Introduction to Teichmüller theory, old and new, IV

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Contents

Like the introductions to the three previous volumes of this Handbook, the present introduction will give the reader an overview of the content of the present volume, and at the same time it will present some classical and recent developments of Teichmüller theory. One of the most fascinating features of this theory is the rich interaction that it establishes between different branches of mathematics: analysis, geometry, dynamics, arithmetic, algebraic geometry, algebraic topology, theoretical physics, etc., and this is exemplified in the various chapters of the volume. Readers will sometimes find here a systematic exposition of a subject that was only sketched in the preceding volumes, and sometimes they will find an exposition of a new subject, or of an application or a connection between Teichmüller theory and some field in mathematics that was or was not treated in the preceding volumes. The chapters differ in length and level of difficulty. Some of them contain proofs and others contain only references to papers that contain proofs, but all the chapters contain motivating material and numerous examples. Several chapters also contain open problems and programs of research.

In the following sections, we consider the chapters separately, we give an overview of each of them and we comment on each subject treated.

I would like to thank Manfred Karbe who read this introduction and made several useful stylistic remarks.

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1.1 Weil–Petersson geometry

The first chapter, written by Sumio Yamada, concerns the Weil–Petersson metric of Teichmüller space. It complements the chapter by Wolpert on the same subject in Volume II of this Handbook.

In this chapter, Yamada starts by studying the local Riemannian structure of the Weil–Petersson metric and he then gives a proof of the Weil–Petersson convexity of the energy of harmonic maps which updates a version that he published a few years ago. He also carefully describes the Weil–Petersson metric as the induced metric from the L^2 pairing of symmetric $(0,2)$ tensors, which constitute the tangent space of the space of all smooth metrics. This work was initiated by Fischer–Marsden and Fischer–Tromba, and Yamada gives a careful description of the L^2-decomposition result, which he calls a Hodge-type theorem because the harmonic part is picked as the intersection of kernels of two linear elliptic operators.

A well-known result of Masur says that the Weil–Petersson completion of the Teichmüller space of a surface S coincides, as a set, with the augmented Teichmüller space, that is, the space obtained by adding to Teichmüller space spaces of nodal marked surfaces obtained by pinching to a point a certain collection of disjoint essential simple closed curves on S. It is also known that the Weil–Petersson completion of the Weil–Petersson metric (which is a Riemannian metric) is not Riemannian at points on the boundary but that it has a $\mathrm{CAT}(0)$-geometry. Chapter 1 contains a survey on this $\mathrm{CAT}(0)$-geometry. The $\mathrm{CAT}(0)$ study of augmented Teichmüller space was initiated by Yamada, and the $\mathrm{CAT}(0)$ geometric techniques in some sense remedy the fact that the Weil–Petersson metric is not complete. The author then discusses metric incompleteness of Teichmüller space versus its geodesic incompleteness. While we have a nice model for the metric completion of the Weil–Petersson metric of Teichmüller space, namely, the augmented Teichmüller space, a model for the geodesic Weil–Petersson completion is proposed by Yamada's Weil–Petersson–Coxeter–Teichmüller complex (which we call for simplicity the Coxeter complex), a complex whose basic simplex is the Weil–Petersson metric completion. This Coxeter complex is obtained by reflecting the Weil–Petersson metric completion along some of its boundary strata.

The construction is based on a result of Wolpert stating that any two intersecting strata of the same dimension in the boundary of augmented space make right angles at their intersection, the angles being understood here in the sense of Alexandrov.

The chapter also contains a proof of the fact that the Teichmüller space of the torus (as a parameter space of marked flat structures) equipped with its Weil-Petersson metric is isometric to the 2-dimensional hyperbolic plane, a result which was inexistent in the literature. The proof given here uses harmonic maps.

The author also surveys the Weil–Petersson geometry of the universal Teichmüller space, based on works of Nag–Verjovsky and Takhtajian–Teo. There is an embedding of the Weil–Petersson Coxeter complex in the universal Teichmüller space. In the context of the universal space, the Weil–Petersson metric tensor is defined as a Hessian of the $\bar{\partial}$-energy of harmonic maps. This result was known for the compact surface case (shown by M. Wolf). Chapter 1 also contains a description of the metric completion of Teichmüller space as a stratified space, and of the expansion of the Weil–Petersson tensor near the boundary strata. This description involves the study of the behavior of the surface when a curve is pinched to a point.

In the last section, Yamada provides a description of Teichmüller space as a Weil–Petersson convex body in the Coxeter complex. Once this is done, he defines an asymmetric metric[1] (respectively a metric) on Teichmüller space, *via* a construction analogous to the one of the Funk asymmetric metric (respectively the Hilbert metric, which is a symmetrization of the Funk asymmetric metric) associated to an open convex subset of a Euclidean space. In this definition of the "Weil–Petersson–Funk metric", the boundary faces of the Weil–Petersson completion act as supporting hyperplanes to the convex body. The author points out relations between this Weil–Petersson–Funk metric, the Funk metric on convex sets, and Thurston's asymmetric metric on Teichmüller space. The embedding of the Weil–Petersson Coxeter complex in the universal Teichmüller space provides an interesting example of a geometric subspace of the universal space which is not the Teichmüller space of a surface.

1.2 Simple closed geodesics in Teichmüller theory

Chapter 2 by Hugo Parlier is the first of a series of chapters whose subject is the study of homotopy classes of simple closed curves on surfaces and how they appear in Teichmüller theory. The chapter concerns more particularly simple closed geodesics on hyperbolic surfaces. The study of the set of (homotopy classes of) simple closed curves is a vast subject, related in several ways to Teichmüller theory, and a lot of work on this theme was done by various people over a span of several decades. We start by recalling some of these works.

The relation between simple closed geodesics and Teichmüller space can be traced back to the works of Fricke, Klein and Vogt, done around the end of the 19th century. Further work on the subject was done by Dehn in the first quarter of the 20th century.

[1] An asymmetric metric is a structure that satisfies all the axioms of a metric except the symmetry action.

More recently, the relation between simple closed geodesics and Teichmüller theory was highlighted by Thurston in the 1970s, when he circulated his manuscript *On the geometry and dynamics of diffeomorphisms of surfaces* (written in 1976). With this work, hyperbolic geometry appeared at the forefront of Teichmüller theory, at the same level of importance as complex analysis. Simple closed geodesics play several roles in this and another manuscript of Thurston, *Minimal stretch maps between hyperbolic surfaces*, written in 1985.

We now review some precise aspects of the relation between simple closed geodesics and Teichmüller theory.

The first important fact that comes to mind in this respect is the fact that geodesic length functions can be used as local parameters for Teichmüller spaces. For surfaces of finite topological type, a finite set of curves suffices, and this is implicit in the work of Vogt, Fricke and Klein that we mentioned, although these authors did not formulate their results in terms of lengths of simple closed curves but in terms of traces of 2×2 matrices acting by isometries on some models of the hyperbolic plane. In these works, a set of parameters originating from these matrix actions was referred to as a set of "moduli". This point of view is treated by Goldman in Chapter 15 of Volume II of this Handbook.

The question of finding the minimal cardinality of a set of simple closed curves whose lengths parametrize (locally or globally) the Teichmüller space of a surface of finite type was thoroughly investigated in the 1990s, and one should mention here the names of Buser, Wolpert, Schmutz Schaller, Seppala-Sorvali, Hamenstädt, and there are certainly others. The question is now settled. In particular, it is known that the cardinality of a set of homotopy classes of simple closed curves whose associated length functions give *global* parameters for Teichmüller space must be at least one unit larger than the dimension of the space, and that such a set with this cardinality exists.

Let us now review other relations between geodesic length functions and Teichmüller spaces.

In 1985, Thurston gave the definition of a metric on Teichmüller space which is based on the length of a random closed geodesic. The idea behind Thurston's definition was that a geodesic length function on Teichmüller space is convex along earthquake paths and behaves in some sense like the square of a distance function. The second derivative at the minimum point of such a function can then be thought of as a metric tensor. Motivated by this idea, Thurston defined a metric on Teichmüller space using the concept of length of a random geodesic. Soon after that, Wolpert showed that this metric is nothing else than the Weil–Petersson metric.

Length functions of simple closed curves also play an important role in the description of the Weil–Petersson symplectic structure of Teichmüller space (works of Wolpert and of Goldman). In some sense, since length functions can be used to define the real and complex part of the Weil–Petersson Kähler metric, they can be used to define the complex structure of Teichmüller space since the complex structure is completely determined by the real and complex part of the metric.

There are other metrics and generalized metrics on Teichmüller space whose definitions use length functions of simple closed geodesics. Two important examples are the asymmetric metric which Thurston introduced in his manuscript *Minimal stretch maps between hyperbolic surfaces*, which we already mentioned, and its symmetrization, the *length spectrum metric*. Thurston's asymmetric metric appears in several chapters of this Handbook and, for the convenience of the reader, we now recall the definition.

If g and h are two complete hyperbolic metrics of finite area on a surface S, we set

$$K(g,h) = \log \sup_{\alpha \in \mathcal{S}} \frac{l_h(\alpha)}{l_g(\alpha)}, \tag{1.1}$$

where $\mathcal{S}$ is the set of homotopy classes of simple closed curves on S which are essential (that is, not homotopic to a point or to a puncture) and where, for each α in $\mathcal{S}$, $l_g(\alpha)$ denotes the length of the unique closed geodesic in the class α for the metric g.

The function K on the set of (homotopy classes of) pairs of hyperbolic metrics induces a function on $\mathcal{T}(S) \times \mathcal{T}(S)$ which satisfies all the axioms of a metric except the symmetry axiom. This is *Thurston's asymmetric metric* on Teichmüller space. Several results on this metric have been recently obtained by various people, in particular, on its geodesics, and on its isometries. Chapter 2 of Volume I of this Handbook, by Théret and the author of this introduction, contains some basic facts about this metric. New results are presented by Walsh in Chapter 7 of the present volume.

There is another asymmetric metric on Teichmüller space, the "dual" K' of K, defined by $K'(x,y) = K(y,x)$ for x and y in $\mathcal{S}$. The metrics K and K' have different characters, and much more is known about Thurston's asymmetric metric than about its dual.

Finally, we mention the *length spectrum metric*, which is a symmetrization of Thurston's asymmetric metric, defined by the formula

$$d(g,h) = \log \sup_{\alpha \in \mathcal{S}} \left\{ \frac{l_h(\alpha)}{l_g(\alpha)}, \frac{l_g(\alpha)}{l_h(\alpha)} \right\}.$$

For surfaces of finite type, this metric has been studied by Sorvali, Li, Shiga, Liu–Sun–Wei and others. It seems that the name "length spectrum metric" is due to Zhong Li.

Let us now mention other instances of the role played by the set $\mathcal{S}$ of homotopy classes of essential simple closed curves in Teichmüller theory. This set is in natural one-to-one correspondence with the set of simple closed geodesics as soon as a hyperbolic metric is chosen on the surface.

Kerckhoff found (1980) the following formula for the Teichmüller metric that is based on extremal length:

$$d(g,h) = \log \sup_{\alpha \in \mathcal{S}} \frac{\mathrm{Ext}_h(\alpha)}{\mathrm{Ext}_g(\alpha)}. \tag{1.2}$$

We recall that given a conformal structure H on the surface S, the *extremal length function* $\alpha \mapsto \mathrm{Ext}_H(\alpha)$ is defined on the set $\mathcal{S}$ of homotopy classes of simple closed curves. (In fact, it is defined on more general families of curves.) This function was introduced by Ahlfors and Beurling in 1960, and the initial function on the space $\mathcal{S}$ has a continuous extension to the space of measured laminations on S. It has been used in several contexts, including in the definition of a boundary for Teichmüller space, namely, the Gardiner–Masur boundary. This boundary is considered in Chapter 8 of this volume by Liu and Su and in Chapter 9 by Miyachi. The extremal length geometry of Teichmüller space is also studied by Miyachi in Chapter 4.

Length functions of simple closed curves are also involved in the description of several boundary structures for Teichmüller space, e.g. Thurston's boundary and the Weil–Petersson completion, which, as we saw, appears in Chapter 1 of this Handbook. Thurston's boundary is defined *via* intersection functions associated to simple closed curves. It is reviewed in Chapter 5 by Ohshika. The Weil–Petersson completion is a stratified space whose strata are encoded by the curve complex.

Length functions of simple closed geodesics also played a crucial role in the work of Kerckhoff on the Nielsen Realization Problem, and in the more recent work done around the quantization theories of Teichmüller space.

