Introduction to Teichmüller theory, old and new, III

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Contents

Surveying a vast theory like Teichmüller theory is like surveying a land, and the various chapters in this Handbook are like a collection of maps forming an atlas: some of them give a very general overview of the field, others give a detailed view of some crowded area, and others are more focussed on interesting details. There are intersections between the chapters, and these intersections are necessary. They are also valuable, because they are written by different persons, having different ideas on what is essential, and (to return to the image of a geographical atlas) using their proper color pencil set.

The various chapters differ in length. Some of them contain proofs, when the results presented are new, and other chapters contain only references to proofs, as it is usual in surveys.

I asked the authors to make their texts accessible to a large number of readers. Of course, there is no absolute measure of accessibility, and the response depends on the sound sense of the author and also on the background of the reader. But in principle all of the authors made an effort in this sense, and we all hope that the result is useful to the mathematics community.

This introduction serves a double purpose. First of all, it presents the content of the present volume. At the same time, reading this introduction is a way of quickly reviewing some aspects of Teichmüller theory. In this sense, the introduction complements the introductions I wrote for Volumes I and II of this Handbook.

1 Part A. The metric and the analytic theory, 3

1.1 The Beltrami equation

Chapter 1 by Jean-Pierre Otal concerns the theory of the Beltrami equation. This is the partial differential equation

$$\bar\partial\phi = \mu\partial\phi, \tag{1.1}$$

where $\phi\colon U \to V$ is an orientation preserving homeomorphism between two domains U of V of the complex plane and where ∂ and $\bar\partial$ denote the complex partial derivativations

$$\partial\phi = \frac{1}{2}\Big(\frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y}\Big) \quad \text{and} \quad \bar\partial\phi = \frac{1}{2}\Big(\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\Big).$$

If ϕ is a solution of the Beltrami equation (1.1), then $\mu = \bar\partial\phi/\partial\phi$ is called the complex dilatation of ϕ.

Without entering into technicalities, let us say that the partial derivatives $\partial\phi$ and $\bar\partial\phi$ of ϕ are allowed to be distributional derivatives and are required to be in $L^2_{\mathrm{loc}}(U)$. The function μ that determines the Beltrami equation is in $L^\infty(U)$, and is called the *Beltrami coefficient* of the equation.

The Beltrami equation and its solution constitute an important theoretical tool in the analytical theory of Teichmüller spaces. For instance, the Teichmüller space of a surface of negative Euler characteristic can be defined as some quotient space of a space of Beltrami coefficients on the upper-half plane. As a matter of fact, this definition is the one commonly used to endow Teichmüller space with its complex structure.

The classical general result about the solution of the Beltrami equation (1.1) says that for any Beltrami coefficient μ satisfying $\|\mu\|_\infty < 1$, there exists a quasiconformal homeomorphism $\phi = f^\mu\colon U \to V$ which satisfies a.e. this equation, and that f^μ is unique up to post-composition by a holomorphic map. There are several versions and proofs of this existence and uniqueness result. The first version is sometimes attributed to Morrey (1938), and there are versions due to Teichmüller (1943), to Lavrentieff (1948) and to Bojarski (1955). In the final form that is used in Teichmüller theory, the result is attributed to Ahlfors and Bers, who published it in their paper *Riemann's* *mapping theorem for variable metrics* (1960). This result is usually referred to as the *Measurable Riemann Mapping Theorem*.

Note that in the case where μ is identically zero, the Beltrami equation reduces to the Cauchy–Riemann equation $\bar{\partial}\phi = 0$, and the result follows from the classical Riemann Mapping Theorem.

Ahlfors and Bers furthermore showed that the correspondence $\mu \mapsto f^{\mu}$ is holomorphic in the sense that if μ_t is a family of holomorphically parametrized Beltrami coefficients on the open set U, with t being a parameter in some complex manifold, then the map $t \mapsto f^{\mu_t}(z)$ (with a proper normalization) is holomorphic for any fixed $z \in U$. This result was used as an essential ingredient in the construction by Bers of the complex structure of Teichmüller space. Indeed, considering the elements of Teichmüller space as equivalence classes of solutions f^{μ} of the Beltrami equation with coefficient μ, the complex structure of Teichmüller space is the unique complex structure on that space satisfying the above parameter-dependence property.

Chapter 1 is an account of recent work on the Beltrami equation. It contains a proof of the Measurable Riemann Mapping Theorem. While the original work on the Beltrami equation, as developed by Morrey, Bojarski and Ahlfors–Bers uses hard analysis (Calderon–Zygmund theory, etc.), the proof presented here should be more accessible to geometers. The existence part in this proof was recently discovered by Alexey Glutsyuk. It concerns the case where the Beltrami coefficient is of class C^{∞}. The general case can be deduced by approximation.

After presenting Glutsyuk's proof, Otal surveys a substantial extension of the theory of the Beltrami equation, namely, the extension to the case where $\|\mu\|_{\infty} = 1$. It seems that such an extension was first studied by Olli Lehto in 1970, with several technical hypotheses on the set of points in U where $\|\mu\|_{\infty} = 1$. The hypotheses were substantially relaxed later on. A major step in this direction was taken by Guy David who, in 1988, proved existence and uniqueness of the solution of the Beltrami equation with $\|\mu\|_{\infty} = 1$, with μ satisfying a *logarithmic growth condition* near the subset $\{|\mu| = 1\}$ of U. This general version of the Beltrami equation led to many applications, in particular in complex dynamics.

There have been, since the work of David, several improvements and variations. In particular, Ryazanov, Srebro & Yakubov introduced in 2001 a condition where the *dilatation function* K_{μ} of μ, defined by $K_{\mu} = \frac{1+|\mu(z)|}{1-|\mu(z)|}$, is bounded a.e. by a function which is locally in the John–Nirenberg space BMO(U) of bounded mean oscillation functions . (We recall that since, in the hypothesis of David's Theorem, $\|\mu\|_{\infty} = 1$ instead of $\|\mu\|_{\infty} < 1$, the dilatation function K_{μ} is not necessarily in L^{∞}.) In this case, the quasiconformal map $f^{\mu}\colon U \to V$ provided by the theorem is not quasiconformal in the usual sense, and it is called a BMO-*quasiconformal homeomorphism* (which explains the title of Chapter 1).

The chapter also contains some useful background material on quasiconformal maps, moduli and extremal length that is needed to understand the proofs of the results presented.

1.2 Earthquakes in Teichmüller space

After the chapter on the existence and uniqueness of solutions of the Beltrami equation, Chapter 2, written by Jun Hu, surveys another existence and uniqueness result, which is also at the basis of Teichmüller theory, namely, Thurston's Earthquake Theorem. The setting here is the hyperbolic (as opposed to the conformal) point of view on Teichmüller theory. The earthquake theorem says that for any two points in Teichmüller space, there is a unique left earthquake path that joins the first point to the second. A "global" and an "infinitesimal" version of this theorem are presented in their most general form, and a parallel is made between this generalization and the general theory of the Beltrami equation and its generalization that is reviewed in Chapter 1.

Earthquake theory has many applications in Teichmüller theory. Some of them appear in other chapters of this volume, e.g. Chapter 3 by Series and Chapter 14 by Krasnov and Schlenker.

Before going into the details of Chapter 2, let us briefly review the evolution of earthquake theory.

The theory originates from the so-called Fenchel–Nielsen deformation of a hyperbolic metric. We recall the definition. Given a hyperbolic surface S containing a simple closed geodesic α, the *time-t left* (*respectively right*) *Fenchel–Nielsen deformation of S along* α is the hyperbolic surface obtained by cutting the surface along α and gluing back the two boundary components after a rotation, or *shear*, "to the left" (respectively "to the right") of amount t. The sense of the shear (left or right) depends on the choice of an orientation on the surface but not on the choice of an orientation on the curve α. The amount of shearing is measured with respect to arclength along the curve.[1] The precise definition needs to be made with more care, so that while performing the twist, one keeps track of the homotopy classes of the simple closed geodesics that cross α. In more precise words, the deformation is one of marked surfaces. In particular, the surface obtained from S after a complete twist (a Dehn twist), as an element of Teichmüller space, is not the element we started with, because its marking is different.

The next step is to shear along a geodesic which is not a simple closed curve. For instance, one can shear along an infinite simple geodesic, that is, a geodesic homeomorphic to the real line. Making such a definition is not straightforward, unless the geodesic is isolated in the surface (for instance, if it joins two punctures, or two points on the ideal boundary). An earthquake deformation is a generalization of a Fenchel–Nielsen deformation where, instead of shearing along a simple closed geodesic, one performs a shearing along a general measured geodesic lamination. Here, the amount of shearing is specified by the transverse measure of the lamination. On order to make such a definition precise, one can define a time-t left (respectively right) earthquake deformation along a measured geodesic lamination μ as the limit of a sequence of time-t left (respectively right) earthquake deformations associated to weighted simple

[1] There is another normalization which is useful in some contexts, where the amount of shear is $t \times \mathrm{length}(\alpha)$. In this case, one talks about a *normalized* earthquake.

closed curves α_n, as this sequence converges, in Thurston's topology on measured lamination space, to the measured geodesic lamination μ. Although this definition is stated in a simple way, one cannot avoid entering into technicalities, because one has to show that the result does not depend on the choice of the approximating sequence α_n.

In any case, it is possible to make a definition of a time-t left (or right) earthquake along a general measured lamination μ. For a fixed μ, varying the parameter t, one obtains a flow on the unit tangent bundle to Teichmüller space: at each point, and in each direction (specified by the measured geodesic lamination) at that point, we have a flowline. This flow is called the *earthquake flow* associated to μ.

Earthquake deformations were introduced by Thurston in the 1970s, and the first paper using earthquakes was Kerckhoff's paper *Nielsen Realization Problem*, published in 1983, in which Kerckhoff gave the solution of the Nielsen Realization Problem. The solution is based on the convexity of geodesic length functions along earthquake paths, and on the "transitivity of earthquakes", that is, the result that we mentioned above on the existence of earthquakes joining any two points in Teichmüller space. The transitivity result is due to Thurston. Kerckhoff provided the first written proof of that result as an appendix to his paper.

A few years later, Thurston developed a much more general theory of earthquakes, in a paper entitled *Earthquakes in two-dimensional hyperbolic geometry* (1986). This included a new proof of the transitivity result. In that paper, earthquake theory is developed in the setting of the *universal Teichmüller space*, that is, the space parametrizing the set of complete hyperbolic metrics on the unit disk up to orientation-preserving homeomorphisms that extend continuously as the identity map on the boundary of the disk. (Note that without the condition on homeomorphisms extending as the identity map on the boundary, all hyperbolic structures on the disk would be equivalent.)

We recall by the way that the universal Teichmüller space was introduced by Ahlfors and Bers in the late 1960s.[2] One reason for which this space is called "universal" is that there is an embedding of the Teichmüller space of any surface whose universal cover is the hyperbolic disk into this universal Teichmüller space.

The universal Teichmüller space also appears as a basic object in the study of the Thompson groups, surveyed in Chapter 10 of this volume.

