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BOOK REVIEW

Piotr Nowak, Guoliang Yu: “Large Scale Geometry”

EMS, 2012, 203 pp.

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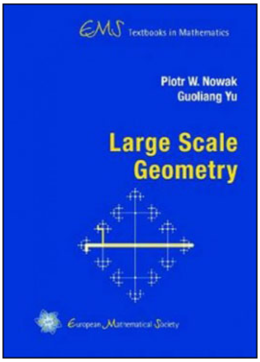

One of the most important invariants associated to a topological space X with base point x_0 is its *fundamental group* $\pi_1(X, x_0)$. It consists of equivalence classes of closed loops in X based at x_0, two loops being identified if they can be deformed into each other through loops based at x_0. Concatenation of loops defines the group structure on $\pi_1(X, x_0)$, which is usually not abelian.

The fundamental group plays an important role in the classification of topological spaces. For example, it can be used to distinguish closed surfaces of different genera. The famous Poincaré conjecture, proved by Perelman, states that closed 3-manifolds with trivial fundamental groups are homeomorphic to spheres.

To what extent are topological, smooth and Riemannian manifolds in arbitrary dimensions determined by their fundamental groups? Homotopy theory shows that *aspherical* manifolds, whose universal covering spaces are contractible by definition, are determined by their fundamental groups up to homotopy equivalence. This is a much weaker notion than homeomorphism, diffeomorphism and isometry, the natural equivalence relations for topological, smooth and Riemannian manifolds.

Hyperbolic manifolds, i.e. Riemannian manifolds of constant sectional curvature -1, form a beautiful and rich class of aspherical manifolds. The famous rigidity theorem of Mostow (1968) states that closed hyperbolic manifolds of dimension $n \geq 3$ are isometric, if their fundamental groups are isomorphic. This is an instance

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of *rigidity*, because a relatively rough invariant, the isomorphism type of the fundamental group, encodes the complete geometric information on the underlying object.

For the proof of Mostow rigidity one lifts the given homotopy equivalence to universal covering spaces, both of which are isometric to the hyperbolic space $\mathbb{H}^n$. This lift is a *quasi-isometry*: It differs from an actual isometry of $\mathbb{H}^n$ by a uniform bound. It can therefore be extended to a quasi-conformal map on the ideal boundary $\partial\mathbb{H}^n$, which can be identified with $\mathbb{S}^{n-1}$. For $n \geq 3$ it can be shown that this map is in fact conformal and induced by an isometry of $\mathbb{H}^n$. This in turn induces an isometry of the original hyperbolic manifolds. This proof illuminates the role of asymptotic properties of hyperbolic space and its group of isometries.

Mostow rigidity is related to the famous *Borel conjecture* stating that closed aspherical manifolds with isomorphic fundamental groups are homeomorphic. This conjecture is known to hold for manifolds of dimension at least 5 and with Gromov hyperbolic fundamental groups (Bartels-Lück 2012), a property which is defined by referring to large scale geometric properties of groups equipped with word length metrics.

Another instance of rigidity is the *Novikov conjecture*, which predicts homotopy invariance of higher signatures. It can be expressed as an injectivity statement of the assembly map relating the topological K-theory of the fundamental group to the K-theory of its group C^*-algebra. By the descent principle in coarse geometry this property is connected to the large scale geometry of the fundamental group. The Novikov conjecture is known to hold, if the fundamental group admits a coarse embedding into Hilbert space (Yu 2000), a property which is again of a large scale geometric nature.

It is remarkable that in these examples the fundamental group enters not only as an algebraic, but also as a geometric object: The *word length metric* on a group defines the distance of two group elements g and h as the minimal length of a word expressing $g^{-1}h$ as a product in a given set of generators and their inverses. For finitely generated groups the resulting metric is uniquely determined up to *quasi-isometry*, i.e. word metrics associated to different finite sets of generators of a given group are bi-Lipschitz equivalent up to a globally bounded difference. With respect to the equivalence relation of quasi-isometry, which can be defined for arbitrary metric spaces, the real line with the usual metric is identified with the integers regarded as a discrete subspace. Intuitively the relevant geometric information is the one which remains visible when seen from far apart, or at "large scales". This concept, which had appeared before in Mostow's book *Strong rigidity of locally symmetric spaces* (1973) and in Gromov's work in geometric group theory, among others, was later generalized and put into an axiomatic setting in John Roe's *coarse geometry*, revealing important connections between large scale geometric properties, index theory and non-commutative geometry.