Thurston circulated, again in the 1980s, a short manuscript entitled *A spine for Teichmüller space*, in which he outlined the construction of a mapping class group-equivariant spine for Teichmüller space. In that manuscript, the length of simple closed geodesics is used to define a stratification of Teichmüller space, the strata being subsets consisting of surfaces on which the set of geodesics of minimal length satisfy certain equalities. The idea behind that construction was further investigated by various people, including Schmutz Schaller, Parlier and Ji.

Another impulse to the study of simple closed geodesics on surfaces was given by a famous formula discovered by G. McShane (in his Warwick PhD thesis, 1991), saying that for any complete hyperbolic metric of finite area on the once-punctured torus, we have

$$\sum_{\gamma} \frac{1}{1+e^{\ell(\gamma)}} = \frac{1}{2},$$

where the sum is taken over all simple closed geodesics γ and where $l(\gamma)$ is the length of γ for a fixed hyperbolic metric of finite area. For the experts in the field, the formula came as a surprise because of its simplicity, regardless of the fact that it does not depend on the chosen hyperbolic metric. This formula soon led to interesting improvements and generalizations by a number of people and groups of people, including McShane himself (to other punctured hyperbolic surfaces), Mirzakhani (to surfaces with boundary), Norbury (extending Mirzakhani's work to non-orientable surfaces with boundary), Bowditch (who gave a new proof of McShane's identity using Markov triples, that is, triples of real numbers that are > 2 and that satisfy the equation $x^2 + y^2 + z^2 = wyz$; Bowditch later on generalized the formula to some quasi-Fuchsian groups and to punctured surface bundles over the circle), Akiyoshi–

Miyachi–Sakuma (extending some work by Bowditch), Tan–Wong–Zhang (to hyperbolic surfaces with cone singularities with cone angles $\leq \pi$, possibly with cups or geodesic boundaries), Tan–Luo (who produced McShane-type identities for hyperbolic closed surfaces whose terms depend on the dilogarithm function), and there are other works, some of which we mention below. Motivated by McShane's formula, new estimates were obtained for the asymptotics of the number of simple closed geodesics on a hyperbolic surface whose length is bounded above by a given constant L as $L \to \infty$. In particular, Mirzakhani showed that for a surface of genus g with n cusps, the asymptotics is of the order $L^{6g-6+2n}$. It should be noted that the exponent in this formula is precisely the dimension of Teichmüller space. Mirzakhani also gave precise values for the other constants involved in the asymptotics. A by-product of Mirzakhani's work is a method for computing Weil–Petersson volumes of moduli spaces using counting functions of lengths of simple closed geodesics. Mirzakhani's volume formulae also involve identities that generalize McShane's identity.

All this shows that it is natural to have in this Handbook this survey on simple closed geodesics and on their relation to Teichmüller theory. The survey is divided into two parts. In the first part, Parlier considers a natural question, namely, how the set of *simple* closed geodesics differs from the set of *all* closed geodesics, and in particular, he examines the question of the sparseness of the former compared to the latter. To illustrate this theme, Parlier studies the behavior of these sets from three different points of view, namely:

(1) the *thickness* of the set of points on a hyperbolic surface that belong to some simple closed geodesic compared to the thickness of the set of points that belong to some (not necessarily simple) closed geodesic;

(2) the asymptotic growth of the number of simple closed geodesics with a given bound on their length compared to the corresponding growth of the number of all closed geodesics;

(3) multiplicity in the simple geodesic length spectrum compared to multiplicity in the full geodesic length spectrum.

Talking about the length spectrum and of multiplicity in this length spectrum, one might recall that the length spectrum of the set of all closed geodesics on a hyperbolic surface is related to the theory of the Laplace operator, but that the existence of a similar relation is unclear in the case of the length spectrum of simple closed geodesics.

Let us now point out a few milestones in the work done on these three questions.

We start with the first question. It is known that on any closed hyperbolic surface (and, more generally, on any closed Riemannian manifold of negative curvature) the set of points that lie on closed (non-necessarily simple) geodesics is dense. This follows from the fact that in the unit tangent bundle of compact manifolds of negative curvature, periodic orbits of the geodesic flow are dense. In contrast, Birman and Series showed in 1985 that on a closed hyperbolic surface the set of points that lie on *simple* closed geodesics (and more generally, on *complete* simple geodesics, closed or not) is nowhere dense and has Hausdorff measure zero. At the same time, the same authors worked out algorithms for recognizing conjugacy classes of simple closed curves in fundamental groups of surfaces. Further work on this topic was done later on by various authors. In particular, Buser and Parlier showed in 2007 that any compact hyperbolic surface of genus g contains a disk of radius c_g which is disjoint from all the simple closed geodesics, where c_g is a positive constant that depends only on g.

Now we consider the second question. It follows from works of Huber and of Selberg in the 1950s on the spectrum of the Laplace operator that the asymptotic growth of the number of closed geodesics of length $\leq L$ on a closed hyperbolic surface of genus ≥ 2 is of order e^L/L. Results on the asymptotic growth rate of *simple* closed geodesics were obtained much later. We already mentioned Mirzakhani's 2007 result stating that the asymptotic growth of the number of *simple* closed geodesics of length $\leq L$ on a closed surface of genus g is of order $c_S L^{6g-6}$, where c_S is a constant that depends only on the surface S. Mirzakhani's work was preceded in the special case of the punctured torus by work by McShane and Rivin. The proof of the growth estimates in this special case uses homology considerations that do not apply to surfaces of higher genera.

Now we consider the third question. Randol proved in 1980 that on a hyperbolic surface, the multiplicity in the set of lengths of closed geodesics is unbounded. The study of multiplicity in the length spectrum and in the simple length spectrum was further extended by Buser, and more recently by Rivin, and in joint work of McShane and Parlier. In particular, McShane and Parlier proved in 2009 that the set of surfaces in which all simple closed geodesics have distinct lengths is dense in Teichmüller space, and that the complement of this set is Baire meagre.

Studying geodesics of shortest lengths naturally leads to the systole function defined on Teichmüller space, and we must say a few words on systoles.

A systole on a surface is a closed geodesic of shortest length. The systole length, as a function on Teichmüller space, has been thoroughly studied in the last few decades, and it was probably Schmutz Schaller who made most of the advertising regarding this theory. Other authors who studied systoles on hyperbolic surfaces include Bavard, Buser, Buser–Sarnak, Babenko, Balacheff–Saboureau, and Massart.

In the second part of Chapter 2, Parlier makes a review of systoles and of a related function which can be described as the *shortest pair of pants length function*. This function can be considered a higher-order analogue of the systole function, pairs of pants being maximal systems of disjoint simple closed curves. The *length* of a pair of pants decomposition is defined as the length of the largest curve in the decomposition.

The systole function and its pants decomposition analogue are bounded by constants that depend only on the topological type of the surface.

Parlier addresses several questions on systoles and their higher-dimensional analogues. As in the study of closed geodesics, there are questions on individual systoles, and others on the systole function defined on Teichmüller space. There are also investigations on the asymptotic behavior of the systole function in terms of the genus of the surface, and there are other interesting growth problems. Furthermore, there are analogous questions regarding the shortest pair of pants function.

An object whose study is closely related to the study of the shortest pair of pants decomposition function is the Bers constant. This constant (which depends only on the genus, the number of cusps and of boundary components of the surface) is defined as the supremum of the length of shortest pants decompositions over all hyperbolic surfaces of the given genus and number of boundary components. Bers showed that the supremum of the Bers constants of all surfaces of some fixed finite hyperbolic type is finite. Buser worked out explicit bounds for this supremum. The Bers constant was used as an essential ingredient in the proof of a result of Brock (2003) stating that the Teichmüller space of a surface of finite type is quasi-isometric to the pants complex of the surface. Balacheff and Parlier (2009) obtained monotonicity results for the Bers constant in terms of the genus. There are no known analogues of these monotonicity results for systoles.

1.3 Curve complexes and buildings

Chapter 3 by Lizhen Ji is also concerned with simple closed curves on surfaces, this time from the point of view of the comparison between curve complexes and buildings.

Before describing the content of this chapter, let us start by recalling a few facts about Tits buildings.

Tits buildings are simplicial complexes associated to semisimple Lie groups and to linear algebraic groups over the field $\mathbb{Q}$ of rational numbers, or over other fields. To such groups are also related symmetric and locally symmetric spaces, and Tits buildings appear as a kind of structure at infinity. They can be used to define (partial) compactifications of these symmetric spaces and locally symmetric spaces. For example, the asymptotic cone of a locally symmetric space associated to an arithmetic group Γ is the cone over the quotient by Γ of the Tits building over $\mathbb{Q}$. Tits buildings provide valuable information about the asymptotic geometry of the symmetric spaces, and, in several instances, their rigidity properties have been used to prove rigidity results for locally symmetric spaces. One of the most famous rigidity results in this area is the so-called Mostow rigidity theorem stating that if Γ_1 and Γ_2 are two isomorphic irreducible lattices in semisimple Lie groups of noncompact type with trivial center (but not Fuchsian groups), then the Lie groups are isometric and the lattices are conjugate. Another famous result is the so-called arithmeticity of lattices result for higher rank symmetric spaces stating that any irreducible lattice in a higher-rank semisimple Lie group is arithmetic. This result is a consequence of the so-called super-rigidity of lattices in real semisimple and p-adic semisimple Lie groups. In the original proof by Mostow of his rigidity theorem the isomorphism between the two lattices is used to induce an equivariant quasi-isometry between the symmetric spaces, and then it is shown that in rank ≥ 2 this induces an isomorphism between Tits buildings. Rigidity of Tits buildings is used to show that the locally symmetric spaces are isometric. Hyperbolic geometers are familiar with a special case of Mostow rigidity through a proof by Thurston contained in his famous Princeton lectures, which says that a finite-volume hyperbolic 3-manifold is determined by its fundamental group.

The other major actors in this chapter, namely, curve complexes, are flag simplicial complexes associated to surfaces whose vertices are the homotopy classes of essential simple closed curves (i.e., the elements of the set we denoted by $\mathcal{S}$) and where for each $k \geq 1$, a collection of $k+1$ vertices forms a k-simplex if and only if the corresponding homotopy classes can be represented by disjoint simple closed curves on the surface. Curve complexes were used to get information about boundary structure of Teichmüller spaces, about mapping class groups through their actions on these complexes and on related spaces, and about the quotient spaces of these actions, namely, moduli spaces. For instance, as we already mentioned, curve complexes parametrize the strata of the Weil–Petersson completion. They are used in the proofs of several rigidity results of mapping class group actions. A well-known instance is Ivanov's use of the action of the mapping class group on the curve complex to obtain a geometric proof of the famous theorem of Royden stating that (except for a small number of surfaces) the isometry group of Teichmüller space coincides with the natural image of the extended mapping class group in that group. Several relations between curve complexes and buildings are analyzed in detail in Chapter 3.

Geometrically, a real Tits building may be defined in terms of a classification of geodesics of the symmetric space associated to the semisimple Lie groups to which they are attached. It also admits a description in terms of proper parabolic subgroups. Buildings are classified into three types: Euclidean, spherical and hyperbolic. The adjectives *spherical*, *Euclidean* and *hyperbolic* refer to the geometry of apartments, which are special sub-simplicial complexes, built in the definitions of the buildings. In the spherical Tits building case, each apartment is a triangulation of a sphere. Euclidean buildings can be described in terms of Euclidean reflection groups and they are associated to Euclidean Coxeter complexes. An important class of Euclidean buildings is the class of Bruhat–Tits buildings that appear in the study of linear semisimple algebraic groups. Such buildings also play a role in the theory of p-adic Lie groups, as analogues of symmetric spaces on which these groups act. Examples of spherical buildings include the spherical Tits buildings that are associated to semisimple Lie groups; they are related to the geodesic compactifications of the associated symmetric spaces and they play an important role in the compactification of the associated locally symmetric spaces. Hyperbolic buildings are described in terms of reflection groups of hyperbolic spaces, and they are associated to hyperbolic Coxeter complexes. A spherical (respectively Euclidean, hyperbolic) building admits a natural complete geodesic metric whose restriction to every top-dimensional apartment is isometric to a unit sphere (respectively a Euclidean space, a hyperbolic space) of appropriate dimension, which depends on the rank of the building. Curve complexes have properties in common with each of the three classes of buildings, and as Ji puts it, from the point of view of their geometric and their topological properties, curve complexes are in some sense combinations of spherical, Euclidean and hyperbolic buildings. Furthermore, Euclidean and hyperbolic buildings are CAT(0) and CAT(-1) spaces respectively, their geometries can be studied in the setting of CAT(0) and CAT(-1) geometry, and as such they have natural geodesic compactifications. The natural simplicial metric of the curve complex is Gromov-hyperbolic and has a natural compactification. The Gromov boundary of the curve complex, which was identified by E. Klarreich as the space of minimal and filling (unmeasured) laminations, was used in the proof of Thurston's ending lamination conjecture by Minsky, Brock and Canary, which is one of the major recent results in geometric 3-manifold theory.