By lifting the earthquake deformations of hyperbolic surfaces to the universal covers, the earthquake deformation theory of any hyperbolic surface can be studied as part of the earthquake deformation theory of the hyperbolic disk. The deformation theory of the disk not only is more general, but it is also a convenient setting for new developments; for instance it includes quantitative relations between the magnitude

[2] There is a relation between the universal Teichmüller space and mathematical physics, which was foreseen right at the beginning of the theory; see Bers's paper *Universal Teichmüller space* in the volume *Analytic methods in Mathematical Physics*, Indiana University Press, 1969, pp. 65–83. In that paper, Bers reported that J. A. Wheeler conjectured that the universal Teichmüller space can serve as a model in an attempt to quantize general relativity. A common trend is to call $\mathrm{Diff}^+(\mathbb{S}^1)/\mathrm{PSL}(2,\mathbb{R})$ the *physicists universal Teichmüller space* and $\mathrm{QS}(\mathbb{S}^1)/\mathrm{PSL}(2,\mathbb{R})$ the *Bers universal Teichmüller space*. Here, $\mathrm{Diff}^+(\mathbb{S}^1)$ denotes the group of orientation-preserving homeomorphisms of the circle and $\mathrm{QS}(\mathbb{S}^1)$ its group of quasi-symmetric homeomorphisms, of which we talk later in this text.

of earthquake maps and distortions of homeomorphisms of the circle, as we shall see below.

Thurston's 1986 proof of existence and uniqueness of left (respectively right) earthquakes between hyperbolic structures in the setting of the universal Teichmüller space is based on a convex hull construction in the hyperbolic plane. In Thurston's words, this proof is "more elementary" and "more constructive" than the previous one.

One may also note here that in 1990, G. Mess gave a third proof of the earthquake theorem that uses Lorentz geometry. In Mess's words, this proof is "essentially Thurston's second (and elementary) proof, interpreted geometrically in anti-de Sitter space".[3]

We finally note that Bonsante, Krasnov and Schlenker gave a new version of the earthquake theorem, again using anti-de Sitter geometry, which applies to surfaces with boundary. Their proof relies on the geometry of "multi-black holes", which are 3-dimensional anti-de Sitter manifolds, topologically the product of a surface with boundary and an interval. These manifolds were studied by physicists. In that case, given two hyperbolic metrics on a surfaces with n boundary components, there are 2^n right earthquakes transforming the first one into the second one.[4] The anti-de Sitter setting has similarities with the quasi-Fuchsian setting; that is, the authors consider an anti-de Sitter 3-manifold which is homeomorphic to the product of a surface times an interval, and the two boundary components of that manifold are surfaces that are naturally equipped with hyperbolic structures.

Now we must talk about the notion of quasi-symmetry, which is closely related to the notion of quasiconformality.

Consider the circle $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. An orientation-preserving homeomorphism $h\colon \mathbb{S}^1 \to \mathbb{S}^1$ is said to be *quasi-symmetric* if there exists a real number $M \geq 1$ such that for all x on $\mathbb{S}^1$ and for all t in $]0, \pi/2[$, we have

$$\frac{1}{M} \leq \left| \frac{h(e^{i(x+t)}) - h(e^{ix})}{h(e^{ix}) - h(e^{i(x-t)})} \right| \leq M. \tag{1.2}$$

The notion of quasi-symmetric map was introduced by Beurling and Ahlfors in 1956, in a paper entitled *The boundary correspondence under quasiconformal mappings*. The main result of that paper says that every quasiconformal homeomorphism of the unit disk $\mathbb{D}^2$ extends to a unique homeomorphism of the closed disk $\overline{\mathbb{D}^2}$, that the induced map on the boundary $\mathbb{S}^1 = \partial\overline{\mathbb{D}^2}$ is quasi-symmetric and that conversely, any quasi-symmetric map of $\mathbb{S}^1$ is induced by a quasiconformal map of $\overline{\mathbb{D}^2}$.

Like the notion of quasiconformality, the notion of quasi-symmetry admits several generalizations, including an extension to higher dimensions and an extension to mappings between general metric spaces. The latter was studied by Tukia and Väisälä.

[3] Mess's work on that subject is reviewed and expanded in Chapter 14 of Volume II of this Handbook by Benedetti and Bonsante.

[4] The number 2^n corresponds to the various ways in which a geodesic lamination can spiral around the boundary components of the surface.

The space of quasi-symmetric maps of the circle considered as the boundary of the hyperbolic unit disk is an important tool in the theory of the universal Teichmüller space. Using the correspondence between the set of quasiconformal homeomorphisms of the open unit disk and the set of quasi-symmetric homeomorphisms of the boundary circle and making a normalization, the universal Teichmüller space can be identified with the space of quasi-symmetric homeomorphisms of $\mathbb{S}^1$ that fix three points.

Thurston noted in his 1986 paper that the fact that any quasiconformal homeomorphism of the circle extends to a homeomorphism of the disk establishes a one-to-one correspondence between the universal Teichmüller space and the set of right cosets $\mathrm{PSL}(2,\mathbb{R})\backslash\mathrm{Homeo}(\mathbb{H}^2)$.

We now recall that the quasiconformal distortion of a homeomorphism of the hyperbolic disk can be defined in terms of distortion of quadrilaterals in that disk. Analogously, the quasi-symmetry of a homeomorphism h of the circle can be defined in terms of distortion of cross ratios of quadruples of points on that circle. The parallel between these two definitions hints to another point of view on the relation between quasi-symmetry and quasiconformality.

Any one of the definitions of a quasi-symmetric map of the circle leads to the definition of a *norm* on the set $\mathrm{QS}(\mathbb{S}^1)$ of quasi-symmetric maps. One such norm is obtained by taking the best constant M that appears in Inequality (1.2) defining quasi-symmetry. Another norm is obtained by taking the supremum over distortions of all cross ratios of quadruples.

More precisely, given a homeomorphism $h\colon \mathbb{S}^1 \to \mathbb{S}^1$, one can define its *cross ratio norm* by the formula

$$\|h\|_{\mathrm{cr}} = \sup_{Q} \left| \ln \frac{\mathrm{cr}(h(Q))}{\mathrm{cr(Q)}} \right|,$$

where Q varies over all quadruples of points on the circle and $\mathrm{cr}(Q)$ denotes the cross ratio of such a quadruple. A homeomorphism is quasi-symmetric if and only if it has finite cross ratio norm.

Now we return to earthquakes.

Thurston calls *relative hyperbolic structure* on the hyperbolic disk a homotopy class of hyperbolic structures in which one keeps track of the circle at infinity.

A left earthquake, in the setting of the universal Teichmüller space, is a transformation of a relative hyperbolic structure of the hyperbolic disk $\mathbb{D}^2$ that consists in cutting the disk along the leaves of a geodesic lamination and gluing back the pieces after a "left shear" along each component of the cut-off pieces. The map thus obtained from $\mathbb{D}^2$ to itself is a "piecewise-Möbius transformation", in which the domain pieces are the complementary components of a geodesic lamination on $\mathbb{D}^2$, where the comparison maps $f_j \circ f_i^{-1}$ between any two Möbius transformations f_i and f_j defined on two such domains is a Möbius transformation of hyperbolic type whose axis separates the two domains and such that all the comparison Möbius transformations translate in the same direction. Such a piecewise-Möbius transformation defined on the unit disk is discontinuous, but it induces a continuous map (in fact, a homeomorphism) of the boundary circle. From this boundary homeomorphism one gets a new relative hyperbolic structure on the unit disk. Thurston proved the following:

(1) Any two relative hyperbolic structures can be joined by a left earthquake.

(2) There is a well-defined transverse measure (called the *shearing measure*) on the geodesic lamination associated to such a left earthquake. This transverse measure encodes the amount of earthquaking (or shearing) along the given lamination.

(3) Two relative hyperbolic structures obtained by two left earthquakes with the same lamination and the same transverse measure are conjugate by an isometry.

Thurston also introduced the notion of a *uniformly bounded measured lamination*, and of an associated *uniformly bounded earthquake*. Here, the notion of boundedness refers to a norm (which is now called *Thurston's norm*) on transverse measures of geodesic laminations of the disk. Specifically, the Thurston norm of a transverse measure σ of a geodesic lamination μ is defined by the formula

$$\|\sigma\|_{\mathrm{Th}} = \sup \sigma(\beta),$$

where the supremum is taken over all arcs β of hyperbolic length ≤ 1 that are transverse to μ. Thurston proved that for any given uniformly bounded measured geodesic lamination μ, there exists an earthquake map having μ as a shearing measure.

Thurston's arguments and techniques have been developed, made more quantitative, and generalized in several directions, by Gardiner, Lakic, Hu and Šarić. A result established by Hu (2001) says that the earthquake norm of a transverse measure σ of a lamination μ of the unit disk and the cross ratio distortion of the circle homeomorphism h induced by earthquaking along μ are Lipschitz-comparable; that is, we have

$$\frac{1}{C}\|h\|_{\mathrm{cr}} \leq \|\sigma\|_{\mathrm{Th}} \leq C\|h\|_{\mathrm{cr}},$$

with C being a universal constant. This is a more explicit version of a result of Thurston saying that a transverse measure σ is Thurston bounded if and only if the induced map at infinity h is quasi-symmetric.

In their work entitled *Thurston unbounded earthquake maps* (2007), Hu and Su obtained a result that generalizes Thurston's result from bounded to unbounded earthquake measures, with some control on the growth of the measures at infinity, that is, on the measure of transverse segments that are sufficiently close to the boundary at infinity of the hyperbolic disk. As the authors put it, this result can be compared to the result by David on the generalized solution of the Beltrami equation, reported on in Chapter 1 of this volume, in which the L^∞-norm of the Beltrami coefficient is allowed to be equal to 1, with some control on its growth near the set where this supremum is attained.

In any case, if μ is a geodesic lamination and σ a bounded transverse measure on μ, then the pair (μ, σ) defines an earthquake map. Introducing a non-negative real parameter t, we get an earthquake curve E_t induced by (μ, σ) and a corresponding 1-parameter family of homeomorphisms h_t of the circle, also called an earthquake curve. The differentiability theory of earthquakes is then expressed in terms of the differentiability of the associated quasi-symmetric maps. For each point x on the circle, the map $h_t(x)$ is differentiable in t and satisfies a certain non-autonomous ordinary differential equation which was established and studied by Gardiner, Hu and Lakic.

This differentiable theory is then used for establishing a so-called *infinitesimal earthquake theorem*. The theory uses the notion of Zygmund boundedness. A continuous function $V\colon \mathbb{S}^1 \to \mathbb{C}$ is said to be *Zygmund bounded* if it satisfies

$$|V(e^{2\pi i(\theta+t)}) + V(e^{2\pi i(\theta-t)}) - 2V(e^{2\pi i\theta})| \leq M|t|$$

for some positive constant M.

The reader will notice that this definition of Zygmund boundedness has some flavor of quasi-symmetry.

The infinitesimal earthquake theorem can be considered as an existence theorem establishing a one-to-one correspondence between Thurston bounded earthquake measures and normalized Zygmund bounded functions. Hu showed that the cross-ratio norm on the set of Zygmund bounded functions and the Thurston norm on the set of earthquake measures are equivalent under this correspondence.

Chapter 2 of the present volume is an account of Thurston's original construction and of the various developments and generalizations that we mentioned. The chapter includes a proof of Thurston's result on the transitivity of earthquakes, an algorithm for finding the earthquake measured geodesic lamination associated to a quasi-symmetric homeomorphism of the circle, a presentation of the David-type extension to non-bounded earthquake measures, an exposition of a quantitative relation between earthquake measures and cross ratio norms, and an exposition of the infinitesimal theory of earthquakes.

1.3 Lines of minima in Teichmüller space

Chapter 3, by Caroline Series, is a survey on lines of minima in Teichmüller space. These lines were introduced by Kerckhoff in the early 1990s. Their study involves at the same time properties of Teichmüller geodesics and of earthquakes.

