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★**Concentration compactness for critical wave maps.**
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A wave map $u\colon \mathbf{R}^{2+1}\to M$ in two spatial dimensions from the Minkowski space $\mathbf{R}^{2+1}$ to a smooth Riemannian manifold $M$ is a (formal) critical point of the Lagrangian $\int_{\mathbf{R}^{2+1}}\partial^\alpha u\cdot\partial_\alpha u$. This is a geometric nonlinear wave equation, with a conserved energy $E(u)$ which is scale-invariant in two spatial dimensions, and is an important example of a critical nonlinear wave equation. Until recently, one of the major open problems in this field was the large data global regularity problem (i.e., existence of smooth global solutions from all large smooth initial data obeying the natural compatibility conditions) for such equations, in the case when the target manifold $M$ had non-negative curvature, or at least when the energy $E(u)$ of the wave map was less than or equal to the energy of the first harmonic map on $M$.

This conjecture (sometimes known as the threshold conjecture for wave maps) was recently solved by J. Sterbenz and D. Tataru in a series of two papers [Comm. Math. Phys. **298** (2010), no. 1, 139–230; MR2657817; Comm. Math. Phys. **298** (2010), no. 1, 231–264; MR2657818] under some mild general hypotheses on the target manifold $M$. However, their proof, while relatively short by the standards of this field, relied on an indirect argument starting with an assumption of singularity formation (either at finite or infinite time) and leading eventually to a contradiction. As such, the Sterbenz-Tataru result is a qualitative one, in that no long-time bounds are provided on the solution constructed.

Simultaneously with the Sterbenz-Tataru result, two other, lengthier, arguments appeared, one by the reviewer (in an unpublished series of preprints), and one by the authors (in the book under review), which were restricted to more symmetric target manifolds $M$, namely those of constant negative curvature targets (which, in the case of the book under review, also need to be two-dimensional), but for which more quantitative bounds were obtained. Namely, in both of these results, the wave map $u$ (and its derivatives) was shown to be globally bounded in certain technical spacetime norms which include in particular Strichartz-type norms, with the bound depending only on the energy $E(u)$ of the wave map. Furthermore, in this book, the authors are able to establish a certain technical concentration-compactness property on sequences $u_n$ of wave maps of bounded energy, which is a (significantly more nonlinear) version of the concentration-compactness results of H. Bahouri and P. Gérard [Amer. J. Math. **121** (1999), no. 1, 131–175; MR1705001]. Roughly speaking, this property asserts that after passing to a subsequence, one can asymptotically express $u_n$ via an inductive procedure in which one builds a wave map from the vacuum solution by inserting frequency-localised components in order of increasing frequency. A major new difficulty, not present in the Bahouri-Gérard work, is that these components do not superimpose linearly upon each other, but rather the low frequency components create a magnetic field which distorts the evolution of the high frequency ones. (Another major new difficulty is that the space of maps into the target manifold $M$ is not a vector space, and so there is no easy way to superimpose two such maps to create a new map.)

Aside from all these (rather monumental) technical difficulties, the basic strategy of proofs in the book are based on the paper of C. E. Kenig and F. Merle [Acta

Math. **201** (2008), no. 2, 147–212; MR2461508]. Namely, one assumes for the sake of contradiction that no global spacetime bound on wave maps in terms of the energy exists, and concludes that there is a critical threshold energy, below which a spacetime bound exists, but for which there exists a sequence of wave maps approaching that energy whose spacetime norms go to infinity. Using Bahouri-Gérard type concentration compactness, one eventually concludes the existence of a limiting profile at the critical energy which is "almost periodic modulo symmetries", so that the wave map takes values in a precompact subset of the state space up to translation and dilation. The analysis then splits up into a number of cases depending on how the position and scale of the wave map evolve in time, but eventually all such cases can be ruled out by a combination of finite speed of propagation, Morawetz or virial type inequalities to control time derivatives of the wave map, and non-existence results for harmonic maps into negatively curved targets. *Terence Tao*