A large part of Chapter 3 concerns applications of curve complexes. The author discusses several results where the curve complex is used as an ingredient. These results include the following: the ending lamination conjecture, which is put in parallel with Mostow strong rigidity; quasi-isometric rigidity of mapping class groups (i.e. the fact that quasi-isometries are uniformly close to isometries induced by left-multiplication by elements of the group, a result obtained by Hamenstädt and others); the relation with the Novikov conjectures for certain classes of S-arithmetic subgroups of algebraic groups; the finiteness of the asymptotic dimension of the mapping class group (works of Bestvina, Bromberg and Fujiwara); the non-hyperbolicity of the Weil–Petersson metric (works of Brock and Farb); the work on the so-called *Hempel distance* on Heegaard splittings of 3-manifolds (works of Hempel, and, more recently, of Moore and Rathbun); the construction of partial compactifications of Teichmüller space (works of Harvey and others); cohomological properties of mapping class groups (works of Harer, Ivanov and Ji); the description of asymptotic cones of moduli space (works of Leuzinger, Farb and Masur); the computation of the simplicial volume of moduli space (work of Ji), and there are others applications. The main goal of Chapter 3 is to put these applications in parallel with analogous results on symmetric spaces.

1.4 Extremal length

Chapter 4 by Hideki Miyachi is a survey on the geometry of Teichmüller space which is based on extremal length. We already recalled that the notion of extremal length, as a conformal invariant of a family of curves on a Riemann surface, was introduced by Ahlfors and Beurling in 1960. Several techniques and results on that theory have been obtained starting in the 1970s, by Thurston, Kerckhoff, Gardiner, Masur and Minsky, and more recently by Miyachi. In the work of Kerckhoff in the late 1970s, the extremal length of a homotopy class of curves, for a given marked Riemann surface, was extended to the extremal length of the equivalence class of a measured foliation. In this way, extremal length became a function on the product of Teichmüller space with measured foliation space.

In Chapter 4, the author starts by recalling some basic material and then he discusses the recent developments. We already recalled in this introduction (Formula (1.2)) that extremal length can be used to measure distances between conformal structures, and that Kerckhoff obtained a formula for the Teichmüller distance which is based on extremal length comparison. We also mentioned the Gardiner–Masur compactification of Teichmüller space, which uses the extremal length function. Gardiner and Masur showed that this boundary is strictly larger than the Thurston boundary when the complex dimension of Teichmüller space is greater than one (and in the case where the dimension is one, the two boundaries coincide). There is a horofunction boundary of Teichmüller space, which has been studied by Liu and Su and which coincides with the Gardiner–Masur boundary. This is reported on in Chapter 8 of this volume. Gardiner and Masur also discovered a relation between extremal length theory and lines of minima. Masur and Minsky found a combinatorial model of a space they called the "electrified Teichmüller space", obtained by modifying the geometry of some "thin sets" of Teichmüller space in order to make them of bounded diameter, and these thin sets are defined by the fact that the extremal lengths of some curves are small. All these results show that the extremal length geometry of Teichmüller space very rich. One unifying framework in Chapter 4 is the notion of intersection number applied to the setting of extremal length. In analogy with work done by Bonahon, Miyachi uses the intersection number function to obtain a hyperboloid model of Teichmüller space using extremal length. In this setting, the extremal length of a measured foliation is regarded as the intersection number between a marked Riemann surface and the given measured foliation. The picture parallels the one obtained by Bonahon using the theory of geodesic currents. A relation between intersection number and the Gromov product with respect to the Teichmüller metric is also established. Miyachi shows as a corollary that the Gromov product extends to the Gardiner–Masur boundary. There is an infinitesimal distance on Teichmüller space which is induced by the intersection number seen as a quadratic form in the new hyperboloid model, which is analogous to an infinitesimal distance that is defined in Bonahon's setting. Miyachi formulates open questions concerning this infinitesimal distance.

1.5 Compactifications of Teichmüller space

In Chapter 5, Ken'ichi Ohshika surveys three compactifications of Teichmüller space: Thurston's compactification, the compactification by Teichmüller rays, and the Bers compactification. The first is purely topological, the second is metrical, and the third one is group-theoretical. In each case, the author discusses the mapping class group action on the compactified space, in particular, whether it is continuous or not.

Thurston's compactification is certainly the most useful one and the most well-known. It is obtained by adjoining to Teichmüller space the space $\mathcal{PMF}$ of projective measured laminations, which is usually described in terms of convergence of geodesic length functions to intersection functions of measured foliations, in the projective sense. This compactification is natural in the sense that the mapping class group action on Teichmüller space extends continuously to the compactified space. Ohshika gives a description of this compactification in terms of limits of earthquake flows and in terms of the action of the surface fundamental group on $\mathbb{R}$-trees.

The Teichmüller compactification is a geodesic ray compactification, obtained by choosing a basepoint in Teichmüller space and adjoining to this space, as boundary at infinity, the space of endpoints of geodesic rays (for the Teichmüller metric) starting at that point. The mapping class group action on Teichmüller space does not extend continuously to this compactification, by a result due to Kerckhoff (1980). However, it was observed by Kerckhoff that if we consider the quotient of this Teichmüller boundary obtained by identifying to one point any pair of projective classes of measured foliations whenever they are topologically equivalent (that is, by forgetting the transverse measure), then the mapping class group action extends continuously to this new compactification. The new boundary obtained is the space $\mathcal{UMF}$ of *unmeasured foliations*. This space is closely related to the space $\mathcal{UML}$ of unmeasured laminations, which appears later in this chapter.

The third compactification that is described in Chapter 5 is the Bers compactification. It also depends on the choice of a basepoint in Teichmüller space, and it is obtained by embedding this space in the space of faithful and discrete representations of the fundamental group of the surface in the Lie group $\mathrm{PSL}(2,\mathbb{C})$. The image of such an embedding (which strongly depends on the choice of a basepoint) is called a *Bers slice*. A Kleinian group in the image is called a *quasi-Fuchsian group*: the limit set of such a group action is a Jordan curve in the complex plane and its domain of discontinuity has two components. Bers, who introduced this compactification (1970), showed that each slice is relatively compact and that the Kleinian groups which are in the image are *b-groups*: their domain of discontinuity is connected. A *Bers compactification* is the closure of such a slice. Ohshika mentions some recent developments of this compactification, namely, the work started by Thurston on parametrizing the boundary of a Bers slice using an invariant consisting of a conformal structure on the surface and an *ending lamination*, and the culmination of this work in the proof of Thurston's *Ending Lamination Conjecture* by Brock, Canary and Minsky. It was proved by Kerckhoff and Thurston (1990) that the mapping class group action does not extend continuously to the Bers boundary. An analysis of the proof of this result shows that the reason why the mapping class group action does not extend continuously to the boundary is the existence of quasi-conformal deformation components in the Bers boundary. Ohshika showed (2011) that if we collapse each such deformation component to a point then the action of the mapping class group extends to the resulting space, which is called a *reduced Bers boundary*. This confirms a conjecture made by Thurston and quoted by McMullen in his talk at the Kyoto ICM (1990). The Bers compactification as well as the result by Ohshika are reported on in Chapter 5. A Kleinian group in the reduced Bers boundary is parametrized by an element of unmeasured lamination space $\mathcal{UML}$, that is, the quotient space of measured lamination space $\mathcal{ML}$ obtained by forgetting the transverse measure. There are subtleties involving two different topologies on $\mathcal{UML}$, the one obtained as the quotient of the topology of $\mathcal{ML}$ and the one induced on the reduced Bers boundary. The chapter ends with a survey of rigidity results of the actions of the mapping class group on $\mathcal{UML}$ and a review of the reduced Bers boundary.

1.6 Operad theory and Teichmüller space

Operads first appeared as a tool in homotopical algebra. They can be traced back to works done in the 1960s by Stasheff, Boardmann–Vogt and May in homotopy theory, and more particularly in the study of loop spaces and iterated loop spaces.[2] Roughly speaking, an operad is a set of relations that are "composable", each of them having as an input some finite number of arguments (instead of only two, as in the familiar case of a binary relation). Such relations appear for instance when one considers multi-linear self-maps of vector spaces. The set of relations defining an operad is equipped with the action of the symmetric group by permuting the variables, and the relations are required to satisfy certain conditions, such as the associativity of composition and the existence of a unit. With such relations one defines a category of algebras that generalize the one that arises in the composition of endomorphisms of vector spaces. Operad theory is also in some sense an analogue, in the setting of representation theory of associative algebras, of the representation theory of groups. For this reason, representations of operads are called operad algebras.

It was realized in the 1990s that operad theory provides a combinatorial setting for the study of moduli spaces of algebraic curves. As a matter of fact, moduli spaces can be thought of in some sense as operad algebras. This establishes a strong relation between operads and Teichmüller theory.

A basic and particularly simple example of an operad is the *rooted tree operad*, whose objects are rooted trees and whose compositions are defined by grafting a root to a leaf.

Another famous operad is the *little-dics* or *little n-dics operad*, defined in terms of ordered collections of disjoint n-discs in the unit disc in $\mathbb{R}^n$, with a certain rule for composition defined by taking disjoint unions and then scaling down to get again an ordered collection of disjoint n-discs in the unit n-disc. An ancestor of this operad is the *little n-cube operad*, which was defined in a similar way by Boardman and Vogt, taking configurations of disjoint n-cubes in the unit cube in $\mathbb{R}^n$. May defined more generally the *little convex bodies operad*.

Since their introduction, operads appeared in several fields of mathematics including, besides homotopy theory, homological algebra, K-theory, category theory, complex algebraic geometry, real algebraic geometry, mathematical physics (string theory and vertex operator algebras) and Teichmüller theory.

In the 1990s, Kontsevich used operads in his work on formal deformation theory and of formal moduli spaces. In a joint paper with Soibelman (2000), he developed the deformation theory of operads and algebras over operads, proving a conjecture of Deligne stating that the Hochschild complex of an associative algebra is equipped with a canonical action of the operad obtained by taking chains of the little discs operad. In their proof, Kontsevich and Soibelman constructed a geometric operad that acts on the Hochschild complex. With this and related works, it became clear that operads are useful in understanding the internal structures that occur in theoretical physics. The work of Kontsevich and Soibelman also pointed out a relationship between operads and the Grothendieck–Teichmüller group. It was realized later on that this group acts

[2]The word "operad" was coined by May, in the early 1970s, and it appears in his book *The Geometry of Iterated Loop Spaces* (1972).

(homotopically) on the moduli space of structures of 2-algebras on the Hochschild complex.

In the work of Kaufmann and collaborators, operads emerge in an essential way in surface theory and mapping class group theory. This is surveyed in Chapter 6 of the present volume, written by Ralph Kaufmann. One setting to start with is the following: Consider a surface with nonempty boundary, with subsets of the boundary called windows. There is also a version with an "in boundary" and an "out boundary" of the surface. The surface can be equipped with a system of arcs going from window to window. A collection of such surfaces can be glued together in various ways, matching the windows, like in the case of the rooted tree operad. The result of this gluing (regarded as a composition relation) is an operad. In fact, there are several operads that one can define in this and similar ways, obtained by requiring special properties for the arcs considered.

A PROP is an object more general than an operad in the sense that it admits several inputs but also several outputs. PROPs appear in the 1960s in the work of Mac Lane. The theory of PROPs is closely related to the theory of operads. PROPs are also mentioned in Chapter 6.

In 2003, Kaufmann, Livernet and Penner defined a particularly interesting operad, denoted by $\mathcal{A}rc$, using weighted arc systems such that arcs emanate from each boundary component. Such a weighted arc system gives rise to a (partial) measured foliations on a surface. This operad makes a link between combinatorial compactifications of moduli spaces and a conjecture that was formulated by Penner in 1996, called the "sphericity conjecture", which says that the arc complex of the surface is homeomorphic to a sphere of a certain dimension. Furthermore, the arc operads, with their foliation description, provide geometric models for several algebraic constructions, including a new model of the little discs operad and its framed version, and they can be used in the study of loop spaces to define natural actions on Hochschild cohomology of associative or Frobenius algebras. There are also obvious relations between these arc operads and string topology, conformal field theory and string field theory.

In conclusion, the combinatorics of arcs on surfaces provides a new example but also a very powerful geometric setting for operad theory. It also sheds new light on several basic constructions in mathematics.