Let us first briefly recall the definition of a line of minima.

Let S be a surface of finite type. For any measured lamination μ on S, let $l_\mu\colon \mathcal{T}(S) \to \mathbb{R}$ be the associated length function on the Teichmüller space $\mathcal{T}(S)$ of S.

Consider now two laminations μ and ν that fill up S in the sense that for any measured lamination λ on S, we have $i(\mu,\lambda) + i(\nu,\lambda) > 0$. Kerckhoff noticed that for any $t \in (0,1)$, the function

$$(1-t)l_\mu + tl_\nu\colon \mathcal{T}(S) \to \mathbb{R} \tag{1.3}$$

has a unique minimum He proved this fact using the convexity of geodesic length functions along earthquakes, and the existence of an earthquake path joining any two points in Teichmüller space.

For any $t \in (0,1)$, let $M_t = M((1-t)\mu, t\nu)$ denote the unique minimum of the function defined in (1.3). The set of all such minima, for t varying in $(0,1)$, is a subset of Teichmüller space called the *line of minima* of μ and ν, and is denoted by $\mathcal{L}(\mu,\nu)$.

It is known that for any two points in Teichmüller space there is a line of minima joining them, but it is unknown whether such a line is unique.

In 2003, Díaz and Series studied limits of certain lines of minima in the compactified Teichmüller space equipped with its Thurston boundary, $\mathcal{T}(S) \cup \mathcal{PML}(S)$. They showed that for any line of minima $(M_t)_{t\in(0,1)}$ associated to two measured laminations μ and ν such that μ is uniquely ergodic and maximal, the point M_t converges as $t \to 0$ to the point $[\mu]$ in Thurston's boundary. They also showed that, at the opposite extreme, if μ is a *rational* lamination in the sense that μ is a weighted sum of closed geodesics, $\mu = \sum_{i=1}^{N} a_i\alpha_i$, then the limit as $t \to 0$ of M_t is equal to the projective class $[\alpha_1 + \cdots + \alpha_N]$; that is, the point M_t converges, but its limit is independent of the weights a_i. In particular, this limit is (except in the special case where all the weights are equal) not the point $[\mu]$. Thus, if μ and ν are arbitrary, then the projective class of μ in Thurston's boundary is not always the limit of M_t as $t \to 0$.

There is a formal analogy between these results and results obtained by Howard Masur in the early 1980s on the limiting behavior of some geodesics for the Teichmüller metric. We also note that Guillaume Théret, together with the author of this introduction, obtained analogous results on the behavior of stretch lines. These lines are geodesic for Thurston's asymmetric metric. The fact that such results hold for lines of minima has a more mysterious character than in the cases of Teichmüller geodesics and of stretch lines, because up to now, unlike Teichmüller lines and stretch lines, lines of minima are not associated to any metric on Teichmüller space.

Series made a study of lines of minima in the context of the deformation theory of Fuchsian groups. She established a relation between lines of minima and bending measures for convex core boundaries of quasi-Fuchsian groups. This work introduced the use of lines of minima in the study of hyperbolic 3-manifolds. Series showed (2005, based on a previous special case studied by herself and Keen) that when the Teichmüller space $\mathcal{T}(S)$ is identified with the space $\mathcal{F}(S)$ of Fuchsian groups embedded in the space of quasi-Fuchsian groups $\mathcal{Q}(S)$, a line of minima can be interpreted as the intersection with $\mathcal{F}(S)$ of the closure of some *pleating variety* in $\mathcal{Q}(S)$. This theory involves the complexification of Fenchel–Nielsen parameters, which combines earthquaking and bending, and it also involves a notion of complex length, defined on quasi-Fuchsian space by analytic continuation of the hyperbolic length function.

More recently (2008), Choi, Rafi and Series discovered relations between the behavior of lines of minima and geodesics of the Teichmüller metric. They obtained a combinatorial formula for the Teichmüller distance between two points on a given line of minima, and they proved that a line of minima is quasi-geodesic with respect to the Teichmüller metric. The latter means that the distance between two points on a line of minima, with an appropriate parametrization, is uniformly comparable (in the sense of large-scale quasi-isometry) to the Teichmüller distance between these points. The proof of that result is based on previous work by Rafi. It involves an analysis of which closed curves get shortened along a line of minima, and the comparison of these curves with those that get shortened along the Teichmüller geodesic whose horizontal and vertical projective classes of measured foliations are the classes of the measured geodesic laminations μ and ν associated to the line of minima.

Summing up, the account that Series makes of lines of minima in Chapter 3 includes the following topics:

(1) The limiting behavior of lines of minima in Teichmüller space compactified by Thurston's boundary.

(2) The relation between lines of minima and quasi-Fuchsian manifolds.

(3) The relation between lines of minima and the geodesics of the Teichmüller metric.

2 Part B. The group theory, 3

2.1 Mapping class groups versus arithmetic groups

In Chapter 4 Lizhen Ji gives a survey of the analogies and differences between mapping class groups and arithmetic groups, and between Riemann's moduli spaces and arithmetic locally symmetric spaces. This subject is vast and important, in particular because a lot of work done on mapping class groups and their actions on Teichmüller spaces (and other spaces) was inspired by results that were known to hold for arithmetic groups and their actions on associated symmetric spaces.

Let us start with a few words on of arithmetic groups.

This theory was initiated and developed by Armand Borel and Harish-Chandra. It is easy to give some very elementary examples of arithmetic groups: $\mathbb{Z}$, $\mathrm{Sp}(n,\mathbb{Z})$, $\mathrm{SL}(n,\mathbb{Z})$ and their finite-index subgroups. But the list of elementary examples stops very quickly, and in general, to know whether a certain group that arises in a certain algebraic or geometric context is isomorphic or not to an arithmetic group is a highly nontrivial question. Important work has been done in this direction. A famous theorem due to Margulis, described as the "super-rigidity theorem", gives a precise relation between arithmetic groups and lattices in Lie groups. Interesting examples of arithmetic groups are some arithmetic isometry groups of hyperbolic space found by E. B. Vinberg, in the early 1970s.

Several analogies between mapping class groups and arithmetic groups were already highlighted in the late 1970s by Thurston, Harvey, Harer, McCarthy, Mumford, Morita, Charney, Lee and many other authors. Several questions on mapping class groups were motivated by results that were known to hold for arithmetic groups, sometimes with the hope that some property of arithmetic groups will not hold for mapping class groups, implying that the latter are not arithmetic.

There are several fundamental properties that are shared by arithmetic groups and mapping class groups. For instance, any group belonging to one of these two classes is finitely presented, it has a finite-index torsion free subgroups, it is residually finite and virtually torsion free, it has only finitely many conjugacy classes of finite subgroups, its virtual cohomological dimension is finite, and it is a virtual duality group in the sense of Bieri and Eckmann. Furthermore, every abelian subgroup of a mapping class group or of an arithmetic group is finitely generated with torsion-free rank bounded by a universal constant, every solvable subgroup of such a group is of bounded Hirsch rank, it is Hopfian (that is, every surjective self-homomorphism is an isomorphism) and co-Hopfian (every injective self-homomorphism is an isomorphism), it satisfies the Tits alternative (every subgroup is either virtually solvable or it contains a free group on two generators), and there are several other common properties. For mapping class groups, all these properties were obtained in the 1980s, gradually and by various people, after the same properties were proved for arithmetic groups.

The question of whether mapping class groups are arithmetic appeared explicitly in a paper by W. Harvey in 1979, *Geometric structure of surface mapping class groups*, at about the same time where mapping class groups started to become very fashionable. In the same paper, Harvey also asked whether these groups are linear, that is, whether they admit finite-dimensional faithful representations in linear groups.

In 1984, Ivanov announced the result that mapping class groups of surfaces of genus ≥ 3 are not arithmetic. Harer provided the first written proof of this result in his paper *The virtual cohomological dimension of the mapping class group of an orientable surface* (published in 1986). The fact that a mapping class group cannot be an arithmetic subgroup of a simple algebraic group of $\mathbb{Q}$-rank ≥ 2 follows from the fact that any normal subgroup of such an arithmetic group is either of finite index or is finite and central. The mapping class group does not have this property since it contains the Torelli group, which is normal and neither finite nor of finite index. Harer solved the remaining case ($\mathbb{Q}$-rank 1) by showing that the virtual cohomological dimension of a mapping class group does not match the one of an arithmetic group. Goldman gave another proof of this fact, at about the same time Harer gave his proof. Ivanov published a proof that the mapping class group is not arithmetic in 1988.

Despite the non-arithmeticity result, several interesting properties of mapping class groups that were obtained later on were motivated by the same properties satisfied by arithmetic groups, or more generally, by linear groups. Some of these properties can be stated in terms that are identical to those of arithmetic groups. For instance, Harer proved a stability theorem of the cohomology for mapping class groups of surfaces with one puncture as the genus tends to infinity, and he showed that mapping class groups are virtual duality groups. Harer and Zagier obtained a formula for the orbifold Euler characteristic of Riemann's moduli space of surfaces with one puncture, and Penner obtained the result for $n \geq 1$ punctures. The formula involves the Bernoulli numbers, as expected from the corresponding formula in the theory of arithmetic groups. Other properties can be stated in similar, although not identical, terms for mapping class groups and arithmetic groups.

One of the most important general properties shared by arithmetic groups and mapping class groups, which gives the key to most of the results obtained, is the existence of natural and geometrically defined spaces on which both classes of groups act. The actions often extend to actions on various compactifications and boundaries, on cell-decompositions of the spaces involved, and on a variety of other associated spaces.

In parallel to the fact that mapping class groups are not arithmetic, one can mention that Teichmüller spaces (except if their dimension is one) are not symmetric spaces in any good sense of the word. Likewise, moduli spaces are not locally symmetric spaces. Meanwhile, one can ask for Teichmüller spaces and moduli spaces several questions about properties that can be shared by symmetric spaces, for instance, regarding their compactifications or, more generally, bordifications.

Borel–Serre bordifications of symmetric spaces were used to obtain results on the virtual cohomological dimension and on the duality properties of arithmetic groups. Similar applications were found for mapping class groups using Borel–Serre-like bordifications of Teichmüller space, which are partial compactifications.

Lizhen Ji, in Chapter 4, makes a catalogue of the various compactifications of Teichmüller space and moduli space. He describes in detail the contexts in which these compactifications arise, and the known relations between the various compactifications. He discusses the question of when a compactification of moduli space can be obtained from a compactification of Teichmüller space, and he points out various analogies between the compactifications of Teichmüller space and moduli space on the one hand and those of symmetric spaces and locally symmetric spaces on the other hand. He addresses questions such as what is the analogue for moduli space of a Satake compactification of a locally symmetric space, in particular, of the quotient of a symmetric space by an arithmetic group.

As we already mentioned, the question of the extent to which mapping class groups are close to being arithmetic is still an interesting question. One can mention the realization of an arithmetic group as a subgroup of a Lie group, that is inherent in the definition of an arithmetic subgroup, leading naturally to the question of the realization of mapping class groups as discrete subgroups of Lie groups.

There are two instances where the mapping class group of a surface is arithmetic, namely, the cases where the surface is the torus or the once-punctured torus. In both cases, the mapping class group is the group $\mathrm{PSL}(2,\mathbb{Z})$. The Teichmüller space in that case is the corresponding symmetric space, namely, the upper-half plane $\mathbb{H}^2$. Furthermore, this identification between the Teichmüller space with $\mathbb{H}^2$ is consistent with the complex structures of the two spaces and the Teichmüller metric on the upper-half plane coincides with the Poincaré metric. The action of the mapping class group on the Teichmüller space corresponds to the usual action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathbb{H}^2$ by fractional linear transformations.

