Preface

Generalizing the theory of classical Teichmüller spaces of compact Riemann surfaces, Lipman Bers introduced the notion of universal Teichmüller space (cf. [4], [5], [6]). It is an infinite-dimensional complex manifold containing all classical Teichmüller spaces as complex submanifolds which explains the name "universal Teichmüller space". This space, denoted by $\mathcal{T}$, is defined as the quotient of the space of quasisymmetric homeomorphisms of the unit circle modulo Möbius group of fractional-linear automorphisms of the unit disk. Recall that a homeomorphism of the circle is called quasisymmetric if it can be extended into the disk as a quasiconformal map.

Apart from the classical Teichmüller spaces, $\mathcal{T}$ contains the space $\mathcal{S}$ of diffeomorphisms of the circle, considered up to Möbius transformations. Both spaces $\mathcal{T}$ and $\mathcal{S}$ are the important examples of infinite-dimensional complex manifolds, having a rich geometric structure. Both of them can be embedded into the "lower hemisphere" of an infinite-dimensional Grassmann manifold and so may be considered as infinite-dimensional analogues of the unit disk. In particular, the space $\mathcal{S}$ can be provided with a Kähler metric, invariant under the action of diffeomorphisms of the circle. This metric has a correctly defined Ricci curvature which is given in a suitable basis by an infinite diagonal matrix. Similar to the unit disk, this curvature is negative meaning that the coefficient of the leading term is equal to $-\frac{26}{12}$. Precisely, this number 26 arises in the bosonic string theory as the "critical dimension" of the Minkowski space where the strings "live".

In fact, relations with string theory were our main motivation to study the spaces $\mathcal{S}$ and $\mathcal{T}$. Recall that $\mathcal{S}$ is the quotient of the diffeomorphism group of the circle modulo Möbius group and the group of diffeomorphisms acts on the strings by the reparameterization. This explains a relation of $\mathcal{S}$ with the theory of smooth strings. The same role is played by the space $\mathcal{T}$ for the theory of non-smooth strings. Again $\mathcal{T}$ is the quotient of the group of quasisymmetric homeomorphisms of the circle modulo Möbius group, acting on half-differentiable strings by the reparameterization.

We note that there are no physical reasons to restrict ourselves to smooth strings, rather such a restriction is explained by the mathematical convenience of working with smooth objects. From our point of view it is much more natural to choose the Sobolev space of half-differentiable vector-functions for the phase manifold of the string theory. Such a choice is motivated by the fact that this space is the largest among all Sobolev spaces on which the symplectic form of string theory is still correctly defined. (We discuss a relation between the universal Teichmüller space and string theory in the afterword to this book.)

To quantize the string theory with the phase manifold given by the space of half-differentiable vector-functions, we have to quantize the space $\mathcal{T}$. Its quantization is considered in the second part of the book. In contrast with the case of the space $\mathcal{S}$, which can be quantized in frames of the conventional Dirac scheme (cf. [27]), the quantization of $\mathcal{T}$ requires dealing with "non-smooth" objects to which the Dirac scheme does not apply. In order to quantize the universal Teichmüller space, we have to use a completely different approach motivated by considerations from the non-commutative geometry.

This book is based on the lecture notes of the course delivered by the author to the students of the Scientific Educational Center of Steklov Mathematical Institute during the spring semester of the year 2011 (the Russian edition of the lecture notes was published by the Steklov Institute in 2013 [26]). Despite the fact that this book is a lecture course, divided into separate lectures, we think of these lectures more like of separate topics rather than "real" lectures (this explains the difference in their volumes). We have included in this book many problems which have been proposed to the listeners of the course. They are not divided into exercises and "difficult" problems, we hope that a reader will understand the difference by himself. We only guarantee that among them there are no unsolved problems.

At the end I want to express my gratitude to Alastair Fletcher and Manfred Karbe who contributed a lot to the publication of this book in the European Lecture Notes Series. I am also grateful to Irene Zimmermann for the excellent typesetting of the text. Last but not least I want to thank all listeners of the course for their questions and remarks

Moscow, July 2014 Armen Sergeev