1.7 The horofunction boundary of Thurston's metric

Chapter 7 by Cormac Walsh concerns Thurston's asymmetric metric on Teichmüller space, introduced by Thurston in his 1986 preprint *Minimal stretch maps between hyperbolic surfaces*, which was mentioned in this introduction. The asymmetric metric was already surveyed in Chapter 2 of Volume I of this Handbook. The present chapter contains new results.

We gave the definition of this metric in (1.1), and it will be useful for what follows to recall another definition.

Let $S = S_{g,n}$ be a surface of finite type and negative Euler characteristic, of genus $g \geq 0$ with $n \geq 0$ punctures. We consider hyperbolic structures on S that are complete and of finite area.

Given two such hyperbolic structures g and h, and given a homeomorphism $\varphi\colon S \to S$ which is isotopic to the identity, the *Lipschitz constant* $\mathrm{Lip}(\varphi)$ of φ is defined by

$$\mathrm{Lip}(\varphi) = \sup_{x \neq y \in S} \frac{d_h\big(\varphi(x), \varphi(y)\big)}{d_g\big(x, y\big)}.$$

The infimum of such constants over the set of all homeomorphisms $\varphi\colon S \to S$ isotopic to the identity is denoted by

$$L(g,h) = \log \inf_{\varphi \sim \mathrm{Id}_S} \mathrm{Lip}(\varphi). \tag{1.3}$$

Replacing g and h by homotopic metrics does not change the value of $L(g,h)$. Therefore the function L induces a function on $\mathcal{T}(S) \times \mathcal{T}(S)$. By a result of Thurston, this function coincides with the asymmetric metric defined by Equation (1.1). The metric is also called *Thurston's metric* (being understood that it is not a genuine metric) on Teichmüller space. We shall denote it by the same letter L:

$$L\colon \mathcal{T}(S) \times \mathcal{T}(S) \to \mathbb{R}_+.$$

Thurston's metric defined using the function L is reminiscent of Kerckhoff's formula for the Teichmüller metric based on extremal length which we recalled in (1.2) above, with the Lipschitz constant of maps replacing their quasiconformal dilatation. This remark is at the basis of a connection that is presented in Chapter 8, on which we comment later on in this introduction.

In Chapter 7, Walsh presents several results on Thurston's metric. The first result concerns its horofunction boundary. Before stating the result, we recall that the horofunction bordification $X(\infty)$ of a (possibly asymmetric) metric space (X,d) is the closure of a set of normalized distance functions in the space of all continuous functions on X. More precisely, we choose a basepoint b in X and we associate to each point z in X the function $\psi_z\colon X \to \mathbb{R}$ defined by

$$\psi_z(x) = d(x,z) - d(b,z).$$

This gives a map ψ from X to the set $C(X)$ of continuous functions on X. We equip the latter with the topology of uniform convergence on bounded sets with respect to the topology defined by d.[3]

The map $\psi\colon X \to C(X)$ defined by $z \mapsto \psi_z$ is continuous and injective and the *horofunction boundary* (or *horoboundary*) of X is the complement of the image of X in the closure of this image in $C(X)$. The elements of the horoboundary are called *horofunctions*.

[3] Some care is needed to define the topology associated to d because the metric d is not necessarily symmetric. The topology on X is (by definition) the topology associated to its symmetrized metric $d(x,y) + d(y,x)$, which is a genuine metric on X.

The horoboundary of a metric space was introduced in 1978 by Gromov, in the setting of symmetric metric spaces.

There is a simple transformation which carries horofunctions on X with respect to one basepoint to horofunctions with respect to another basepoint, and the resulting map between horoboundaries is a homeomorphism. Thus, one can talk about the horoboundary of (X, d) without reference to a basepoint. This space is denoted by $X(\infty)$.

In the case where the topology on X (defined by the symmetrized metric) is proper (that is, if all closed balls are compact), the closure of the image of X in $C(X)$ is compact, and the horoboundary $X(\infty)$ is a *horofunction compactification*.

The results by Walsh that are presented in Chapter 7 include the following:

(1) There is a natural identification between the horoboundary of Teichmüller space for Thurston's asymmetric metric and Thurston's boundary of that space, defined topologically.

(2) Any almost-geodesic for Thurston's asymmetric metric converges in the forward direction to a point in Thurston's boundary.

(3) If S is not a torus with at most four punctures or a torus with at most two punctures, then any isometry of Thurston's asymmetric metric is induced by an element of the extended mapping class group of S.

Item (2) contrasts with the situation of Teichmüller geodesics. We note in this respect that Kerkhoff (1980) and then Lenzhen (2008) showed that there are Teichmüller geodesics which do not have limits in Thurston's boundary. Item (3) is an analogue of Royden's result (1971) concerning the isometries of the Teichmüller metric, and of a result of Masur and Wolf (2002) concerning the Weil–Petersson metric. It answers a question that was raised in Chapter 2 (Problem IV, p. 198) of Volume I of this Handbook. Walsh also obtained a concrete formula that describes horofunctions for Thurston's asymmetric metric.

1.8 The horofunction boundary of the Teichmüller metric

Chapter 8 by Lixin Liu and Weixu Su is in some sense parallel to Chapter 7 by Walsh. Instead of the horofunction boundary for Thurston's asymmetric metric, the authors study here the horofunction boundary of Teichmüller's metric. For a surface of finite type, they show the following:

(1) There is a natural identification between the horofunction compactification of the Teichmüller space equipped with the Teichmüller metric and the Gardiner–Masur compactification of that space.

(2) Every (almost-)geodesic ray for the Teichmüller metric converges to a point in the Gardiner–Masur boundary.

(3) The action of the mapping class group on Teichmüller space extends continuously to the Gardiner–Masur boundary.

The Gardiner–Masur boundary compactification was introduced by Gardiner and Masur in 1991, in a paper called *Extremal length geometry of Teichmüller space*. They showed in particular that this new boundary contains the Thurston boundary. Later on, this compactification was studied by several authors. For instance, Miyachi studied the asymptotic behavior of Teichmüller geodesic rays under the Gardiner–Masur embedding, and he also showed that the action of the mapping class group on Teichmüller space extends continuously to the Gardiner–Masur boundary. Some other work of Miyachi is surveyed in Chapter 9 of this volume.

There is a formal analogy between the Gardiner–Masur compactification and Thurston's compactification which is apparent right in the definitions: Thurston's compactification is defined by embedding Teichmüller space in the space of functions on the set of homotopy classes of essential simple closed curves on the surface, while using the hyperbolic length functions, and taking the closure of the image, the Gardiner–Masur boundary is defined in a similar way, using the (square root) of the extremal length functions instead of the hyperbolic length functions. Thus, whereas the developments of the Teichmüller compactification use hyperbolic geometry, those of the Gardiner–Masur boundary use its conformal geometry, and it is useful to compare in detail the results and the formal statements that hold in both settings.

The horofunction boundary of the Teichmüller metric is also considered in the next chapter.

1.9 The Lipschitz algebra on Teichmüller space

In Chapter 9, Hideki Miyachi initiates a new object of study in Teichmüller theory, namely, the Lipschitz algebra associated to the Teichmüller metric. It turns out that this Lipschitz algebra is closely related to the notion of extremal length, a relation which stems again from Kerckhoff's formula for the Teichmüller metric which we already recalled (1.2). The results that are presented in Chapter 9 include a proof of a Stone-Weierstrass type theorem for this Lipschitz algebra and a definition of a new boundary structure of Teichmüller space that arises from this Lipschitz algebra. The Stone-Weierstrass type theorem asserts that any norm-closed subalgebra of the algebra of real- or complex-valued Lipschitz functions on Teichmüller space which vanish at a certain point and which satisfy certain algebraic conditions coincides with the whole algebra.

Following a construction of Loeb, Miyachi introduces the notion of a Q-compactification of a non-compact Hausdorff metric space M associated to a nonempty set Q of bounded continuous functions on M. This is a compactification which has the property that every function in Q extends to a continuous function on the compactification. Miyachi then introduces the notion of a *Lipschitz compactification* of M, whose definition depends on the choice of a basepoint. Applying this notion to the Lipschitz algebra of Teichmüller space, he gets the *Lipschitz compactification* of that space.

There is a strong relation between the Lipschitz compactification and the Gardiner–Masur compactification, a relation which originates in the fact that the Lipschitz algebra can be defined in terms of extremal length. More precisely, Miyachi shows that there is a canonical continuous surjection from the Lipschitz compactification to the Gardiner–Masur compactification. He then describes a certain subalgebra Q of the Lipschitz algebra on Teichmüller space for which the Gardiner–Masur compactification becomes a Q-compactification. As a corollary, he obtains the following naturality property of the Gardiner–Masur compactification: Any Lipschitz map from a metric space M to Teichmüller space extends continuously from the Lipschitz compactification of M to the Gardiner–Masur compactification. Making a relation with more classical notions, Miyachi notes that in general Lipschitz mappings on Teichmüller space do not extend continuously to the horofuction boundary. In some sense, this makes the Gardiner–Masur boundary of Teichmüller space more canonical. He obtains the following corollary, which concerns the complex geometry of Teichmüller space: If M is a complex manifold which is Kobayashi hyperbolic (that is, if the Kobayashi pseudo-distance on M is a genuine distance), then any holomorphic mapping from M to Teichmüller space extends continuously to a map from the Lipschitz compactification of M to the Gardiner–Masur compactification of Teichmüller space.

1.10 Geodesics in infinite-dimensional Teichmüller spaces

The Teichmüller space of a surface of infinite type, equipped with its Teichmüller metric, is an infinite-dimensional non-separable Banach manifold. The metric is complete and it is geodesically convex. Chapter 10 by Zhong Li concerns properties of geodesics in general infinite-dimensional Teichmüller spaces, with a stress on properties that do not hold in finite-dimensional ones. The author notes that the following properties, in which distances, geodesics, etc. refer to the Teichmüller metric, hold in any infinite-dimensional Teichmüller space but not in any finite-dimensional Teichmüller space.

(1) There exist local geodesics that are not global geodesics.

(2) There exist self-intersecting geodesics.

(3) There exist pairs of points with more than one straight line joining them.

(4) There exist closed geodesics of arbitrarily short length.

(5) The Teichmüller distance function is not differentiable.

(6) The Finsler norm function of the Teichmüller metric is not of class C^1.

(7) There exist pairs of points with infinitely many geodesic disks containing them. (A geodesic disk in Teichmüller space is an isometric image of the hyperbolic plane.)

Other properties of infinite-dimensional Teichmüller spaces that are presented in Chapter 10 include the fact that these spaces admit isometric holomorphic embeddings of the infinite-dimensional polydisk D^{∞} (a result of Earle and Li). Furthermore, it is known that in the infinite-dimensional case, no sphere in Teichmüller space is geodesically strictly convex, whereas in the finite-dimensional case the response is unknown.

Several of the above properties can be deduced from each other, although originally they were proved by independent means.

Another theme that arises in Chapter 10 is that there is a characterization of the points in infinite-dimensional Teichmüller spaces which satisfy the above properties. Indeed, the author gives several characterizations of points and pairs of points where Properties (3) and (7) hold. In this setting, the points in Teichmüller space are seen as equivalence classes $[\mu]$ of Beltrami differentials on the base surface. In particular, there is a characterization of the set of points $[\mu]$ for which the geodesic joining such a point to the base surface (that is, the equivalence class of the zero Beltrami differential) is unique. Earle and Li (1999) gave such a characterization in terms of a property that involves the boundary dilatation of the Beltrami differential μ. Lakic proved (1997) that in an infinite-dimensional Teichmüller space the "good points" (i.e., the points for which the geodesic joining them to the origin is unique) form an open and dense subset. These good points coincide with the so-called *Strebel points*, or *Strebel differentials*, a terminology introduced by Earle and Li in their 1999 paper.[4] In the case of finite-dimensional Teichmüller spaces, all points are good points (a result that follows from the work of Teichmüller). A theorem by Gardiner and Lakic states that the set of Strebel points is dense and open in Teichmüller space.

One of the facts that the reader can learn in Chapter 10 is that several of the geometric features of infinite-dimensional Teichmüller spaces – non-uniqueness of geodesics joining two points, existence of closed geodesics, non-strict convexity of spheres, etc. – have infinitesimal analogues (and in fact they can be deduced from their infinitesimal analogues) in the tangent space to Teichmüller space at the basepoint, that is, the space dual to the space of holomorphic quadratic differentials on the base surface, equipped with the metric induced from the sup norm.