Lizhen Ji makes in Chapter 4 a list of notions that are inherent in the theory of arithmetic groups and that have been (or could be) adapted to the theory of mapping class groups. This includes the notions of irreducibility, rank, congruence subgroup, parabolic subgroup, Langlands decomposition, existence of an associated symmetric space, Furstenberg boundaries and Tits buildings encoding the asymptotic geometry, reduction theory, the Bass–Serre theory of actions on trees, and there are many others. All these questions from the theory of arithmetic groups gave already rise to very rich generalizations and developments that were applied to the study of mapping class groups and their actions on various spaces.

The curve complex is an important ingredient in the study of mapping class groups. It was introduced as an analogue for these groups of buildings associated to symmetric spaces and locally symmetric spaces. Curve complexes turned out to be useful in the description of the large-scale geometry and the structure at infinity of mapping class groups and of Teichmüller spaces. Volume IV of this Handbook will contain a survey by Lizhen Ji, entitled *Curve complexes versus Tits buildings: structures and applications*, that explores in great detail the relation between curves complexes and Tits buildings.

Another topic of interest in both theories is the study of fundamental domains.

It is well known that producing a good fundamental domain for an action and understanding its geometry gives valuable information on the quotient space. An idea that appears in the survey by Lizhen Ji is to make a relation between Minkowski reduction theory and mapping class group actions on Teichmüller spaces, from the point of view of producing intrinsically defined fundamental domains. In a generalized form, reduction theory can be described as the theory of finding good fundamental domains for group actions. This theory was developed by Siegel, Borel and Harish-Chandra and others. Gauss worked out the reduction theory for quadratic forms. We recall in this respect that the theory of quadratic forms is related to that of moduli spaces by the fact that $\mathbb{H}^2 = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ is also the space of positive definite quadratic forms of determinant 1. Poincaré polyhedra and Dirichlet domains are examples of good fundamental domains. The Siegel domain for the action of $\mathrm{SL}(2,\mathbb{Z})$ on the hyperbolic plane is a prototype for both theories, arithmetic groups and mapping class groups. The upper-half plane $\mathbb{H}^2$ is the space of elliptic curves in algebraic geometry, and at the same time it is the Teichmüller space of the torus equipped with the mapping class group action.

In the case where there is no obvious good fundamental domain, one may try to find rough fundamental domains. In the sense used by Ji in this survey, this means that the natural map from the fundamental domain to the quotient space is finite-to-one. Finding a good fundamental domain, or even a rough fundamental domain, in the case where the quotient is non-compact, is not an easy matter. Motivated by reduction theory, Ji addresses the question of the existence of various kinds of fundamental domains (geometric, rough, measurable, etc.), and of studying finiteness and local finiteness properties of such domains in relation to questions of finite generation and of bounded generation, and other related questions on group actions.

2.2 Simplicial actions of mapping class groups

Chapter 5, written by John McCarthy and myself, is a survey of several natural actions of extended mapping class groups of surfaces of finite type on various simplicial complexes.

The earliest studies of actions of mapping class groups on combinatorial complexes that gave rise to substantial results are the actions on the pants complex and on the cut system complex. These studies were done by Hatcher and Thurston in the mid 1970s, at the time Thurston was developing his theory of surface homeomorphisms. This work paved the way for a theory that included a variety of other simplicial actions of mapping class groups.

The curve complex was introduced slightly later (in 1977) by Harvey.

While the main motivation of Hatcher and Thurston for studying the actions on the pants complex and the cut system complex was to get a finite presentation of the mapping class group, the original motivation of Harvey in studying the curve complex was to construct some boundary structure for Teichmüller space.

After the curve complex was introduced, several authors studied it from various points of view. Ivanov proved in the 1990s the important result stating that (except for a few surfaces of low genus and small number of boundary components) the simplicial automorphism group of the curve complex coincides with the natural image of the extended mapping class group in that group.[5] Later on, Ivanov used this action to give a new and more geometric (as opposed to the original analytic) proof of the celebrated theorem obtained by Royden in 1971 saying that (again, except for a few surfaces of low genus and small number of boundary components) the natural homomorphism from the extended mapping class group to the isometry group of the Teichmüller metric is an isomorphism. Ivanov's proof is based on a relation between the curve complex and some boundary structure of Teichmüller space, a relation that was already suspected by Harvey.

Masur and Minsky (1996) studied the curve complex, endowed with its natural simplicial metric, from the point of view of large-scale geometry. They showed that this complex is Gromov hyperbolic. Klarreich (1999) identified the Gromov boundary of the curve complex with a subspace of *unmeasured lamination space* $\mathcal{UML}$, that is, the quotient space of measured lamination space obtained by forgetting the transverse measure. The Gromov boundary of the curve complex is the subspace of $\mathcal{UML}$ consisting of minimal and complete laminations. Here, a measured lamination is said to be complete if it is not a sublamination of a larger measured lamination, and it is called minimal if there is a dense leaf (or, equivalently, every leaf is dense) in its support.

Now we mention results on the other complexes.

[5] Ivanov's original work did not include the case of surfaces of genus 0 and 1, and this was completed by Korkmaz. The work of Korkmaz also missed the case of where the surface S is a torus with two holes, which was completed by Luo. Luo also gave an alternative proof of the complete result.

The pants graph is the 1-skeleton of the Hatcher–Thurston pants complex. The hyperbolicity of the pants graph was studied by Brock and Farb (2006). Brock (2003) proved that the pants graph of S is quasi-isometric to the Teichmüller space of S endowed with its Weil–Petersson metric. Margalit (2004) proved that (again, with the exception of a few surfaces of low genus and small number of boundary components) the simplicial automorphism group of the pants graph coincides with the natural image of the extended mapping class group in that automorphism group.

Other complexes with vertex sets being homotopy classes of compact subsets of the surface that are invariant by the extended mapping class group action were studied by various authors. We mention the arc complex, the arc-and-curve complex, the ideal triangulation complex, the Schmutz graph of non-separating curves, the complex of non-separating curves[6], the complex of separating curves, the Torelli complex, and there are other complexes. All these actions were studied in detail, and each of them presents interesting features. The study of mapping class group actions on simplicial complexes is now a large field of research, which we may call the subject of "simplicial representations of mapping class groups".

The aim of Chapter 5 is to give an account of some of the simplicial actions, with a detailed study of a complex that I recently introduced with McCarthy, namely, the *complex of domains*, together with some of its subcomplexes.

The complex of domains is a flag simplicial complex which can be considered as naturally associated to the Thurston theory of surface diffeomorphisms. The various pieces of the Thurston decomposition of a surface diffeomorphism in Thurston's canonical form, which we call the *thick* domains and *annular* or *thin* domains, fit into this flag complex. Unlike the curve complex and the other complexes that were mentioned above and for which, for all but a finite number of exceptional surfaces, all simplicial automorphisms are geometric (i.e. induced by surface homeomorphisms), the complex of domains admits non-geometric simplicial automorphisms, provided the surface has at least two boundary components. As a matter of fact, if the surface has at least two boundary components, then the simplicial automorphism group of the complex of domains is uncountable. The non-geometric automorphisms of the complex of domains are associated to certain edges of this complex that are called *biperipheral*, and whose vertices are represented by *biperipheral pairs of pants* and *biperipheral annuli*. A biperipheral pair of pants is a pair of pants that has two of its boundary components on the boundary of the surface. A biperipheral annulus is an annulus isotopic to a regular neighborhood of the essential boundary component of a biperipheral pair of pants.

The complex of domains can be projected onto a natural subcomplex by collapsing each biperipheral edge onto the unique vertex of that edge that is represented by a regular neighborhood of the associated biperipheral curve. In this way, the computation of the simplicial automorphism group of the complex of domains is reduced to the computation of the simplicial automorphism group of this subcomplex, called the

[6]The one-skeleton of the complex of non-separating curves is different from the Schmutz graph of non-separating curves.

truncated complex of domains. With the exception, as usual, of a certain finite number of special surfaces, the simplicial automorphism group of the truncated complex of domains is the extended mapping class group of the surface. From this fact, we obtain a complete description of the simplicial automorphism group of the complex of domains.

Besides the interesting fact that the automorphism groups of most of the complexes mentioned are isomorphic to extended mapping class groups, it turns out that the combinatorial data (links of vertices, links of links of vertices, etc.) are sufficient, in many cases, to reconstruct the topological objects that these vertices represent. Thus, in many ways, the combinatorial structure of the complexes "remembers" the surface and the topological data on the surface that were used to define the complexes. This is another theme of Chapter 5, and it is developed in detail in the case of the complex of domains and the truncated complex of domains.

In Chapter 6, Valentina Disarlo studies the coarse geometry of the complex of domains $D(S)$ equipped with its natural simplicial metric. She proves that for any subcomplex $X(S)$ of $D(S)$ containing the curve complex $C(S)$, the natural simplicial inclusion $C(S) \to X(S)$ is an isometric embedding and a quasi-isometry. She also proves that with the exception of a few surfaces of small genus and small number of boundary components, the arc complex $A(S)$ is quasi-isometric to the complex $P_\partial(S)$ of peripheral pairs of pants, and she gives a necessary and sufficient condition on S for the simplicial inclusion $P_\partial(S) \to D(S)$ to be a quasi-isometric embedding. She then applies these results to the study of the arc and curve complex $AC(S)$. She gives a new proof of the fact that $AC(S)$ is quasi-isometric to $C(S)$, and she discusses the metric properties of the simplicial inclusion $A(S) \to AC(S)$.

2.3 Minimal generating sets for mapping class groups

Chapter 7 by Mustafa Korkmaz is a survey on generating sets of minimal cardinality for mapping class groups of surfaces of finite type.

Three types of generating sets are considered: Dehn twists, torsion elements and involutions.

Let us first discuss the case of orientable surfaces.

It is well known that Dehn twists generate the mapping class group. Such generators were first studied by Dehn in the 1930s, who showed that a finite number of them suffice. Humphries (1979) found a minimal set of Dehn twist generators.

Maclachlan (1971) showed that the mapping class group is generated by a finite number of torsion elements, and he used this fact to deduce that moduli space is simply connected.

McCarthy and Papadopoulos (1987) showed that the mapping class group is generated by involutions. Luo (2000), motivated by the case of $\mathrm{SL}(2, \mathbb{Z})$ and by work of Harer, showed that torsion elements of bounded order generate the mapping class group of a surface with boundary, except in the special case where the genus of the surface is 2 and the number of its boundary components is of the form $5k + 4$ for some integer k. In this exceptional case, Luo showed that the torsion elements generate a subgroup of index 5 of the mapping class group. Brendle and Farb (2004) solved Luo's question in the case of closed orientable surfaces, by showing that there is a finite generating set of involutions whose cardinality does not depend on the genus.

Besides surveying minimal generating sets, Korkmaz provides some background material on the set of relations between Dehn twist elements in mapping class groups.