The monograph under review offers a systematic introduction to the subject of large scale geometry, with an emphasis on geometric group theory and functional analytic methods, some of which are relevant for Yu's theorem mentioned before.

The book is divided into eight chapters. The first one outlines general concepts like metric spaces, word metrics on groups, quasi-isometries, the Švarc-Milnor lemma and coarse equivalences. It ends with a short introduction to Gromov hyperbolic spaces and groups, which share essential features with the hyperbolic space $\mathbb{H}^n$ on a large scale.

The following five chapters discuss several asymptotic properties of metric spaces and groups. Asymptotic dimension (Gromov 1993) and decomposition complexity (Guentner-Tessera-Yu 2012), which are the theme of Chap. 2, play important roles in recent progress concerning the Novikov- and stable Borel conjectures.

Chapter 3 introduces amenability, a fundamental concept going back to von Neumann (1929), and growth conditions for groups. The latter occur in the polynomial growth theorem of Gromov (1981), one of the cornerstones of geometric group theory.

Property A (Yu 2000), a non-equivariant analogue of amenability, is defined and discussed in Chap. 4. This notion is put into the context of coarse embeddability into Hilbert space (Gromov 1991) in Chap. 5, which also discusses expanders and their use for the construction of metric spaces not coarsely embeddable into Hilbert space.

An important class of groups which embed coarsely into Hilbert space are a-T-menable groups (Haagerup 1979; Gromov 1991). This concept is explained in Chap. 6, which deals with group actions on Banach spaces. A-T-menability is presented as an equivariant analogue of coarse embeddability into Hilbert space, in a similar way as amenability can be viewed as an equivariant analogue of property A. This chapter also introduces Kazhdan's property (T), its spectral properties and applications to the construction of expanders.

Chapter 7 is devoted to concepts of coarse algebraic topology. It starts with an introduction to coarse homology (Roe 1993) and its bounded refinement, uniformly finite homology (Block-Weinberger 1992). The equivalence of the vanishing of 0-th uniformly finite homology with non-amenability, the characterization of quasi-isometries that are close to bi-Lipschitz maps in terms of uniformly finite homology and the use of uniformly finite homology for a construction of aperiodic tilings are presented. The chapter ends with a short introduction to coarse generalized homology theories and an application of coarse homology to the determination of upper bounds on the asymptotic dimension of bounded geometry metric spaces.

The last chapter contains a rather brief account of applications to topological rigidity, largeness properties of non-compact manifolds, index theory, the Baum-Connes and Novikov conjectures, existence of metrics of uniformly positive scalar curvature, and the zero-in-the-spectrum conjecture.

Each of the first seven chapters ends with a set of exercises and with some very informative “Notes and remarks” on the history of the subject, important results and open problems.

After Gromov's papers and books on geometric group theory, Roe's texts on coarse geometry and Higson-Roe's book on analytic K-homology this monograph serves as an up-to-date introductory guide to the active research field of large scale geometry. The emphasis lies on a presentation of concepts and their interrelation, illustrated by plenty of examples, rather than on a complete exposition of the theory. Many important results, classical and recent, are mentioned, but not discussed in detail, such as, remarkably, Yu's proof of the Novikov conjecture for groups that coarsely embed into Hilbert space. For a more thorough study of these topics, that motivate the concepts treated in the monograph, one must refer to other sources including the research literature.

In summary Nowak-Yu's "Large scale geometry" serves both as a nice complement to Roe's "Lectures on coarse geometry" (2003) and as a valuable survey of some of the more modern aspects of the field.