The notion of extremality is an important notion for finite and infinite-dimensional Teichmüller spaces, but in the latter setting it is much more intricate. In this respect, Li also presents a result by Bozin, Lakic, Markovic and Mateljević (1998) establishing a characterization of good points $[\mu]$ in terms of extremality of the Beltrami differential μ. Here, one says that μ is *extremal* in its class $[\mu]$ if for any other μ' in the same class, one has $|\mu|_{\infty} \leq |\mu'|_{\infty}$. In the case of surfaces of finite type, there exists a unique extremal Beltrami differential in each class, and it is defined in terms of a quadratic differential. In an infinite-dimensional Teichmüller space, a theorem due to Reich, Strebel, Kra and Krushkal' states a criterion for extremality of a quadratic differential in terms of a notion of infinitesimal extremality. This result reduces a question of finding extremal objects up to homotopy to a question of extremality in the tangent space. Another important notion is the one of unique extremality. A Beltrami differential is

[4] This notion is not to be confused with the notion of a Strebel or Jenkins–Strebel quadratic differential, which is well known to Teichmüller theorists.

said to be uniquely extremal if it is the unique extremal differential in its class. The theorem of Bozin, Lakic, Markovic and Mateljević says that a Beltrami differential μ is uniquely extremal if and only if it is infinitesimally uniquely extremal.

Lets us mention that one of the most important recent results in this theory is the theorem of Markovic stating that if two infinite-dimensional Teichmüller spaces are isometric, then the corresponding base surfaces are quasi-conformally equivalent.

1.11 Holomorphic families

Chapter 11 by Hiroshige Shiga is about holomorphic families of Riemann surfaces. A holomorphic family of Riemann surfaces is a family of Riemann surfaces parametrized by some holomorphic parameter. More precisely, a holomorphic family of type (g, n) is a triple (M, π, B) where B is a complex manifold called the *base*, M is a complex manifold, and $\pi : M \to B$ is a surjective holomorphic map of maximal rank at each point of M such that for every t in the base, $S_t = \pi^{-1}(t)$ is a conformally finite type surface of genus n with p punctures and the map $t \mapsto S_t$ is holomorphic. To a holomorphic family of Riemann surfaces (M, π, B) is associated a holomorphic "classifying map" from the base into the moduli space of Riemann surfaces and a homomorphism from the fundamental group of the base to the mapping class group of a fiber S_t called the *monodromy* of the family. The monodromy is well defined up to conjugation.

Holomorphic families of Riemann surfaces, in the particular case where the base space is the punctured disk, were already studied by Imayoshi in Chapter 3 of Volume II of this Handbook. In Chapter 11, of the present volume, the author considers the case where B is an arbitrary Riemann surface. The association $t \mapsto S_t$ is a map ϕ from B to the moduli space of Riemann surfaces of type (g, n). There is a natural notion of isomorphic holomorphic families of Riemann surfaces (biholomorphisms of the total space respecting the projections), and a rigidity theorem states that two holomorphic families over the same base surface are isomorphic if and only if their monodromies are conjugate. A result by Imayoshi and Shiga says that for a fixed base, there are only finitely many holomorphic families (except the locally trivial ones). The proof is based on a generalized form of the Schwarz Lemma, on the fact that a holomorphic family is determined by its monodromy and on a result of Shiga stating that the image of the monodromy is always an irreducible subgroup (in the sense of Thurston's classification) of the mapping class group of the fibre.

In Chapter 11, the author considers applications of holomorphic families in several geometrical settings. One setting is the problem of solving Diophantine equations over function fields. This problem consists in finding meromorphic functions satisfying some homogeneous polynomial equations with coefficients in some function field of meromorphic functions on some Riemann surface. The rigidity result by Imayoshi and Shiga that we mentioned above is considered as the geometric Shafarevich conjecture. Relations are made with the Mordell conjecture asserting that some diophantine equations have a finite number of solutions. The author then discusses the relation between holomorphic families and holomorphic motions, and with the finiteness properties of Teichmüller curves (in particular Veech surfaces) in moduli space.

2 Part B. Representation spaces and generalized structures, 2

2.1 Flat affine structures

Chapter 12 by Oliver Baues is a survey on flat affine structures on surfaces. A flat affine structure is a geometric structure defined by an atlas with values in the Euclidean plane $\mathbb{R}^2$ and coordinate changes in the group $\mathrm{Aff}(2)$ of affine transformations of the plane. By a theorem of Benzécri (1960), a closed orientable surface which carries a flat affine structure is necessarily the two-torus. Thus, this chapter concerns only the torus. Despite this restriction, the theory is rich and has many interesting facets. In a way that parallels the corresponding notions for Teichmüller space, one studies the holonomy maps of flat affine structures, the representation space as a subset of the character variety $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{Aff}(2))/\mathrm{Aff}(2)$, the surface fundamental group action on this representation space, the resulting moduli space, one can study discreteness criteria for such an action, and so on.

The work on flat affine structures can be traced back at least to 1953, when Kuiper started a classification of these structures on surfaces. This classification was completed in the 1970s by Nagano–Yagi, and by Furness–Arrowsmith. More recent activity on the subject was carried out in the 1990s by Baues, Baues–Goldman, and Benoist. The main results obtained so far include a complete classification of affine structures on the torus, a description of the deformation space and of the moduli space of such structures (the analogues of the Teichmüller space and the moduli space of a surface of negative Euler characteristic), with a study of various group actions and flows associated to these spaces.

Chapter 12 contains a detailed account of the classical results as well as some new results on flat affine structures and their deformation spaces. The survey starts with a proof of Benzécri's theorem and of the classification theory of flat affine structures on the torus, together with a rich variety of examples and methods of construction of such structures: taking quotients of affine Lie groups, gluing patches of affine space along their boundary (in particular, gluing polygons and annuli along their sides), examples obtained from linear algebra, and others. An affine version of the classical Poincaré polygon gluing theorem is given. The author considers families of flat affine two-tori obtained as quotients of quadrilaterals glued along their boundaries by affine maps. He studies the dependence of such flat affine tori on the shape of the quadrilaterals and on the annuli that are used to define them. He shows that the flat affine tori that are obtained by gluing affine quadrilaterals along their sides are all homogeneous (that is, their groups of affine automorphisms act transitively on these spaces). He then describes in detail a construction of affine tori with developing image in $\mathbb{A}^2 - \{0\}$. The chapter also contains a review of more recent results on discontinuous affine actions with compact quotients, and a survey of the structure of the deformation and of the moduli spaces. The reader will learn that even though the definition of the deformation space of flat affine structures on the two-torus is comparable to that of the Teichmüller space of a surface of higher genus, the basic properties of the two spaces differ in a substantial manner. For instance, while the latter is a Hausdorff and metrizable space, the former is not Hausdorff. An important feature of the action of $\mathrm{Aff}(2)$ on $\mathbb{R}^2$ that makes things different from the group action that defines Teichmüller space is that the $\mathrm{Aff}(2)$-action has non-compact stabilizers.

The author shows that the development map of a flat affine structure on the two-torus is always a covering map. He gives a description of the topological local structure of the space of deformations of flat affine structures, proving that the holonomy map defined on that space is a local homeomorphism onto an open connected subset of the character variety. He then provides examples that show that this property does not always hold for other locally homogeneous structures. In other words, there exist deformation spaces of locally homogeneous structures on the torus whose holonomy maps at some points are not locally injective.

Furthermore, Chapter 12 contains all the necessary introductory material on general locally homogeneous structures on manifolds and their deformation spaces, and on Thurston's treatment of development maps and holonomy homomorphisms.

The theory of affine structures can be developed in dimension greater than two. One of the main conjectures in this respect is the so-called Chern conjecture, stating that the Euler characteristic of a compact flat affine manifold is zero. (This would be the analogue of Benzécri's Theorem in dimension two.)

2.2 Higher Teichmüller theory

Chapter 13 by Marc Burger, Alessandra Iozzi and Anna Wienhard is a survey on higher Teichmüller theory. This is an extension of Teichmüller theory to representations of fundamental groups of surfaces into Lie groups which are not $\mathrm{PSL}(2,\mathbb{R})$ (which is the setting of the classical theory). In the classical case, one considers the space of homotopy classes of marked hyperbolic structures on, say, a closed surface S, as a subset of the character variety $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$. This is obtained by considering $\mathrm{PSL}(2,\mathbb{R})$ as the isometry group of the upper half-space model of the hyperbolic plane and associating to each hyperbolic structure its holonomy homomorphism, an element of $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))$. The group $\mathrm{PSL}(2,\mathbb{R})$ acts by conjugation on the space $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))$, and the image of Teichmüller space in $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ coincides with a connected component of this variety. This component consists entirely of discrete and faithful representations.

It is also well known that one can obtain a description of Teichmüller space by working with the projective special unitary group $\mathrm{PSU}(1,1)$ instead of the projective special linear group $\mathrm{PSL}(2,\mathbb{R})$, identifying $\mathrm{PSU}(1,1)$ with the orientation-preserving isometry group of the Poincaré disc. The theory obtained is identical.

Higher Teichmüller theory was developed as a generalization of the representation theory of $\pi_1(S)$ to other simple Lie groups. The natural questions that appear right at the beginning of the theory are whether the results that hold in the classical representation theory still hold in the generalized setting. It turns out that for some groups the situation is quite different from the one in the case of $\mathrm{PSL}(2,\mathbb{R})$. There are still several interesting questions that remain unanswered in the general theory.

A *higher Teichmüller space* is defined accordingly as a connected component (or a union of connected components) of a space $\mathrm{Hom}(\pi_1(S), G)/G$ that consists of equivalence classes of discrete and faithful representations of $\pi_1(S)$ into some simple Lie group G. Some of the natural questions that one would like to answer in this theory are the following:

(1) Study individual representations $\rho\colon \pi_1(S) \to G$, find invariants of these representations and detect when such a representation is discrete and faithful.

(2) Figure out whether there exist connected components in the representation variety that consist entirely of discrete and faithful representations.

(3) Study the geometry of such components, equipping them with various structures (metric, symplectic, complex, combinatorial, etc.), in analogy with the structures that are known to exist on Teichmüller space.

(4) Study the dynamics of the surface mapping class group actions on such components.

(5) Find geometric structures on surfaces that correspond to these new conjugacy classes of representations, possibly *via* the holonomy representation of the fundamental group of the surface, and in analogy with the fact that homotopy classes of hyperbolic structures correspond to conjugacy classes of representations in $\mathrm{PSL}(2,\mathbb{R})$ or in $\mathrm{PSU}(1,1)$.

Some of these questions are already settled and some others are under thorough investigation, by several people. For instance, it is known that the answer to (2) is *no* for some Lie groups. We shall comment below on the other questions.

A few words on the development of higher Teichmüller theory may be appropriate here.

It is usually acknowledged that higher Teichmüller theory started with the work of Hitchin (1992), who studied representations of fundamental groups of closed surfaces in $\mathrm{PSL}(n,\mathbb{R})$ (generalizing the classical case where $n = 2$) and, more generally, in the adjoint group of a split real simple Lie group, namely, the groups which are in the following list: $\mathrm{PSL}(n,\mathbb{R})$, $\mathrm{PSp}(2n,\mathbb{R})$, $\mathrm{PSO}(n,n)$, $\mathrm{PSO}(n,n+1)$ (and there are some other such groups, called exceptional). Using techniques of Higgs bundles, Hitchin showed the existence of a component in the representation variety of such a group that has properties which resemble those of Teichmüller space. Higgs bundles are holomorphic vector bundles, equipped with so-called "Higgs fields" that appeared in works of Hitchin and of Simpson. Their study brought new tools in surface group representation theory. In the special case where S is a closed surface of genus $g \geq 2$ and where $G = \mathrm{PSL}(n, \mathbb{R})$, the component highlighted by Hitchin is homeomorphic to $\mathbb{R}^{(2g-2)(n^2-1)}$; it is now called *Hitchin component*. Hitchin also studied the other components. One year later (1993), Choi and Goldman showed that in the case where $G = \mathrm{PSL}(3, \mathbb{R})$, the Hitchin component consists precisely of the holonomy representations of convex real projective structures on the surface, thus answering Questions (2) and (5) above in the particular case of $\mathrm{PSL}(3, \mathbb{R})$. This component contains the usual Teichmüller space as a subset.