Mapping class groups of non-orientable surfaces are also discussed in Chapter 7. In this case, the mapping class group is defined as the group of all homotopy classes of homeomorphisms (there is no orientation involved). Lickorish (1963) showed that Dehn twists generate a subgroup of index two in this mapping class group, and he produced a system of generators for it: Dehn twists along two-sided curves and the isotopy class of a homeomorphism called a "cross-cap slide", and supported on a Klein bottle embedded in the surface. Chillingworth (1969) showed that the mapping class group is generated by finitely many elements. Korkmaz (2002) extended Chillingworth's result to the case of surfaces with boundary. Motivated by the work done in the orientable case, Szepietowski obtained results on involutions in mapping class groups of non-orientable surfaces. He showed that the mapping class group of a closed non-orientable surface is generated by four involutions.

The chapter ends with some open questions.

2.4 Mapping class groups and 3-manifold topology

Chapter 8 by Kazuo Habiro and Gwénaël Massuyeau, and Chapter 9 by Takuya Sakasai concern relations between mapping class groups and 3-manifolds. The two chapters are complementary to each other. In each of them, the authors study a monoid that arises in 3-manifold topology and that is an extension of the mapping class group. The elements of this monoid are called *homology cobordisms* by Habiro and Massuyeau, and *homology cylinders* by Sakasai.[7]

The results of these two chapters especially apply to a surface $S = S_{g,1}$, that is, a compact oriented surface of genus $g \geq 1$ with one boundary component.[8] The mapping class group $\Gamma = \Gamma_{g,1}$ in this context is defined as the group of isotopy classes of orientation-preserving homeomorphisms that fix the boundary pointwise. The basepoint of the fundamental group $\pi_1(S)$ is chosen on the boundary, and in this way the mapping class group Γ acts naturally on $\pi_1(S)$. This fundamental group is free on $2g$ generators, and by a result attributed to Dehn, Nielsen and Baer, the natural homomorphism $\Gamma \to \mathrm{Aut}(\pi_1(S))$ is injective. Thus, we have a natural monomorphism

[7] Habiro and Massuyeau call *homology cylinder* an object that is more special than the homology cylinder in the sense of Sakasai. Likewise, Sakasai uses the term *homology cobordism* in a different sense than the one used by Habiro and Massuyeau, namely, he uses it in association with an equivalence relation involving 4-manifolds. This is a very unfortunate inconsistency in the mathematics literature. There was no obvious way to make things uniform in this Handbook, and I decided to leave the authors stick to the terminology used in the papers referred to in their contribution.

[8] We note however that most of the constructions in Chapter 8 by Habiro and Massuyeau also apply to closed surfaces.

from the mapping class group of S into the automorphism group of a free group. This is an instance of the general fact that the theory of free groups is much more present in the study of mapping class groups of surfaces with boundary than in that of closed surfaces.

The theory of the monoid of homology cobordisms (respectively homology cylinders) is based on surgery techniques that were introduced by Goussarov and Habiro independently in the second half of the 1990s.[9] The aim of these techniques was to prove general properties of finite type invariants for 3-manifolds and for links in these manifolds.

We recall that the expression "finite type invariant" in 3-manifold theory refers to invariants that behave polynomially with respect to some surgery (that is, cut-and-paste) operations.

Examples of such invariants are the cohomology ring of a manifold (which is a degree-one finite type invariant), the Rochlin invariant for closed spin 3-manifolds (also degree-one finite type invariant) and the Casson invariants (degree-two invariants). The Johnson and Morita theories of the Torelli group of surfaces also involve finite type invariants of 3-manifolds. It seems that Ohtsuki was the first to introduce the notion of finite type invariant, in the setting of integral homology spheres, and he constructed the first examples. Goussarov and Habiro extended this notion to all 3-manifolds, and they developed the necessary techniques to study the general case.

The surgery techniques introduced by Goussarov and Habiro are called *clover* and *clasper* techniques respectively.[10] These theories are essentially equivalent to each other. They originate in a surgery theory called *Borromean surgery*, due to Matveev. Clasper calculus can also be seen as a topological analogue of commutator calculus in groups. Like Matveev's surgery, clasper surgery does not affect the homology of the underlying 3-manifold. Using the techniques they introduced independently, Goussarov and Habiro obtained results similar to each other. These techniques were also used to obtain a topological interpretation of "Jacobi diagrams", which may be compared to Feynman diagrams and which appear in the theory of universal finite type invariants.

The Johnson homomorphisms, the Magnus representation and several other algebraic notions that pertain to mapping class group theory extend to the setting of the homology cobordism (respectively homology cylinder) monoid.

To present in more precise terms the chapter by Habiro and Massuyeau, we recall a few definitions.

A *cobordism* of a surface $S_{g,1}$ is a pair (M, m) where M is a compact connected oriented 3-manifold and where $m\colon \partial\left(S_{g,1} \times [-1,1]\right) \to \partial M$ is an orientation-preserving homeomorphism. The homeomorphism m is regarded as a parametrization of ∂M, and the two inclusions of $S_{g,1}$ into ∂M obtained by restricting m to the upper and lower factor of $S_{g,1} \times [-1,1]$ allow one to talk about the *top* and the *bottom* boundary of M.

[9] Habiro wrote his thesis, on this theory, in 1997. Goussarov did not publish much. He passed away in a drowning accident in 1999. In both cases, the first papers on the theory appeared in print around the year 2000.

[10] It seems that the word *clasper* is the one that is mostly used today, and in this introduction we shall use it.

Two cobordisms (M, m) and (M', m') of the same surfaces are said to be *homeomorphic* if there exists an orientation-preserving homeomorphism $f: M \to M'$ such that $f|_{\partial M} \circ m = m'$. Composition of cobordisms (M, m) and (M', m') of S is defined by gluing the bottom boundary of M' to the top boundary of M.

A *homology cobordism* (M, m) of S is a cobordism whose top and bottom inclusions induce isomorphisms between the homology groups $H_*(S)$ and $H_*(M)$. Homology cobordisms are stable under composition. This operation makes the set of homology cobordisms (up to the homeomorphism relation defined above) a monoid, called the *homology cobordism monoid* and denoted by $\mathcal{C}(S)$. The unit in that monoid is the homology cobordism $(S \times [-1, 1], \mathrm{Id} \times \{-1\}, \mathrm{Id} \times \{1\})$.

The definition of homology cobordism is due to Goussarov and Habiro (independently), and it was used by Garoufalidis and Levine in the study of finite-type invariants of 3-manifolds. The recent developments in the theory of homology cobordisms are due to Garoufalidis and Levine, Habiro, Massuyeau, Meilhan, Habegger, Sakasai, Morita, and there are certainly other authors.

Denoting as before by $\Gamma(S)$ the mapping class group of S, there is an embedding

$$\Gamma(S) \to \mathcal{C}(S)$$

obtained by the *mapping cylinder construction*, in which the 3-manifold M is defined as the product $S \times [-1, 1]$, the top boundary homeomorphism being the given element of $\Gamma(S)$ and the lower boundary homeomorphism being the isotopy class of the identity map of S.

The map $\Gamma(S) \to \mathcal{C}(S)$ is not surjective. Surgery along claspers provides examples of homology cylinders that are not obtained as images of elements of the mapping class group.

Since we are dealing with the homology of the surface, the Torelli subgroup of the mapping class group plays a central role in this theory. We recall that the Torelli group is the subgroup of $\Gamma(S)$ that consists of the elements that induce the identity on homology.

A *homology cylinder* over S (in the sense of Habiro and Massuyeau) is a cobordism that has the same homology type as the trivial cobordism, that is, $(S \times [-1, 1], \mathrm{Id})$.

Like the set of homology cobordisms, the set of homology cylinders is stable under composition, and it forms a submonoid $\mathcal{IC}(S) \subset \mathcal{C}(S)$. It turns out that the image of the Torelli group $\mathcal{I}(S)$ by the embedding $\Gamma(S) \to \mathcal{C}(S)$ is contained in the submonoid $\mathcal{IC}(S)$ of homology cylinders. The map $\mathcal{I}(S) \to \mathcal{IC}(S)$ is injective, and therefore $\mathcal{IC}(S)$ can be thought of as an extension of the Torelli group. The image of $\Gamma(S)$ (respectively of $\mathcal{I}(S)$) in $\mathcal{C}(S)$ (respectively in $\mathcal{IC}(S)$) is the group of units (that is, the group of invertible elements) of $\mathcal{C}(S)$ (respectively of $\mathcal{IC}(S)$). The study of the inclusion $\mathcal{IC}(S) \subset \mathcal{C}(S)$ can be done using finite type invariants of 3-manifolds, in particular clasper calculus.

In Chapter 8, Habiro and Massuyeau present the recent developments in the theory of the monoid $\mathcal{C}(S)$ of homology cobordisms, with special attention given to the submonoid $\mathcal{IC}(S)$ of homology cylinders, and to its relation to the Torelli group. There is also a strong relation to the Johnson homomorphisms.

We recall that the Johnson homomorphisms are defined on a filtration of the mapping class group, and they give a kind of measure of the unipotent part of the action of the Torelli group $\mathcal{I}(S)$ on the second nilpotent truncation of $\pi_1(S)$.

The first Johnson homomorphism was introduced in the early 1980s by Johnson. In 1993, Morita studied in detail a sequence of homomorphisms that extend the first Johnson homomorphism. These homomorphisms are sometimes referred to as the "higher Johnson homomorphisms".[11] In 2005 Garoufalidis and Levine published a paper[12] in which the Johnson homomorphisms and their generalization by Morita were extended to the setting of homology cobordisms. There is a Johnson filtration $\{\mathcal{I}_g(k)\}_k$, and the k-th Johnson homomorphism is a homomorphism $\tau_g(k)$ from $\mathcal{I}_g(k)$ to a certain finitely-generated free abelian group arising from the k-th graded piece of a certain graded Lie algebra.

Chapter 8 also contains a report on an "infinitesimal version" of the Dehn–Nielsen representation, first defined by Massuyeau, and of which the homomorphisms of Johnson and Morita become special cases. This work uses the Malcev Lie algebra of $\pi_1(S)$ instead of the group $\pi_1(S)$ itself. We recall here that by work of Malcev published in 1949, every torsion-free finitely generated nilpotent group can be embedded as a discrete co-compact subgroup of a Lie group. From this, one can associate to any finitely generated group π a tower of nilpotent Lie groups. To this tower is then associated a tower of corresponding Lie algebras. Applied to the case where $\pi = \pi_1(S)$, this gives a tower whose inverse limit is the *Malcev Lie algebra* associated to $\pi_1(S)$.

In 1988, Le, Murakami and Ohtsuki, based on the Kontsevich integral and using surgery presentations in the 3-sphere, constructed an invariant of closed oriented 3-manifolds, which is now called the LMO invariant. This invariant is particularly interesting for the study of homology spheres. In 2008, Cheptea and Habiro–Massuyeau extended the LMO invariant to compact oriented 3-manifolds with boundary. In this work, this extension is presented as a functor defined on a certain cobordism category, which the authors called the *LMO functor*, and which is a kind of TQFT theory. This cobordism category contains the homology cylinder monoid. In 2009, Habiro and Massuyeau defined the LMO homomorphism on $\mathcal{IC}(S)$ by restriction of the LMO functor. They obtained and studied a monoid homomorphism, which they called the "LMO homomorphism", from $\mathcal{IC}(S)$ to the algebra of Jacobi diagrams. This homomorphism provides a diagrammatic representation of the monoid $\mathcal{IC}(S)$, and it is useful in the study of the action of $\mathcal{IC}(S)$ on the Malcev Lie algebra of $\pi_1(S)$. It is

[11] Chapter 7 of Volume I of this Handbook, written by Morita, is a survey on mapping class groups and related groups, and it contains a section on the Johnson homomorphisms. Let us mention by the way that some ideas that are at the basis of the Johnson homomorphisms can be found in the work of Andreadakis (1965) who introduced and studied the filtration on $\{\mathrm{Aut}(F_n)(k)\}_k$ that is induced from the action of $\mathrm{Aut}(F_n)$ on nilpotent quotients of F_n.

