomorphisms which preserve the space of solutions, but it is not clear how such a notion can be used. There is a *differential Galois theory* which is parallel to the Galois theory of polynomial equations. In the differential theory, the question “what are the symmetries of a (linear or nonlinear, partial or ordinary) differential equation?” is considered as the central question. Chapter 6 also contains reviews of the notions of jets and jet spaces and other constructions to explain the right setup for formulating the question of symmetry, with the goal of providing a uniform framework for the study of nonlinear partial differential equations. The author is critical of the widely held view that each nonlinear partial differential equation arising from geometry or physics is special and often requires its own development. He believes that the general approach based on symmetry is the right one.

The author mentions developments of these ideas that were originally formulated by Lie and Klein in works of E. Noether, Bäcklund, É. Cartan, Ehresmann and others. A lot of questions in this domain remain open, and this chapter will certainly give the reader a new perspective on the geometric theory of nonlinear partial differential equations.

In Chapter 7, Charles Frances surveys the modern developments of geometric structures on manifolds in the lineage of Klein and Lie. The guiding idea in this

⁵Lie has had a course at Oslo by Sylow on Galois and Abel theory before he meets Klein, but it is clear that Klein also brought some of his knowledge to Lie.

chapter is the following question: When is the automorphism group of a geometric structure a Lie group, and what can we say about the structure of such a Lie group?

The author considers the concept of *Klein geometry*, that is, a homogeneous space acted upon by a Lie group, and a generalization of this notion, leading to the concept of a *Cartan geometry*. (Cartan used the expression *espace généralisé*.) Besides the classical geometries, like constant curvature spaces (Euclidean, Lobachevsky and spherical) as well as projective geometry which unifies them, the notion of Cartan geometry includes several differential-geometric structures. These notions are defined using fiber bundles and connections. They describe spaces of variable curvature and they also lead to pseudo-Riemannian manifolds, conformal structures of type (p, q) , affine connections, CR structures, and the so-called *parabolic geometries*. The author presents a series of important results on this subject, starting with the theorem of Myers and Steenrod (1939) saying that the isometry group of any Riemannian manifold is a Lie group, giving a bound on its dimension, and furthermore, it says that this group is compact if the manifold is compact. This result gave rise to an abundance of developments and generalizations. The author also explains in what sense pseudo-Riemannian manifolds, affine connections and conformal structures in dimensions ≥ 3 are rigid, symplectic manifolds are not rigid, and complex manifolds are of an intermediate type.

Thus, two general important questions are addressed in this survey:

- What are the possible continuous groups that are the automorphism groups of a geometry on a compact manifold?
- What is the influence of the automorphism group of a structure on the topology or the diffeomorphism type of the underlying manifold?

Several examples and recent results are given concerning Cartan geometries and in particular pseudo-Riemannian conformal structures.

Chapter 8, by Norbert A'Campo and Athanase Papadopoulos, concern transitional geometry. This is a family of geometries which makes a continuous transition between hyperbolic and spherical geometry, passing through Euclidean geometry. The space of transitional geometry is a fiber space over the interval $[-1, 1]$ where the fiber above each point t is a space of constant curvature t^2 if $t > 0$ and of constant curvature $-t^2$ if $t < 0$. The fibers are examples of *Klein geometries* in the sense defined in Chapter 7. The elements of each geometry are defined group-theoretically, in the spirit of Klein's Erlangen program. Points, lines, triangles, trigonometric formulae and other geometric properties transit continuously between the various geometries.

In Chapter 9, by Athanase Papadopoulos and Sumio Yamada, the authors introduce a notion of cross ratio which is proper to each of the three geometries: Euclidean, spherical and hyperbolic. This highlights the relation between projective geometry and these geometries. This is in the spirit of Klein's view of the three constant curvature geometries as part of projective geometry, which is the subject of Chapter 5 of the present volume.

Chapter 10, by Yuri Suris, concerns the Erlangen program in the setting of discrete differential geometry. This is a subject which recently emerged, whose aim is to develop a theory which is the discrete analogue of classical differential geometry.

It includes discrete versions of the differential geometry of curves and surfaces but also higher-dimensional analogues. There are discrete notions of line, curve, plane, volume, curvature, contact elements, etc. There is a unifying transformation group approach in discrete differential geometry, where the discrete analogues of the classical objects of geometry become invariants of the respective transformation groups. Several classical geometries survive in the discrete setting, and the author shows that there is a discrete analogue of the fact shown by Klein that the transformation groups of several geometries are subgroups of the projective transformation group, namely, the subgroup preserving a quadric.

Examples of discrete differential geometric geometries reviewed in this chapter include discrete line geometry and discrete line congruence, quadrics, Plücker line geometry, Lie sphere geometry, Laguerre geometry and Möbius geometry. Important notions such as curvature line parametrized surfaces, principal contact element nets, discrete Ribeaucour transformations, circular nets and conical nets are discussed. The general underlying idea is that the notion of transformation group survives in the discretization process. Like in the continuous case, the transformation group approach is at the same time a unifying approach, and it is also related to the question of “multi-dimensional consistency” of the geometry, which says roughly that a 4D consistency implies consistency in all higher dimensions. The two principles – the transformation group principle and consistency principle – are the two guiding principles in this chapter.

Chapter 11 by Catherine Meusburger is an illustration of the application of Klein’s ideas in physics, and the main example studied is that of three-dimensional gravity, that is, Einstein’s general relativity theory⁶ with one time and two space variables. In three-dimensions, Einstein’s general relativity can be described in terms of certain domains of dependence in three-dimensional Minkowski, de Sitter and anti de Sitter space, which are homogeneous spaces. After a summary of the geometry of spacetimes and a description of the gauge invariant phase spaces of these theories, the author discusses the question of quantization of gravity and its relation to Klein’s ideas of characterizing geometry by groups.

Besides presenting the geometrical and group-theoretical aspects of three-dimensional gravity, the author mentions other facets of symmetry in physics, some of them related to moduli spaces of flat connections and to quantum groups.

Chapter 12, by Jean-Bernard Zuber, is also on groups that appear in physics, as group invariants associated to a geometry. Several physical fields are mentioned, including crystallography, piezoelectricity, general relativity, Yang–Mills theory, quantum field theories, particle physics, the physics of strong interactions, electromagnetism, sigma-models, integrable systems, superalgebras and infinite-dimensional algebras. We see again the work of Emmy Noether on group invariance principles in variational problems. Representation theory entered into physics through quantum mechanics, and the modern theory of quantum group is a by-product. The author comments on Noether’s celebrated paper which she presented at the occasion of Klein’s academic Jubilee. It contains two of her theorems on conservation laws.

⁶We recall by the way that Galileo’s relativity theory is at the origin of many of the twentieth century theories.

Today, groups are omnipresent in physics, and as Zuber puts it: “To look for a group invariance whenever a new pattern is observed has become a second nature for particle physicists”.

We hope that the various chapters of this volume will give to the reader a clear idea of how group theory, geometry and physics are related to each other, the Erlangen program being a major unifying element in this relation.

Lizhen Ji and Athanase Papadopoulos