Following the work of Hitchin in the case of $\mathrm{PSL}(n, \mathbb{R})$, an extensive research on higher Teichmüller theory was conducted in the last decade by several authors, including Burger, Iozzi, Wienhard, Labourie, Fock, Goncharov, Guichard, Loftin, Bradlow, García-Prada, Mundet i Riera, Gothen, and others. In particular, Fock and Goncharov developed a higher Teichmüller theory which is parallel to the theory of Hitchin representations, using a notion of positivity of representations of the fundamental group of the surface into a split semisimple algebraic group G with trivial center. A similar notion of positivity was already introduced by G. Lusztig in his theory of canonical bases that appeared in the 1990s. Fock and Goncharov gave a more geometric version of that theory, where positivity of a representation of a surface fundamental group into a real split simple Lie group is defined in terms of the familiar Thurston shear coordinates on the edges of surface hyperbolic ideal triangulations. They proved that these positive representations are faithful and discrete, and that their moduli space is an open cell in the space of all representations, generalizing the classical case where $G = \mathrm{PSL}(2, \mathbb{R})$. Furthermore, they developed a relation between these positive representations and cluster algebras and with the quantization theory of Teichmüller space. The work of Fock and Goncharov was motivated in part by the theory of quantum representations of mapping class groups.

Labourie, in 2006, associated to every representation in a Hitchin component a curve in a projective space, which he called a *hyperconvex Frenet curve*, generalizing the Veronese embedding from $P(\mathbb{R}^2)$ into $P(\mathbb{R}^n)$. He also showed that a Hitchin component in the case of $\mathrm{PSL}(n, \mathbb{R})$ consists entirely of discrete and faithful representations, and that a representation in such a component is a quasi-isometric embedding. He also showed that the mapping class group acts properly discontinuously on the Hitchin component. He introduced the concept of *Anosov representations*. These are also discrete and faithful representations, and they are quasi-isometric embeddings. An Anosov representation generalizes the notion of a convex cocompact representation. Anosov representations form a set on which the mapping class group acts property discontinuously. In the case where $G = \mathrm{PSL}(n, \mathbb{R})$, Hitchin representations are Anosov representations, but the converse does not hold. In its original version, an Anosov representation arises as the holonomy of an Anosov structure on the underlying surface. This concept became the basis of a new dynamical framework for the study of representations in the Hitchin components.

In 2008, Guichard and Wienhard gave a characterization of the representations in the Hitchin components in the case where $G = \mathrm{PSL}(4, \mathbb{R})$ as properly convex foliated projective structures on the unit tangent bundle of the surface, thus answering Question (5) in that particular case. Guichard and Wienhard gave a characterization of convex Anosov representations in the case of irreducible representations. They also developed an analogue of the Labourie notion of Anosov representation theory of fundamental groups of surfaces in the context of representation theory of hyperbolic groups. Labourie, Guichard and Wienhard showed that in the case where G is a rank one semisimple Lie group, a representation is Anosov if and only if it is a quasi-isometric embedding, and moreover that this representation is Anosov if and only if its image is a convex cocompact group.

To sum up, there are, so far, two large classes of simple Lie groups G for which higher Teichmüller theory has been developed:

- the class of split real simple Lie groups that were mentioned above, i.e., $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SO}(n, n+1)$, $\mathrm{SO}(n, n)$;
- the class of Lie groups of Hermitian type, that is, those that carry an invariant complex structure, or, equivalently, those whose associated symmetric space is Hermitian.

(Note that the Lie groups $\mathrm{Sp}(2n, \mathbb{R})$ belong to both classes.)

In the case of Lie groups of Hermitian type, there is a bounded integer-valued function defined on the character variety, called the *Toledo invariant*. D. Toledo, in 1989, studied such a function in the particular setting of representations of fundamental groups of surfaces into the groups of isometries of complex hyperbolic spaces. Burger, Iozzi and Wienhard introduced the Toledo invariant in the general setting of representations into Lie groups of Hermitian type.

The Toledo invariant is reminiscent of the *Euler number*, an integer-valued function defined on the character variety $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))/\mathrm{PSL}(2, \mathbb{R})$ and which is constant on the connected components of this variety. The classical Teichmüller space is precisely a connected component with maximal Euler number (Goldman's thesis, 1980).

In the setting of Lie groups of Hermitian type, the level set of the maximal value of the modulus of the Toledo invariant is a union of connected components of the representation variety whose elements are called *maximal representations*, and such a component is an instance of a higher Teichmüller space. Global properties of the components of the representation variety that have maximal Toledo invariant were also studied (in the case of closed surfaces) by Bradlow, García-Prada and Gothen, using Higgs bundles. The geometric properties of the representations that lie in these components were studied by Burger, Iozzi and Wienhard, who also considered the case of surfaces with boundary. These authors obtained a structure theorem for maximal representations as well as information about the representation variety. They associated to maximal representations boundary maps with monotonicity (positivity) properties expressed in terms of maximal triples of points in the Shilov boundary of the symmetric space associated to the underlying Lie group G, and they gave a formula for the Toledo invariant in terms of the bounded Euler class of a group action on a circle, a notion that generalizes the classical Poincaré rotation number for circle homeomorphisms. Chapter 13 is a survey of all these results.

2.3 Quasiconformal mappings in higher dimensions

It is well known that the development of the theory of quasiconformal mappings in dimension two was largely motivated by Teichmüller theory. Conversely, Teichmüller theory (especially the complex-analytic part of it) relies substantially on two-dimensional quasiconformal mapping theory. In this sense, the theory of quasiconformal mappings in higher dimensions can be considered as a *higher Teichmüller theory*. Therefore it is not unreasonable to include a survey on that theory in the present section of the Handbook. This is the subject of Chapter 14 of this volume, by Gaven Martin. In this chapter, the author outlines the major ideas and results in higher-dimensional quasiconformal theory, starting from its developments in the early 1960s, in particular in works of Reshetnyak, Gehring and Väisälä. The exposition also highlights the connections of quasiconformal theory with other theories such as geometric function theory, the calculus of variations, nonlinear partial differential equations, differential and geometric topology, and the dynamics of iteration of holomorphic functions in higher dimensions.

There are common features of quasiconformal mappings in dimension two and in higher dimensions. For instance, quasiconformal mappings, in any dimension, are Hölder continuous, they are solutions of a Beltrami-type partial differential equation, they satisfy a generalized version of the Schwarz Lemma and a normal family-type compactness property. But it is also important to know that the theory of quasiconformal mappings in higher dimensions is not simply a generalization of quasiconformal theory in dimension two; there are severe differences between the two theories, and even at the level of the most basic principles, the theory in dimension ≥ 3 has its proper features. As a matter of fact, one might first ponder on the difference in the case of *conformal* mappings, where in dimension ≥ 3 there are no conformal – sufficiently smooth – mappings from a domain $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^n$, except restrictions on Ω of Möbius transformations. This is the so-called Liouville theorem, a phenomenon that is very different from what occurs the case of dimension two where there are many conformal mappings.

We recall by the way that the word *quasiconformal*, in dimension two, first appeared in Ahlfors' 1935 paper *Zur Theorie der Überlagerungsflächen*[5] (On the theory of covering surfaces). Before that, in papers published in 1929 and 1932, Grötzsch had used a similar notion for maps that deviate from conformality (without giving them a name). In 1935, the year where Ahlfors' paper appeared, Lavrentieff published a

[5]This is the paper for which Ahlfors, in 1936, was awarded the Fields medal; it concerns a generalization of Nevanlinna theory, which will be mentioned later in this introduction.

paper in which he introduced a weaker notion of functions deviating from conformality, which he called *almost-analytic* (the paper is in French and Lavrentieff called these functions “fonctions presque-analytiques”).

Now we pass to dimension ≥ 3.

There are several definitions of quasiconformal mappings in higher dimensions which generalize the definitions in dimension two, and each of them highlights some important aspect of the theory. We now recall some of them.

If $f : \Omega \to \Omega'$ is a *diffeomorphism* between open subsets of $\mathbb{R}^n$, then one possible measure of the deviation of f from conformality is the quantity

$$\sup_{x\in\Omega} k(f'(x)),$$

where for each x in Ω, $f'(x)$ is the derivative (infinitesimal linear map) of f at x, and where $k(f'(x))$ is the value of the ratio of the major axis to the minor axis of the ellipsoid $f'(S^n_x)$, with S^n_x being the unit sphere in the tangent space $T_x\Omega$ of Ω at x. (Recall that the image by a linear map of a sphere centered at the origin is an ellipsoid.) This notion of quasiconformality was used by Grötzsch and by Teichmüller. It is geometrically very appealing, but it has the disadvantage of applying only to C^1 maps, or to maps which are C^1 except on a very small set.

We recall a second definition. It involves moduli of families of curves. Given a family of curves Γ in a domain Ω, the *modulus* $M(\gamma)$ of Γ is defined by the formula

$$M(\Gamma) = \inf_{\rho} \int_{\Omega} \rho^n(x)|dx|,$$

where the infimum is taken over all Borel functions on Ω satisfying

$$\int_{\gamma} \rho ds \geq 1 \quad \text{for all } \gamma \in \Gamma.$$

A homeomorphism $f : \Omega \to \Omega'$ is then said to be K-quasiconformal for some $K \in [0,\infty]$ if we have

$$\frac{1}{K} M(\Gamma) \leq M(f(\Gamma)) \leq KM(\Gamma) \tag{2.1}$$

for every family of curves Γ in Ω.

Such a definition of quasiconformal mappings was given by Ahlfors in his paper *On quasiconformal mappings* (1954). Grötzsch had already shown that Property (2.1) is satisfied by *quadrilaterals* under the first definition of K-quasiconformal maps, that is, when the letter Γ in (2.1) denotes the family of curves joining the two vertical sides of a quadrilateral (and in this case the quantity $M(\Gamma)$ is called the *modulus of the quadrilateral*). Thus, Ahlfors used Grötzsch’s property to get a new definition of quasiconformality. He defined a homeomorphism $f : \Omega \to \Omega'$ to be K-quasiconformal if Property (2.1) holds for every quadrilateral Γ in Ω.

The modulus of a family of curves is a conformal invariant, and it follows directly from this second definition that the notion of K-quasiconformality of a function is invariant by pre- and post-composition by conformal mappings. In this sense, quasiconformality can be seen as a conformal invariant.

A third definition, valid in an arbitrary metric space, requires that a homeomorphism $f : \Omega \to \Omega'$ between two domains in $\mathbb{R}^n$ has bounded infinitesimal distortion. Here, one defines the *infinitesimal distortion* $H(x, f)$ of f at a point x in Ω by

$$H(x, f) = \lim \sup_{|h| \to 0} \frac{\max |f(x+h) - f(x)|}{\min |f(x+h) - f(x)|},$$

and the map f is said to be *quasiconformal* in Ω if its infinitesimal distortion is bounded, that is, if $\sup_{x \in \Omega} H(x, f) < \infty$.

It turns out that in higher dimensions there is an important generalization of quasiconformal mappings to non-injective mappings, called *quasiregular* mappings.[6] Quasiconformal mappings and quasiregular mappings are solutions of partial differential equations which are analogous to the Beltrami equation satisfied by quasiconformal mappings in the plane. The bases of the theory of quasiregular mappings was developed by Reshetnyak, Martio, Rickman and Väisälä.

In Chapter 14, after surveying the general theories of quasiconformal and quasiregular mappings in domains in $\mathbb{R}^n$, the author presents the important results on the quasiconformal theory of manifolds. There are several interesting results in this setting. It is known that the topological and the smooth categories of manifolds in dimension four are very different. The theory of quasiconformal 4-manifolds lies between the two. The author in Chapter 14 surveys in particular Sullivan's important uniformization theorem stating that except in dimension 4, every topological manifold admits a unique quasiconformal structure. Donaldson and Sullivan, in joint work, showed that several results of Donaldson in the smooth category hold in the quasiconformal one. The results they obtained shed new light on the fact that the (quasiconformal) theory of manifolds of 4-manifolds is very different from the theory in other dimensions. The following are two of their main results:

I. There are topological 4-manifolds that do not admit any quasiconformal structure. This results is in line with a result of Freedman. (In contrast, a result by Sullivan says that any topological manifold in dimension $\neq 4$ admits a quasiconformal structure.)

II. There are smooth (and in particular quasiconformal) compact 4-manifolds that are homeomorphic but not quasiconformally homeomorphic.