[12] The paper, entitled *Tree-level invariants of three manifolds, Massey products and the Johnson homomorphism*, was published in 2005 in the proceedings of a conference, but it seems that the results were obtained before 2001, when Levine published a paper on related work.

injective on the image of the Torelli group. It is also a useful tool in the study of the Johnson and Morita homomorphisms, and more generally, in the study of the way the Torelli group embeds into the monoid $\mathcal{IC}(S)$.

In Chapter 8, Habiro and Massuyeau also report on a filtration, called Y-filtration and defined by clasper surgeries, of the monoid of homology cylinders. The Y-filtration is an analogue of the lower central series of the Torelli group. The graded abelian group associated to this filtration is computed (in the case of rational coefficients) diagrammatically using the LMO homomorphism and the clasper calculus. The first quotient of this graded abelian group, that is, the quotient $\mathcal{IC}(S)/Y_2$, is computed in a way analogous to the way Johnson computed the abelianization of the Torelli group, that is, using the (first) Johnson homomorphism and the Birman–Craggs homomorphism. The authors in Chapter 8 also report on this generalized Birman–Craggs homomorphism, defined on $\mathcal{IC}(S)$.

Garoufalidis and Levine introduced a group $\mathcal{H}(S)$ whose elements are homology cobordism classes of homology cobordisms. The mapping class group still embeds in the group $\mathcal{H}(S)$. Habiro and Massuyeau present some recent work on this group, and this group is also studied in Chapter 9 by Sakasai.

Chapter 9 by Sakasai provides another point of view on the theory of homology cobordisms, which are called there homology cylinders (and we shall adopt from now on the latter terminology). The author reviews the classical theory of the Magnus representation and its extension to the setting of these homology cylinders.

The extension of the Magnus representation to homology cylinders was introduced by Sakasai. In this work, Sakasai heavily used various localization and completion techniques of groups and rings that are due to Vogel, Le Dimet, Levine, Cohn and others. These techniques had previously been used in the algebraic theory of knots and links.

We recall that the Magnus representation of the mapping class group $\Gamma_{g,1}$ is a crossed homomorphism from $\Gamma_{g,1}$ into the group $\mathrm{GL}(2g, \mathbb{Z}[\pi_1(S_{g,1})])$. The definition of this representation is usually presented using Fox calculus and Fox derivation. Chapter 9 includes the necessary background on Fox calculus.

We recall that a Fox derivation (or Fox derivative) on a free group F_n with a free generating set $\gamma_1, \dots, \gamma_n$ is a map denoted, for $i = 1, \dots, n$, by

$$\frac{\partial}{\partial \gamma_i} \colon F_n \to \mathbb{Z}[F_n].$$

This notation and the name "Fox derivative" reflect the fact that Fox derivation satisfies rules which look formally like the rules of partial derivation on differentiable functions. For instance, one has $\frac{\partial \gamma_i}{\partial \gamma_j} = \delta_{ij}$ where δ_{ij} is the Kronecker delta; there is a "chain rule" for Fox derivatives, a "Leibniz rule" for the Fox derivative of products, and so on.

The Fox differential calculus produces matrix representations of free groups of finite rank, of automorphism groups of these free groups, and of subgroups of these automorphism groups.

In its original form, the Magnus representation is a matrix representation of free groups and their automorphism groups.

The Magnus representation, which was first defined as a representation of the automorphism group of a free group, was later on adapted to the setting of the mapping class groups by Morita, and it plays an important role in the study of the Johnson homomorphisms. Using the Dehn–Nielsen–Baer theorem which injects the group $\Gamma_{g,1}$ into the group $\mathrm{Aut}(F_n)$ by using the natural action of $\Gamma_{g,1}$ on the fundamental group of the surface $S_{g,1}$, one obtains the *Magnus representation* (which is a crossed homomorphism),

$$r\colon \Gamma_{g,1} \to \mathrm{GL}(2g, \mathbb{Z}[\pi_1(S_{g,1})]).$$

By restriction and after reduction of the coefficients $\pi_1(S) \to H$, where $H = \pi_1/[\pi_1, \pi_1]$, one has also a Magnus representation of the Torelli group $\mathcal{I}_{g,1}$ (which is a genuine homomorphism)

$$\mathcal{I}_{g,1} \to \mathrm{GL}(2g, \mathbb{Z}[H]).$$

The Magnus representation of the Torelli group was studied by various authors, with the hope of better understanding that group. Morita was the first who used the Magnus representation $\mathcal{I}_{g,1} \to \mathrm{GL}(2g, \mathbb{Z}[H])$ defined through Fox derivation, to get results about the mapping class group. Suzuki showed in 2002 that this representation of the Torelli group in $\mathrm{GL}(2g, \mathbb{Z}[H])$ is not faithful for $g \geq 2$. Church and Farb obtained in 2009 that the kernel of this representation is not finitely generated, and that the first homology group of that kernel has infinite rank. Morita proved that the Magnus representation of the mapping class group is symplectic in some twisted sense.

Chapter 9 by Sakasai also contains some algebraic background which should be useful for geometers, namely, a quick survey of group homology and cohomology, a short exposition of the Fox calculus and of other concepts and tools that are used in the definitions of the Magnus representation and its various extensions. Furthermore, Sakasai reviews some invariants of homology cylinders that are obtained through the Magnus representation. He also describes several abelian quotients of the monoid and of the homology cobordism groups of homology cylinders.

2.5 Thompson's groups

Chapter 9, by Louis Funar, Christophe Kapoudjian and Vlad Sergiescu, is on Thompson's groups. These are finitely presented groups that were introduced by Richard Thompson in 1965, originally in connection with certain questions in mathematical logic. The theory of these groups was later on developed in several directions, in relation to word problems, combing properties of groups, Dehn functions, normal form theory, automaticity and to other questions. It also turned out that Thompson's groups are related to braid groups, to surfaces of infinite type and their mapping class groups, to asymptotic Teichmüller spaces, and to quantization of Teichmüller spaces. In fact, Thompson's groups are in some precise sense mapping class groups of some infinite type surfaces. For all these reasons it seemed natural to have a chapter on Thompson's groups in this Handbook.

First, let us recall the definitions.

There are three classes of Thompson's groups and, classically, they are denoted by F, T and V.

The elements of the group F are the piecewise linear homeomorphisms f of the unit interval [0, 1] satisfying the following properties:

(1) The homeomorphism f is locally linear except at finitely many points which are dyadic rational numbers. (A dyadic rational number is of the form $p/2^q$ where p and q are positive integers.)

(2) On each subinterval of [0, 1] on which f is linear, its derivative is a power of 2.

The elements of the group T are the piecewise-linear homeomorphisms f of the circle $S^1 = [0,1]/0 \sim 1$ with the following properties:

(1) The homeomorphism f preserves the images in S^1 of the set of dyadic rational numbers.

(2) The homeomorphism f is differentiable except at a finite set of points contained in the image by the natural projection $[0,1] \to S^1$ of the dyadic rational numbers.

(3) On each interval where f is linear, the derivative of f is a power of 2.

The elements V are right-continuous bijections f of $S^1 = [0,1]/0 \sim 1$ that have the following properties:

(1) The map f preserves the images in S^1 of the set of dyadic rational numbers.

(2) The map f is differentiable except at a finite set of points contained in the image of the dyadic rational numbers.

(3) On each maximal interval where f is differentiable, f is linear and its derivative is a power of 2.

In all cases (F, T and V), the composition of two bijections in the given class is in the same class. This makes F, T and V groups, the group operation being composition of maps.

The three classes of groups turned out to be important. They provided counter-examples to several natural conjectures in group theory. For instance, McKenzie and Thompson used these groups to show that there are finitely presented groups that have unsolvable word problems (1973). The same authors proved that the groups T and V are infinite, simple and finitely presented, providing the first examples of such groups.

The group F has a standard two-generator presentation. Brin and Squier proved that this group does not contain any non-abelian free group (1985). There are several open questions about the group F, e.g. whether it is amenable (Geoghegan conjectured in 1979 that the answer is yes).

A low-dimensional topologist will surely notice that such piecewise-linear actions with constraints on the nonlinearity set appear in the theory of the action of the mapping class group on Thurston's space of measured foliations. Thurston introduced the notion of piecewise integral projective transformation, as a property satisfied by the action of the mapping class group on measured lamination space. He showed that the Thompson groups F and T have this property. In fact, Thurston was interested in Thompson's groups in more than one way. He proved that the group T has a representation as a group of C^∞ diffeomorphisms of the circle. Ghys and Sergiescu proved later on the stronger result saying that T is conjugate (by a homeomorphism) to a group of C^∞ diffeomorphisms of the circle (1987).

In 1991, Greenberg and Sergiescu discovered a relation between Thompson's groups and braid groups, by studying an action of the derived subgroup F' of the Thompson group F. Using this action they defined a morphism $F' \to \mathrm{Out}(B_\infty)$, where B_∞ is the *stable braid group*, a braid group on a countable number of strands. They deduced the existence of an acyclic extension of F' by the stable braid group B_∞.

In 2001, de Faria, Gardiner and Harvey showed that Thompson's group F can be realized as a mapping class group of an infinite type surface in the quasiconformal setting. Here, the surface is the complement in the complex plane of a Cantor set, and the Teichmüller space is the space of asymptotically conformal deformations of that surface. In this setting, the marking that defines the elements of Teichmüller space is an asymptotically conformal homeomorphism, meaning that it is quasiconformal and that for every $\epsilon > 0$ there exists a compact subset of the surface such that the complex dilatation of the surface is bounded by $1 + \epsilon$ on the complement of this compact set.[13] The result says that Thompson's group F admits an embedding into the group of isotopy classes of orientation-preserving homeomorphisms of a surface $S_{0,\infty}$ of genus 0 and of infinite topological type. In 2005, Kapoudjian and Sergiescu obtained a similar result for the group T. Whereas Faria, Gardiner and Harvey worked in the quasiconformal setting, Kapoudjian and Sergiescu worked in the topological setting. They introduced in 2004 the notion of asymptotically rigid homeomorphism in the study of Thompson groups, and this notion was extensively used in later works by Funar and Kapoudjian.

All Thompson groups have interesting finite and infinite presentations. Some of these presentations use surface homeomorphisms, which make another relation to mapping class groups.

There is also a description of each element in the three classes of Thompson's groups in terms of operations on objects called *rooted binary tree pair diagrams*. Here, a pair of trees associated to a group element describes the subdivision of the domain and range into subintervals on which the element acts linearly. The tree interpretation makes the Thompson groups related to the so-called Ptolemy groupoids, a category whose objects are marked Farey tessellations and which are also closely related to mapping class groups.

[13] We recall that there are various non-equivalent definitions of Teichmüller space in the case of surfaces of infinite type. The quasiconformal setting provides one possible definition, and the hyperbolic setting provides other definitions which in general are not equivalent.