The author in Chapter 14 also touches upon the Donaldon–Sullivan approach to Yang–Mills theory on quasiconformal 4-manifolds. He also presents an outline of Nevanlinna theory on the growth rate of meromorphic function, a theory which is very powerful in complex analysis. He then reviews the appearance of quasiconformal mappings in non-linear potential theory and the dynamics of quasiregular mappings,

[6] It seems that this notion is due to Reshetnyak, who called these maps "maps of bounded distortion". The name "quasiregular mappings was introduced by Martio, Rickman and Väisälä. (See Vuorinen's review of Reshetnyak's book *Space mappings with bounded distortion*, Bulletin of the AMS, 1991.) In some precise sense, quasiconformal mappings in Euclidean n-space generalize plane conformal mappings whereas quasiregular mappings generalize analytic functions of one complex variable.

highlighting the analogies between this theory and the theory of iteration of rational maps on the Riemann sphere. The survey contains a section on quasiconformal group actions, introduced by Gehring and Palka, and also developed by Sullivan, Tukia and others. The author then presents the solution of the Hilbert-Smith conjecture for quasiconformal actions (Martin,1999) stating that any locally compact group of quasiconformal homeomorphisms acting effectively on a Riemannian manifold is a Lie group. He formulates a "Lichnerowicz problem" for rational maps of manifolds, asking for a classification of closed n-manifolds that admit a non-injective rational map; for injective mappings, the problem was solved in the 1970s. The author also reports on a "Negative Curvature Theorem" implying that a branched quasiregular mapping between closed hyperbolic manifolds cannot induce an injection at the level of fundamental groups. He mentions works of Gromov, Varopoulos, Saloff-Coste and Coulhon in this connection. He also presents several topological rigidity results for quasiregular mappings between hyperbolic manifolds, works on quasiconformal groups by Tukia, Gromov, Sullivan, Gehring, Freedman, Skora and others, as well as the theory of quasiregular semigroups developed by himself and Mayer.

3 Part C. Dynamics

3.1 Dynamics on Teichmüller spaces of surfaces of infinite type

Chapter 15 by Katsuhiko Matsuzaki is a survey on the dynamics of actions of mapping class groups of surfaces of topologically infinite type on the corresponding Teichmüller spaces. Let us recall some facts on Teichmüller spaces of surfaces of infinite type. Like in the case of surfaces of finite type, we take a base Riemann surface R and define its Teichmüller space $\mathcal{T}(R)$ as the space of homotopy classes of quasiconformal homeomorphisms from R to another (varying) Riemann surface. Unlike the definition in the case of surfaces of finite type, the definition here depends on the choice of the base surface, for several reasons. First of all, not all surfaces of infinite type are homeomorphic, and therefore the associated spaces are *a priori* different. Secondly, it is known that there exist pairs of Riemann surfaces of infinite type that are homeomorphic but not related by any quasiconformal homeomorphism. Therefore, the Teichmüller space $\mathcal{T}(R)$ of R is restricted by the choice of the base conformal structure R.

The Teichmüller space of any Riemann surface of infinite type is infinite-dimensional. A result by Fletcher, which is reported on in Volume II of this Handbook, says however that all these spaces, equipped with their Teichmüller metric, are locally bi-Lipschitz equivalent; in fact, they are locally bi-Lipschitz equivalent to the Banach space l^∞.[7]

[7] Fletcher's result is stated in the setting of the so-called *non-reduced* Teichmüller theory, that is, the theory where the equivalence relation that defines the elements of the space is homotopy that is the identity on the ideal boundary of the marked surfaces.

We also recall that in the theory of mapping class group actions on Teichmüller spaces, there are severe differences between the cases of surfaces of finite type and surfaces of infinite type. First of all, while in the case of surfaces of finite type there is one commonly used definition of the mapping class group, a definition which is purely topological (with some variations, regarding the actions on the boundary components, or the fact that the mapping classes considered preserve the punctures pointwise, and so on), in the case of surfaces of infinite type there are many possibilities. For instance, for the actions on Teichmüller spaces defined using quasiconformal maps that we just recalled, one usually considers only homotopy classes of quasiconformal homeomorphisms.[8] The mapping class group of R, which is also called in Chapter 15 the Teichmüller modular group (because it depends on the choice of the Teichmüller space), is defined as the space of homotopy classes of quasiconformal homeomorphisms of R. This chapter is a survey of the dynamics of the action of the mapping class group of R on the Teichmüller space $\mathcal{T}(R)$.

Another difference between the case considered here and the case of surfaces of topologically finite type is that the mapping class group of a surface of infinite type is generally uncountable. Furthermore, in the case of surfaces of finite type, the action of the mapping class group on Teichmüller space is properly discontinuous, and therefore the study of the dynamical properties of such an action has a rather limited scope. In the setting of surfaces of infinite type, this is not the case, and there are different types of orbit behavior under the corresponding actions.[9]

Besides the dynamical properties of the action of the mapping class group itself, several dynamical properties of actions of *subgroups* of the mapping class groups of surfaces of infinite type on the corresponding Teichmüller spaces are highlighted in Chapter 15. Such actions can be classified at a given point, with respect to the behavior of the orbit of that point. For instance, such an action might be (in the terminology used by Matsuzaki):

- discontinuous: the orbit of the point is discrete and its stabilizer is finite;
- weakly discontinuous: the orbit of the point is discrete;
- stable: the orbit of the point is closed and the stabilizer is finite;
- weakly stable: the orbit of the point is closed.

There are other dynamical properties that enter the scene. For instance, Matsuzaki makes a distinction between subgroups of bounded type (i.e., those for which orbits

[8] There are other settings than the quasiconformal one, and they are not considered in this chapter. For instance, one can study mapping class group actions on Teichmüller spaces where the base surface is equipped with a hyperbolic metric (and not only a conformal structure); Teichmüller space can be taken here to be the set of homotopy classes of hyperbolic metrics that are bi-Lipschitz equivalent to the base structure by a homeomorphism that is homotopic to the identity. In this setting one defines the mapping class group (or, what is more appropriately called in this setting, the bi-Lipschitz mapping class group) as the group of homotopy classes of bi-Lipschitz homeomorphisms. There are several other possibilities.

[9] The fact that in the infinite-dimensional Teichmüller spaces the action of the mapping class group is not properly discontinuous was known since a long time (it is mentioned in Bers' 1964 ETH lecture notes *On moduli of Riemann surfaces*).

of points are bounded) and of divergent type (i.e., the intersection of the orbit with any bounded subset is finite). To such actions are associated limit points, limit sets, regions of discontinuity (the complements of limit sets) and regions of stability (the set of points where the Teichmüller modular group acts stably). Some of these concepts are imported into this setting from the dynamical theory of Kleinian groups.

At the level of the surface itself, a useful notion of a *surface of bounded geometry* is introduced. Here, a surface equipped with a Riemannian metric is said to be of bounded geometry if the injectivity radius at each point is bounded from above and from below by uniform positive constants, except for neighborhoods of cusps.

The *region of stability* of a group action is the set of points whose orbits are closed. In the present setting, this region is a dense open subset of Teichmüller space. This leads to a useful variation on the notion of moduli space which is proper to the case of surfaces of infinite type, namely the *stable moduli space* is defined as the metric completion of the quotient of the region of stability by the Teichmüller modular group.

Matsuzaki also considers the *asymptotic Teichmüller space*. This is a parameter space for complex structures whose dilatation with respect to the base complex structure is arbitrarily small in the neighborhoods of the topological ends of the surface. The asymptotic Teichmüller space, like the Teichmüller space itself, is infinite-dimensional and non-separable. This space was introduced by Gardiner and Sullivan in the case where the underlying surface is the hyperbolic plane,[10] and it was then studied by Earle, Gardiner and Lakic for more general surfaces. This space admits a complex structure, and its group of biholomorphic automorphisms is called the *asymptotic Teichmüller modular group*. The action of the asymptotic Teichmüller modular group on the asymptotic Teichmüller space was studied by Fujikawa.

Teichmüller space can be regarded as a fiber space over the asymptotic Teichmüller space. The mapping class group acts on Teichmüller space preserving the fibers, and this action is studied separately on the asymptotic Teichmüller space and on the fibers. Notions of asymptotically elliptic modular transformations and of asymptotically elliptic subgroups of modular groups are introduced.

Several particular subgroups of mapping class groups in this setting of surfaces of topological infinite type are highlighted. Examples include the following:

- The *stable mapping class group* is the group of mapping classes that are supported on surfaces of finite type. This group acts naturally on the asymptotic Teichmüller space, and, in the case where the surface is of bounded geometry, the action is discontinuous. We note that the stable mapping class group had already appeared in other contexts; for instance, Tillman worked on the homotopy of the stable mapping class group, and Weiss studied the cohomology of that group.

- The *pure mapping class group* is the group of mapping classes that fix all surface ends except the cuspidal ones. It is a closed normal subgroup of the mapping class group which contains the stable mapping class group.

[10] Here, one considers again the non-reduced Teichmüller theory of surfaces with nonempty ends.

- A *stationary group* of the mapping class group is a subgroup G for which there exists a compact sub-surface V of S such that $g(V) \cap V \neq \emptyset$ for every homeomorphism g representing an element of G.
- The *asymptotically trivial mapping class group* is the kernel of the homomorphism from the mapping class group to the group of biholomorphic isometric automorphisms of the asymptotic Teichmüller space.

  The asymptotically trivial mapping class group contains the stable mapping class group.
- Countable subgroups of the mapping class group have special dynamical properties. For instance, any countable and closed subgroup of the mapping class has a nonempty region of discontinuity.

Matsuzaki points out in Chapter 15 a relation between quasiconformal mapping classes that act trivially on the asymptotic Teichmüller space and the *asymptotic Nielsen realization problem*, which concerns the asymptotic Teichmüller modular group.

Another theme that is developed in this chapter is the fact that in the setting of surfaces of infinite type there are interesting *intermediate* moduli spaces, obtained by quotienting Teichmüller spaces by appropriate subgroups of the Teichmüller modular group. One reason for studying these intermediate spaces is that, as we already pointed out, the mapping class group does not act properly discontinuously on Teichmüller space, and consequently the quotient of this action does not inherit geometric structures in the usual sense from those of Teichmüller space. But some interesting subgroups of mapping class groups act properly discontinuously on Teichmüller space, and the quotients by these actions are examples of intermediate spaces. They lie between Teichmüller spaces and moduli spaces. The list of interesting intermediate moduli spaces that are considered in Chapter 15 includes the following:

- The *stable moduli space*, which is a metric completion of the quotient of the region of stability by the Teichmüller modular group.
- The *enlarged moduli space*, which is the quotient of Teichmüller space by the stable modular group.
- The *moduli space of stable points*, which is the quotient of the region of stability by the Teichmüller modular group.
- The *moduli space of discontinuous points*, that is, the quotient of the region of discontinuity by the Teichmüller modular group.
- The *geometric moduli space*, that is, the quotient of Teichmüller space by the equivalence relation of *closure equivalence*, where two points are considered equivalent if one of them is contained in the closure of the orbit of the second one. The geometric moduli space is a quotient of the topological moduli space, and it is equipped with a complete metric.
- The quotient of Teichmüller space by the subgroup consisting of mapping classes that act trivially on the asymptotic Teichmüller space.

All these spaces as well as the relations between them are reviewed in Chapter 15.

3.2 Teichmüller theory and complex dynamics

A rational map is a map of the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ which is of the form $f(z) = \frac{p(z)}{q(z)}$ where p and q are polynomial mappings. The theory of iteration of rational maps was developed in the 1920s by Pierre Fatou (1878–1929) and Gaston Julia (1893–1978). In this theory, a rational map f is considered as a discrete dynamical system acting on S^2. If $f : S^2 \to S^2$ is a rational map, then one is interested in the behavior of the orbits of points in S^2 under iteration by f. The sphere, under this action, is decomposed into two subsets: a closed subset called the Julia set, on which the dynamics of the iterates of a point is "chaotic", and its complement, called the Fatou domain, on which the dynamics is "predictable". More precisely, the Fatou domain of a rational map f is the set of points z on the Riemann sphere such that the family of iterates $\{f^n\}$ is a normal family in a neighborhood of z. (We recall that a family of analytic maps defined on an open subset U is said to be *normal* if every sequence has a convergent subsequence; convergence is in the sense of uniform convergence on compact sets of U). The Fatou domain is also the largest subset of the Riemann sphere on which the family of iterates of f is a normal family. That the behavior of the iterates of f on the Julia set is "chaotic" is expressed by the fact that for any open set U meeting this set, the closure of the union of the forward orbits of U, that is, the set $\overline{\bigcup_{n\geq 0} f^n(U)}$, is equal to the whole Riemann sphere. There are several other features of chaotic behavior of points in the Julia set. For instance, the Julia set is the closure of the *repelling* periodic points of f, that is, the periodic points whose multiplier is > 1. The backward orbit $\bigcup_{n\geq 0} f^{-n}(z)$ of every point z in the Julia set is dense in this set. More generally, the backward orbit of any point of the Riemann sphere accumulates on the Julia set. Fatou and Julia, among other things, worked out a classification of the eventually periodic components of the Fatou domains.

In the 1980s, the theory of iteration of rational maps became again fashionable, due in large part to the work of Sullivan, Douady, Hubbard and their collaborators. Sullivan proved in 1985 a conjecture stating that there are no "wandering domains" for rational maps, that is, every component of the Fatou domain of a rational map is eventually periodic and there are only finitely many periodic components. This question had been left open by Fatou and Julia in the 1920s. Perhaps more importantly, in the same paper, Sullivan developed a correspondence between the theory of iteration of rational maps and the theory of Kleinian groups, that is, discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$ acting by conformal automorphisms on the Riemann sphere S^2 and by isometries on hyperbolic three-space $\mathbb{H}^3$, S^2 being regarded as the boundary at infinity of $\mathbb{H}^3$. He stated his results in the form of a dictionary translating notions, techniques, and results from one theory into the other one. The analogy between the two theories was not obvious from the beginning since the actions are *a priori* of different kinds: on the one hand, the dynamics is obtained by iterating a single rational map, and on the other hand it is induced on the sphere by the action of a group acting on hyperbolic three-space. The analogy immediately turned out to be most fruitful; it inspired several geometers, and it gave a huge impulse to research in both fields. Let us mention a few entries of the dictionary established by Sullivan. We recall:

(1) Fatou domains of rational maps correspond to domains of discontinuity of Kleinian groups. (We recall that the domain of discontinuity of a Kleinian group Γ is the set of points $z \in S^2$ which have a neighborhood $U(z)$ such that the set $\{\gamma \in \Gamma | \gamma(U) \cap U \neq \emptyset\}$ is finite.)

(2) Julia sets correspond to limit sets, that is, complements of domains of discontinuity (both sets being loci of chaotic behavior of the given dynamical system).

(3) The Mandelbrot set, a parameter space for (degree two) polynomials, corresponds to Teichmüller space, and more precisely to a Bers slice representation of Teichmüller space in quasi-Fuchsian space.

(4) Sullivan's "no wandering domain" theorem for Fatou domains which we already mentioned corresponds to Ahlfors' finiteness theorem (1964) saying that the quotient of the domain of discontinuity of a finitely-generated torsion-free Kleinian group by the group action has a finite number of components, and that each such component is a compact Riemann surface with a number of points removed. The measurable Riemann Mapping Theorem (1960), developed by Ahlfors and Bers as a fundamental tool in Teichmüller theory, was the essential ingredient used by Sullivan in the solution of the problem of wandering domains.

There are other entries in the dictionary, and some of the most important ones have been formulated by McMullen. We shall further comment on this below.

The proof of Ahlfors' measure zero conjecture (stated in 1966) saying that the limit set of a finitely-generated Kleinian zero either is the whole Riemann sphere or it has measure zero was recently completed by Agol, Calegari and Gabai. This conjecture was put in parallel with a conjecture stating that the Julia set of a rational map either has full measure or has nonempty interior. The latter was recently shown to be false by Buff and Chéritat. Thus, in that particular case, the dictionary is not completely faithful. But it is probable that the formulation of the conjecture on the measure of Julia sets was motivated by Ahlfors' measure zero conjecture, and this is also an instance of how the two fields influenced each other.

Some of the open problems that were raised by Sullivan concern the local structure of limit sets (respectively Julia sets), for instance, local connectivity and fractal behavior. Other more general problems were expected to have several possible answers and to remain open for years. One such problem asks for associating a 3-manifold to a rational map of the sphere, in the same way a 3-manifold is associated to a Kleinian group (the quotient of the action of that group on hyperbolic three-space, seen as the interior of the Riemann sphere).

McMullen proposed in 1994 a list of open problems on the dynamics of rational maps that are inspired from the theory of Kleinian groups, in a paper entitled *Rational* *maps and Teichmüller space: Analogies and open problems*.[11] One of these problems asks for the description of a boundary structure for the space of polynomials of degree n with an attracting fixed point with all critical points in its immediate basin, in analogy with the construction of boundaries for Teichmüller space. Another problem concerns the analogy between the mating operation of proper holomorphic maps from the unit disk to itself and the mating of quasi-Fuchsian groups provided by Bers' simultaneous uniformization theorem.

At the 1990 ICM in Kyoto, McMullen gave a communication titled *Rational maps and Kleinian groups*. The whole subject of that communication was to highlight three items in the dictionary between iteration of rational maps and Kleinian groups. These items concern particularly the (then developing) theory of hyperbolization of 3-manifolds. More precisely, they ask the following:

(1) Make a parallel between the combinatorics of critically finite rational maps and the geometrization of Haken 3-manifolds *via* iteration on Teichmüller theory.

(2) Make a parallel between renormalization of quadratic polynomials and 3-manifolds which fiber over the circle.

(3) Make parallels between the notions of boundaries and laminations in both theories, in particular between Teichmüller space in the Bers embedding and the Mandelbrot set.

Several developments of the theory of iteration of rational maps took place during the last three decades. The aim of some of these works was to find new invariants of post-critically finite rational maps and to define new conformal dynamical systems by combining known ones. In particular, these works led to the notions of decomposition and of mating, and to other surgery techniques in complex dynamics.

A cornerstone of the dictionary between the two theories is Thurston's theorem on the topological characterization of (conjugacy classes of) post-critically finite rational maps among orientation-preserving branched covering maps of the Riemann sphere. The theorem and its proof made a profound link between iteration of rational maps and Teichmüller theory.

Chapter 16 of this volume, by Xavier Buff, Guizhen Cui and Tan Lei concerns Thurston's theorem. To state this theorem, we recall a few points of vocabulary. A *critical point* of an orientation-preserving branched covering $F\colon S^2 \to S^2$ is a point at which F is not locally injective; the *critical set* of F is the set of critical points of F; the *post-critical set* of F is the closure of the set of forward images of the critical set. The post-critical set is also the smallest forward-invariant closed set containing the critical values of f.

Thurston's theorem concerns equivalence classes of orientation-preserving branched coverings $F\colon S^2 \to S^2$ whose post-critical set is finite. The theorem gives a characterization of post-critically finite branched coverings $F\colon S^2 \to S^2$ that are equivalent to rational maps, and it tells us to what extent such a rational map is unique

[11] In: V. P. Havin and N. K. Nikolskii, editors, Linear and Complex Analysis Problem Book, volume 1574 of Lecture Notes in Math., p. 430-433. Springer, 1994.

when it exists. The theorem is expressed in terms of a combinatorial obstruction for the existence of a rational map representing a post-critically finite branched covering. The obstruction consists in the existence of an invariant system of essential and pairwise non-homotopic curves (the system is called a "multicurve") to which is associated a certain incidence matrix, with an eigenvalue ≥ 1 equal to its spectral radius. The theory uses the familiar Perron–Frobenius Theorem from linear algebra. As such, the obstruction has the flavor of a well-known obstruction for a mapping class to be of pseudo-Anosov type. In other words, Thurston's result says that the post-critically finite map F is equivalent to a rational map if and only if every nonnegative eigenvalue of some induced action of this map on a vector space whose basis is the elements of the multi-curve is < 1. In the context of mapping classes, one has a much similar characterization of pseudo-Anosov maps: non-existence of stable curves for any iterate of the map, with several criteria formulated in terms of the action of an incidence matrix on some combinatorial data (train tracks) and the use of the Perron–Frobenius Theorem. McMullen established a relation between Thurston's characterization and the geometrization of 3-manifolds.

Although the relation with Teichmüller theory is not obvious from the statement, the proof of Thurston's theorem uses an iteration on Teichmüller space, and the existence result provided by the theorem is obtained as a fixed point of an action on that space. Thurston obtained this theorem in 1982; he lectured on it on several occasions, and he circulated notes on the proof. A detailed proof was published in 1983 by Douady and Hubbard.

After giving a proof of Thurston's theorem, the authors of Chapter 16 present several recent applications of this theorem. The theory of iteration of rational maps combines hyperbolic geometry and complex analysis, as the classical Teichmüller theory does. The tools that are used involve quasiconformal maps, the Teichmüller metric and quadratic differentials. Associated to a rational map f of the Riemann sphere, there is a *moduli space*, a group $\mathrm{QC}(f)$ consisting of the quasiconformal automorphisms of the sphere that commute with f, a *modular group* $\mathrm{Mod}(f)$, which is the quotient of $\mathrm{QC}(f)$ by the quasiconformal automorphisms that are isotopic to the identity, and there is a *Teichmüller space* $\mathcal{T}(f)$ of f, which is a space of rational maps equipped with a marking which is a quasiconformal conjugacy with f. The theory of Teichmüller spaces associated to holomorphic dynamical systems was developed by McMullen and Sullivan.

A new ingredient in the theory, compared to the classical theory of surface homeomorphism actions on Teichmüller spaces, is the use of the action of the branched cover on the set of homotopy classes of essential multi-curves. The tools that are presented in Chapter 16 include actions on cotangent spaces to Teichmüller spaces (identified with spaces of integrable meromorphic quadratic differentials), estimates of norms of maps between cotangent spaces and between tangent spaces, and a search for contraction properties of such maps, which are in fact major tools in analytical Teichmüller theory.

We should mention again that in Chapter 14 of this volume, the author highlights several analogies between the theory of iteration of rational maps of the sphere and the theory of iteration of quasiconformal maps in $\mathbb{R}^n$

4 Part D. The quantum theory, 2

Chapter 17 of this volume concerns the quantization theory of Teichmüller space, and it is written by Ren Guo. This chapter can be considered as a sequel to the four chapters on quantization, written by Chekhov–Penner, Fock–Goncharov, Teschner and Kashaev that are contained in Volume I of this Handbook.

We recall that the quantization theory of the Teichmüller space of a punctured surface is a theory of deformations of the C^*-algebra of functions on that space. It was first developed independently by Chekhov–Fock and by Kashaev. The theory was motivated by theoretical physics, namely, by the physical interpretation of $2+1$-dimensional gravity as a Chern–Simons quantum field theory with noncompact gauge group. From the analytic point of view, both works of Chekhov–Fock and of Kashaev make use of self-adjoint operators on Hilbert spaces and of the quantum dilogarithm function. While the geometric setting of the work of Chekhov and Fock uses Thurston's shear coordinates for Teichmüller spaces, the setting of Kashaev employs the notion of decorated ideal triangulation, that is, an ideal triangulation in which the set of triangles is equipped with a total order and where in each ideal triangle there is a mark at one of its corners. Kashaev's theory also uses the lambda-length parameters that were introduced by Penner.

More recently, a purely algebraic version of the Chekhov–Fock algebra was worked out by X. Liu, and a similar description was done for the Kashaev algebra in joint work by R. Guo and X. Liu. In this work, Guo and Liu established a natural link between the Chekhov–Fock algebra and an appropriate generalization of the Kashaev algebra by examining the relationship between the two kinds of coordinates for Teichmüller space that we mentioned: the shear coordinates and the Kashaev coordinates on decorated ideal triangles.

In Chapter 17, Guo makes a survey of the algebraic aspect of this circle of ideas, and in particular on the relationship between the Chekhov–Fock algebra and the generalized Kashaev algebra. He also presents some recent work by Bonahon and Liu on the representation theory of the Chekhov–Fock algebra.

5 Part E. Sources

Chapter 18 consists of a translation, by Annette A'Campo-Neuen, of Teichmüller's paper *Veränderliche Riemannsche Flächen* (Variable Riemann Surfaces), published in 1944. This paper is the last one that Teichmüller wrote on the problem of moduli. He presents in it a construction of Teichmüller space which is completely new compared with the construction he introduces in his first seminal paper on the subject, *Extremale quasikonforme Abbildungen und quadratische Differentiale* (1939) and its sequel *Bestimmung der extremalen quasikonformen Abbildungen bei geschlossenen orientierten Riemannschen Flächen* (1943). In these two papers, Teichmüller laid down the foundations of the metric theory of Teichmüller space, showing that this space is homeomorphic to a ball of a certain dimension and proving in particular the existence and uniqueness of the extremal map in each homotopy class of homeomorphisms that was later on given the name *Teichmüller extremal map*. In the present paper, Teichmüller equips the space with a complex analytic structure, and he characterizes it by a certain universal property. At the same time, he introduces a fibre bundle which today is called the *Teichmüller universal curve*. The paper is rather sketchy, and it contains the following results:

(1) The existence and uniqueness of the universal Teichmüller curve and the idea of a fine moduli space, that is, a fiber space where the isomorphism type of the fibre determines the point below it.

(2) The proof of the fact that the automorphisms group of the universal Teichmüller curve is the extended mapping class group.

(3) The idea that Teichmüller space represents a functor. (We are using a language that Grothendieck introduced a few years later.)

(4) The idea of using the period map to define a complex structure on Teichmüller space.

The paper is rather unknown in the mathematical community and it was read only by very few specialists. This is the reason why we decided to include it here in English translation, together with a commentary (Chapter 18 of this volume) written by Annette A'Campo-Neuen, Norbert A'Campo, Lizhen Ji and the author of this introduction.