Based on work of Penner on the universal Teichmüller space, Funar and Kapoudjian showed in 2004 that Thompson's group T is isomorphic to a "universal mapping class group", a finitely presented group of mapping classes that are "asymptotically rigid" of the surface $S_{0,\infty}$, which is itself equipped with a certain rigid structure associated to a hexagon decomposition. The universal mapping class group contains all mapping class groups of compact surfaces of genus zero, and it also encodes the mutual relations between these groups. The results are formulated in terms of the Ptolemy groupoid and for this reason the group T also carries the name the *Ptolemy–Thompson group*. The Ptolemy–Thompson group T is seen as the analogue of the mapping class group of the hyperbolic plane. A dilogarithmic representation of the Ptolemy groupoid induces a representation of the Ptolemy–Thompson group. In the same work, Funar and Kapoudjian discovered a relation between the Ptolemy groupoid and a pants decomposition complex associated to the surface $S_{0,\infty}$ of genus 0 and of infinite topological type which generalizes the Hatcher–Thurston pants decomposition complex of compact surfaces. The pants decomposition complex of $S_{0,\infty}$ is defined here as an inductive limit of the pants decomposition complexes of compact subsurfaces of $S_{0,\infty}$. To a pants decomposition is associated a hexagon decomposition which is intuitively defined by distinguishing a "visible" and a "hidden" side of $S_{0,\infty}$ (this pants decomposition defines the rigid structure that we alluded to above, and this rigid structure is used to define the notion of asymptotic rigidity) and a result by Funar and Kapoudjian says that the Thompson group T is the group of asymptotically rigid mapping classes of that surface which preserve the decomposition into hidden/visible sides. Using the genus-0 infinite-type Hatcher–Thurston complex, and by a method which parallels the work of Hatcher and Thurston on mapping class groups of surfaces of finite type, the authors showed that a certain group defined as the group of asymptotically rigid mapping classes of $S_{0,\infty}$ is finitely presented.

Funar and Kapoudjian then introduced in 2008 a group T^* called the *braided Ptolemy–Thompson group*, which is another extension of T by the stable braid group B_∞. They showed that the group T^* and therefore the group T, are asynchronously automatic, a result that is an analogue of a result by Mosher saying that mapping class groups of surfaces of finite type are automatic. The complete analogue of Mosher's result is presented as an open problem.

By quantization, a projective representation of T (called a *dilogarithmic representation*) is obtained. The authors of Chapter 10 also present a recent result by Funar and Sergiescu saying that this representation comes from a central extension of T whose class is 12 times the Euler class generator. The relation to clusters is also made.

In addition, several extensions of Thompson groups are presented in Chapter 10; for instance, the extension of V by the so-called braided Thompson group of Brin–Dehornoy, its extension by the so-called universal mapping class group and its extension by the asymptotically rigid mapping class group in infinite genus. The authors include all such extensions in a unified setting arising from a functorial algebraic construction, defined on a category whose objects are called *cosimplicial symmetric group extensions*. This algebraic formalism is also used to describe the action of the

Grothendieck–Teichmüller group acts on some group completions. Other works on the relation to the Grothendieck–Teichmüller theory were done by Lochak, Nakamura and Schneps.

The authors also present the relation between this theory and the theory of the so-called *braided Houghton groups*, studied by Degenhardt and Dynnikov, and which are also mapping class groups of surfaces of infinite type.

We finally note that besides sharing properties with mapping class groups, Thompson's groups have connections with arithmetic groups.

3 Part C. The algebraic topology of mapping class groups and moduli spaces

3.1 The intersection theory of moduli space

Intersection theory is a classical subject in algebraic geometry. Its main object of study is the intersection of subvarieties in an algebraic variety. The theory can be traced back to works by eighteenth century mathematicians on the intersection of hypersurfaces in $\mathbb{R}^n$. The theorem of Bézout, which states that the number of intersection points of two plane algebraic curves is equal to the product of their degrees, is considered as belonging to intersection theory. In the modern theory, intersections are computed in the cohomology ring.

The moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points, together with its Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$, are generalized algebraic varieties. More precisely, they are algebraic stacks of complex dimension $3g-3+n$. A stack is the analogue, in algebraic geometry, of an orbifold in the analytic setting. In the algebro-geometric setting, Riemann surfaces are called *curves* (manifolds of complex dimension one) and the elements of the compactification are surfaces with nodes called *stable curves*. An element of $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$, that is, a stable curve, is a singular complex algebraic curve whose singularities are nodes, which are the isolated singularities of the simplest possible kind: the local model of a node is the neighborhood of the origin of the plane algebraic set defined by the equation $xy = 0$. Topologically, a node is the singularity of two real 2-dimensional disks identified at their centre. The compactification $\overline{\mathcal{M}}_{g,n}$ is naturally considered as the moduli space of surfaces with nodes. As algebro-geometric orbifolds, the moduli space and its compactification have an intersection theory, with its associated algebro-geometric apparatus on homology and cohomology; it is equipped with complex vector bundles which have their Chern classes, a Grothendieck–Riemann–Roch formula, and so on.

Chapter 11 by Dmitry Zvonkine contains a review of the intersection theory of the moduli space of curves $\mathcal{M}_{g,n}$ and of its Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$, also called the moduli space of stable curves.

A strong impetus to the study of the intersection theory of moduli spaces that especially caught the attention of topologists and geometers was given in 1991 by Witten, who conjectured the existence of a generating recursion formula for all intersection numbers of some special elements of $H^2(\bar{\mathcal{M}}_{g,n}, \mathbb{Q})$, called ψ-classes. These ψ−classes, $\psi_1, \dots, \psi_n$, also known as *tautological classes*, are the first Chern classes of some natural line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n$ that are themselves "tautological" in the sense that the fiber at each point is precisely the cotangent line to the corresponding curve or stable curve representing the point. The tautological classes are natural with respect to forgetful maps and attaching maps performed at the level of Riemann surfaces and of stable curves. They generate a subring called the *tautological cohomology ring*. Witten's conjecture was proved by Kontsevich in 1992, and several other proofs of this conjecture were given later on. In 2004, Mirzakhani made a relation between the intersection numbers of the ψ-classes and the Weil–Petersson volume of moduli space.

In Chapter 11 of this volume, Zvonkine starts by introducing the basic objects in the theory, namely, the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points, its Deligne–Mumford compactification $\bar{\mathcal{M}}_{g,n}$, the universal curve $\mathcal{C}_{g,n}$ over $\mathcal{M}_{g,n}$, and the universal curve $\bar{\mathcal{C}}_{g,n}$ over $\bar{\mathcal{M}}_{g,n}$. He gives a description of the smooth orbifold structure of the spaces $\mathcal{M}_{g,n}$ and $\bar{\mathcal{M}}_{g,n}$. He then introduces the tautological cohomology classes on $\bar{\mathcal{M}}_{g,n}$. He gives a wide class of explicit examples of tautological classes and he computes intersection numbers between them. The computations are based on the Grothendieck–Riemann–Roch Theorem, and on a study of pull-backs of such classes under attaching and forgetful maps. In order to make the exposition self-contained, the author gives a short introduction to the theory of characteristic classes of vector bundles. He also motivates Witten's conjecture, which turned out to be a major question for research done in the last two decades. Elements of the proof given by Kontsevich are mentioned. In particular, the string and dilation equations as well as the KdV equations are discussed.

In some sense, this chapter complements a chapter by G. Mondello in Volume II of this Handbook which gives a detailed account of the use of ribbon graphs in the intersection theory of moduli space, in relation to the Witten conjecture.

3.2 The generalized Mumford conjecture

In Chapter 12, Ib Madsen gives a survey of the proof of a generalized version of the Mumford conjecture which he obtained in joint work with M. Weiss. The original Mumford conjecture states that the stable rational cohomology of the moduli space $\mathcal{M}_g$ is a certain polynomial algebra generated by the Mumford–Morita–Miller cohomology classes of even degrees. The conjecture can also be formulated in terms of the cohomology of a classifying space of mapping class groups. The Madsen–Weiss result generalizing Mumford's conjecture states that a certain map between some classifying spaces which a priori have different natures induces an isomorphism at the level of integral homology. The result, obtained in 2002, was published in 2007 in a paper entitled *The stable moduli space of Riemann surfaces: Mumford's conjecture.*

This solution of the Mumford conjecture is considered as spectacular progress in the question of understanding the homotopy type of moduli space. This question is one of the most fundamental questions in Teichmüller theory. We are still very far from having a response to it, except for some special surfaces. It is in relation to this question that Mumford started in the early 1980s a study of the cohomology ring of moduli space.

There is an intimate relation between Riemann's moduli space and the classifying space of the mapping class group: the rational cohomology rings of the two spaces coincide. It seems that up to now, the only closed orientable surfaces for which we have a complete description of this rational cohomology ring are the surfaces of genus ≤ 4. At the other extreme, we have information about the *stable* cohomology, which can be considered as information about the cohomology ring of moduli spaces of surfaces with very large genus.

By definition, the stable rational cohomology ring of moduli space is the direct limit of rational cohomology rings of moduli spaces of a class of surfaces of increasing genus. More concretely, these surfaces are compact with one boundary component, embedded into one another, that is, $S_{g+1,1}$ is obtained by attaching to $S_{g,1}$, along its boundary component, a torus with two disks removed. Mumford's conjecture (which appeared in print in 1983) states that the stable rational cohomology of the moduli space $\mathcal{M}_g$ is a polynomial algebra generated by certain tautological cohomology classes which Mumford defined in the context of the Chow ring of the Deligne–Mumford compactification of moduli space. The same classes were re-introduced from a more topological point of view by Miller and by Morita, in 1986 and 1987. Miller and Morita defined the tautological cohomology classes as cohomology classes of the classifying space $B\Gamma_g$ of the mapping class group Γ_g. These tautological classes, usually denoted by κ_i, are now called Mumford–Morita–Miller classes. Mumford's conjecture states that the rational cohomology of the stable moduli space is a polynomial algebra generated by the Mumford–Morita–Miller classes κ_i of dimension $2i$.

Mumford's conjecture can also be formulated in terms of the cohomology of the classifying space $B\Gamma_\infty$ of the mapping class group. The conjecture seems to have been motivated by a stability result in the context of the Grassmannian of d-dimensional linear subspaces of $\mathbb{C}^n$, stating that the cohomology of that space stabilizes as $n \to \infty$ to a polynomial algebra in the Chern classes of the tautological d-dimensional vector bundles.

The generalized form of Mumford's conjecture, proved by Madsen and Weiss, says that the integral cohomology ring of the infinite genus mapping class group is equal to the cohomology ring of a certain space associated with the Pontryagin–Thom cobordism theory. This result is important because the algebraic topology of the space associated with cobordism theory is well understood. This theory had already been used as a basic tool in establishing several major results in geometry, for instance

Milnor's construction of exotic spheres and the early proofs of the Atiyah–Singer theorem.

In their proof of the generalized conjecture, Madsen and Weiss computed the rational stable cohomology of the mapping class group. But the generalized conjecture may also be used to calculate the mod p cohomology of the stable mapping class group for all primes p, and hence the integral cohomology.

The generalized conjecture was formulated by Madsen around the year 2000, after Tillmann discovered, using Harer's stability theorem, that Quillen's plus construction applied to the classifying space $B\Gamma_\infty$ of the mapping class group makes this space an infinite loop space. In her work, Tillmann was motivated by string theory, and one can consider these developments as an instance of the fact that ideas in theoretical physics can have a major impact in geometry.

The proof by Madsen and Weiss of the generalized Mumford conjecture uses techniques from high-dimensional manifold theory (the Pontryagin–Thom theory of cobordisms of smooth manifolds that we already mentioned) and singularity theory. There is an identification of the rational cohomology of Riemann's moduli space with what Madsen and Weiss call the *embedded* moduli space $\mathcal{S}(2)$, the space of differentiable subsurfaces of a high-dimensional Euclidean space. The rational homology isomorphism between the two spaces is obtained by assigning to each differentiably embedded surface its induced Riemann surface structure. The embedded moduli space is then used to classify smooth embedded surface bundles, and one recovers characteristic classes of surface bundles (like the Mumford–Morita–Miller classes) from $\mathcal{S}(2)$ cohomology classes.

Chapter 12 also contains a report on a new proof of the generalized Mumford conjecture that was given by Galatius, Madsen, Tillmann and Weiss in 2009. Let us note that in the same year, Eliashberg, Galatius and Mishachev gave another proof of the generalized Mumford conjecture, in a paper entitled "Madsen–Weiss for geometrically minded topologists". The title indicates that the proof is more geometrical than the original one. It is based on Madsen and Weiss's original ideas, and it uses a new version of Harer's stability result which the authors formulate in terms of folded maps. They attribute the idea of such a geometrical proof to Madsen and Tillmann who suggested it in their paper *The stable mapping class group and* $Q(\mathbb{C}P^\infty_+)$, published in 2001.

3.3 The L^p-cohomology of moduli space

Chapter 13 by Lizhen Ji and Steven Zucker concerns the L^p-cohomology of moduli space.

The definition of the L^p-cohomology of a non-compact manifold depends on the choice of a metric (usually, a Riemannian metric) on that space. Thus, one has to choose a metric on Teichmüller space. But since L^p-cohomology is a quasi-isometry invariant of the manifold, the results presented here are valid for Teichmüller and moduli spaces with respect to several of their known metrics.

Before stating the results that are reviewed in this chapter, let us first say a few words on L^p-cohomology.

Historically, the theory started with the case $p = 2$, that is, L^2-cohomology. This theory was developed independently, at the end of the 1970s, by Cheeger and Zucker, as a cohomology theory for non-compact manifolds which is defined in a way parallel to de Rham cohomology, but where one uses, instead of general differential forms, square-integrable forms, with respect to a Riemannian metric on the ambient manifold. As usual, the L^2-theory has an advantage over the general L^p-theory, because L^2-norms define Hilbert space structures.

It turned out that the L^2-cohomology of a space is related to the intersection cohomology of a suitable compactification of that space. This was first realized by Zucker, who conjectured in 1982 that the L^2-cohomology of an arithmetic Hermitian locally symmetric space is isomorphic to the intersection cohomology of its Baily–Borel compactification, for what is called the middle perversity. Two independent proofs of that conjecture were given by Loojienga in 1988 and by Saper and Stern in 1990, and although it became a theorem, the result is still called the "Zucker conjecture".

Near the end of the 1970s, Cheeger proved that the L^2-cohomology of a Riemannian manifold with cone-like singularities is isomorphic to its intersection cohomology.

The theory of L^p-cohomology was developed later on for all $p > 1$, using the Banach space of L^p-differential forms equipped with a natural L^p-norm. The L^p-cohomology of a Riemannian manifold is invariant by bi-Lipschitz diffeomorphisms. For Riemannian manifolds with finite area and cusps, it was expected that an analogue of the Zucker conjecture is true, that is, that the L^p-cohomology coincides with the cohomology of some appropriate compactification. There have been several works in that direction, by Zucker.

To what metrics on moduli space does this theory apply? The moduli space $\mathcal{M}_{g,n}$ of algebraic curves of genus g with n punctures carries several complete Riemannian metrics, including the so-called Kähler–Einstein metric, the McMullen metric, the Ricci metric and the Liu–Sun–Yau metric. In contrast with the Weil–Petersson metric, of which we have now a better understanding (but which has the disadvantage of being incomplete), these metrics are still not well studied. But it is known that they are all quasi-isometric and hence they have the same L^p-cohomology. Ji and Zucker showed that for all $1 < p < \infty$, the L^p-cohomology of moduli space $\mathcal{M}_{g,n}$ is isomorphic to the ordinary cohomology of its Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}^{\mathrm{DM}}$. This result is an analogue of the Zucker conjecture for Hermitian locally symmetric spaces equipped with their Baily–Borel compactification. It also shows that the L^p-cohomology does not depend on p. Ji and Zucker consider this as a rank-one property of moduli space, because in the case of symmetric spaces of rank > 1, the L^p-cohomology in general depends on the value of p.

The result concerning the L^p-cohomology of the Weil–Petersson metric is of a different nature. In this case, Ji and Zucker showed that for $4/3 < p < \infty$, the L^p-cohomology is isomorphic to the cohomology of $\overline{\mathcal{M}}_{g,n}^{\mathrm{DM}}$ whereas for $1 \leq p \leq 4/3$, the L^p-cohomology is isomorphic to the cohomology of the space $\mathcal{M}_{g,n}$ itself.

Chapter 13 contains a proof of these results. It also contains survey sections on L^p-cohomology, on the intersection cohomology of projective algebraic varieties, and on the Hodge decomposition of compact Kähler manifolds. This will give the reader a complete and self-contained account of the subject treated. The chapter ends with several open problems concerning the various complete metrics on Teichmüller spaces and moduli spaces.

Chapter 13 is somehow in the same spirit as Chapter 4 of this volume in the sense that it presents some analogies between Teichmüller spaces (resp. moduli spaces) and symmetric spaces of non-compact type (resp. non-compact locally symmetric spaces).

4 Part D. Teichmüller theory and mathematical physics

This volume had started with two fundamental tools in the deformation theory of Riemann surfaces, namely, the Beltrami equation and the earthquake theorem. We already discussed at length these two tools, the Beltrami equation being at the basis of the analytic deformation theory of Riemann surfaces, while the earthquake theorem is at the basis of the deformation theory of hyperbolic metrics. Now the volume ends with a part on the relation between Teichmüller theory and physics, and the two chapters that constitute this part use a third basic tool in uniformization theory, namely the Liouville equation (1853).

We start by recalling the definition.

Let h_0 be a Riemannian metric on a closed surface S. Any other Riemannian metric which is conformal to h_0 can be written as $h = e^{2\phi} h_0$, where ϕ is a real-valued function on S. Let Δ_0 be the Laplacian and K_0 the Gaussian curvature function on S, both with respect to h_0. The metric h is hyperbolic (i.e. it is a Riemannian metric of constant curvature -1) if and only if it satisfies the following equation (called Liouville equation):

$$\Delta_0 \phi - K_0 = e^{2\phi}.$$

In principle, the existence theory of solutions to the Liouville equation can be considered as a precise version of Riemann's uniformization principle, and it was used by Poincaré in his first attempts to prove the uniformization theorem. But in practice, this approach to uniformization is considered to be too difficult for being useful.

The work done on the Liouville equation is mostly due to theoretical physicists, and it is interesting to make this work accessible to mathematicians. Chapter 14 by Kirill Krasnov and Jean-Marc Schlenker and Chapter 15 by Rinat Kashaev should be useful in this respect. The two chapters provide a review of some applications of this equation, from different points of view. Both chapters highlight the connection between the Liouville equation and Teichmüller theory through various recent works.

The Liouville equation has been extensively used in several domains of theoretical physics, including two-dimensional gravity (Jackiw, 1983), non-critical string theory (Polyakov, 1981), conformal field theory (Belavin, Polyakov & Zamolodchikov, 1984), in the quantization theory of Teichmüller space (Kashaev, 1988 ca. and Teschner 2003 ca.), and more recently in work on $N = 2$ supersymmetric gauge theories in 4 dimensions (Gaiotto, 2009).

Let us mention a few of the developments in theoretical physics.

The Liouville equation gives rise to a functional on the moduli space of metrics, called the *Liouville functional*, which can be defined as

$$S[h_0, \phi] = \frac{1}{8\pi} \int d\,\mathrm{vol}_0(|\nabla\phi|^2 + e^{2\phi} - 2\phi K_0).$$

In this form, Liouville theory appeared as a tool in non-critical relativistic string theory. Takhtajan and Zograf (1987), showed that the Liouville functional provides a Kähler potential for the Weil–Petersson metric on Schottky space. Several developments followed that discovery, and they are again due to theoretical physicists. In particular, a Liouville action defined as a functional on moduli spaces provided a relation between the Liouville theory and the renormalized volume of hyperbolic manifolds (works of Krasnov, 2000, of Takhtajan & Teo, 2003, and others). One should also mention that the work of Takhtajan & Zograf motivated McMullen in his construction of a Kähler hyperbolic metric on moduli space, and in his discovery of the so-called *quasi-Fuchsian reciprocity* law, a duality formula which expresses the fact that the tangent maps at the Fuchsian complex projective structure of the Bers embedding of the two boundary components of a quasi-Fuchsian manifold are adjoint linear operators.

4.1 The Liouville equation and normalized volume

Chapter 14 presents the ideas of renormalized volume of hyperbolic 3-manifolds. The relation between the geometry of a hyperbolic 3-manifold and moduli spaces of Riemann surfaces can be conceived most clearly in the case where the manifold is a product $S \times [0, 1]$ of a surface with $\mathbb{R}$, that is, the context of quasi-Fuchsian manifolds. More precisely, these are the complete hyperbolic 3-manifolds that are homeomorphic to the product of a surface with an interval and that contain a non-empty convex set. (We already encountered these manifolds in Chapter 3 of this volume.) A useful ingredient in this relation is the fact that the space of hyperbolic structures on $S \times [0, 1]$ is closely related to the space of projective structures on the boundary. A quasi-Fuchsian 3-manifold has infinite volume, but following an idea that is due to theoretical physicists (Witten, 1998), one can define a *renormalized volume*, using a foliation by equidistant surfaces in the complement of a compact convex subset N of M. The renormalized volume is thereby defined in terms of the asymptotic behavior of the volume of the set of points at distance at most ρ from N as $n \to \infty$. The renormalized volume appears as the constant term of the asymptotic expansion of the volume in terms of the parameter ρ. Krasnov and Schlenker gave an alternative approach to that theory that is based on simple differential geometry arguments and that is more suited to mathematicians than the one used by physicists. They obtained a definition of renormalized volume in terms of the volume of the convex manifold N and the total mean curvature of its boundary, and they derived a variational formula which is an analogue of the Schläfli formula for volumes of spherical and hyperbolic tetrahedra. Krasnov and Schlenker also gave an interpretation of renormalized volume as a function on Teichmüller space, and they obtained a new proof of the fact that the renormalized volume of quasi-Fuchsian (or more generally geometrically finite) hyperbolic 3-manifolds provides a Kähler potential for the Weil–Petersson metric on Teichmüller space. The theory can be developed in the general setting of convex co-compact hyperbolic 3-manifolds where the complex projective structure on the boundary plays a central role, and in which the renormalized volume can be expressed in terms of the Liouville functional at the corresponding projective structure.

4.2 The discrete Liouville equation and the quantization theory of Teichmüller space

Chapter 15 by Kashaev is a review of a discrete version of the Liouville equation, interpretating it as a mapping class group dynamical system in the Teichmüller space of an annulus with marked points on its boundary. The theory makes use of Thurston's shear coordinates on the annulus. The discrete version of the Liouville equation that is reported on in this chapter was first defined on the integer lattice $\mathbb{Z}^2$ by Fadeev and Volkov. Kashaev reviews elements of quantum discrete Teichmüller theory and its relation to the usual quantum Teichmüller theory. Both theories are based on the so-called *non-compact dilogarithm function*, and for the convenience of the reader, the basic properties of this function are reviewed in this chapter.