

# Preface

Teichmüller theory succeeds in describing and classifying geometric structures on surfaces. It was born in the work of O. Teichmüller using techniques of complex analysis [2], [65] and was transformed under W. Thurston’s influence [161], [42] using techniques of hyperbolic geometry.

Decorated Teichmüller theory is an essentially combinatorial treatment of the Teichmüller theory of surfaces using techniques of hyperbolic geometry, where the surface is required to have punctures and/or boundary, and the punctures or boundaries often come equipped with a further “decoration” typically given by a real or positive real parameter which may be assigned to punctures, to boundary components, to distinguished points on boundary components, or to subsets of these sets. For example in the case of punctured surfaces, the decoration may describe a tuple of “horocycles”, one about each puncture.

One studies in each case an appropriate so-called “decorated Teichmüller space”, which is typically a trivial bundle over the Teichmüller space with fiber  $\mathbb{R}^N$  or  $\mathbb{R}_{>0}^N$ , for some  $N \geq 1$ , and the “mapping class group” action on Teichmüller space extends by permuting the parameters to an action on the decorated space itself. Thus, very little is lost in passing to the decorated space, but one must study punctured and/or bordered surfaces to get started.

A main point of passing to the decorated space is that decorated Teichmüller space (more precisely, its quotient by the diagonal action of  $\mathbb{R}_{>0}$  on decorations) admits a mapping class group-invariant “ideal simplicial decomposition”, by which we mean a decomposition into open simplices together with only certain of their boundary faces. Moreover, the cells in the decomposition are described sufficiently succinctly by an elaboration of graphs called “fatgraphs” so as to allow computations of invariants, for instance, presentations of the mapping class groups and calculations of cocycles. Our approach to the ideal simplicial decomposition assigns to each “decorated hyperbolic structure” a decomposition into polygons of the underlying surface, and it depends upon a convex hull construction in Minkowski space. The Poincaré dual of such a polygonal decomposition of the surface is a fatgraph, and one assigns a real number to each edge of the fatgraph called a “simplicial coordinate” using Minkowski geometry.

Rather than the hyperbolic version of this ideal simplicial decomposition treated here, one may instead derive an analogous one in the setting of conformal (rather than hyperbolic) geometry relying on the foundational work of K. Strebel [153]. An explicit construction assigns to a fatgraph together with a tuple of positive real numbers, one number for each edge of the fatgraph, a “Jenkins–Strebel differential”, i.e., a meromorphic quadratic differential  $q$  whose horizontal trajectories foliate the underlying surface-minus-fatgraph by simple closed curves with residues of  $\sqrt{q}$  assigned at the punctures. Deep work of Strebel plus further results of Hubbard–Masur [66] shows

that this assignment of conformal structure to a fatgraph-with-numbers establishes an isomorphism of Teichmüller space decorated by residues at the punctures with the natural space of all isotopy classes of fatgraphs-with-numbers embedded in the underlying surface. This decomposition agrees combinatorially with the hyperbolic one, but the two differ as point sets in decorated Teichmüller space.

It is a basic distinction that the explicit constructions in the two theories “go in opposite directions” in the sense that the convex hull construction produces a fatgraph-with-numbers from a decorated hyperbolic structure, and the Strebel theory produces a conformal structure from a fatgraph-with-numbers<sup>1</sup>. In particular, the inverse of the Strebel construction depends upon solving the “Beltrami equation” while the inverse of our construction amounts to the solution of an explicit family of “arithmetic problems”, one such system of integral algebraic equations for each trivalent graph. Whereas the solution to the Beltrami equation is highly transcendental, the solutions to our arithmetic problems are algebraic.

It was D. Mumford who first observed the application of Strebel’s work to the combinatorics of Riemann’s moduli space as described in J. Harer’s landmark papers [57], [58], which gave the first substantial applications of the conformal version of the triangulation to the geometry of Riemann’s moduli space. At roughly the same time, the decorated Teichmüller theory gave the hyperbolic version described here by specializing to dimension two the general convex hull construction [41] of the author with D. Epstein for complete but non-compact finite-volume hyperbolic manifolds of any dimension. This is enough to describe the polygonal decomposition of the surface, but more work is required to show that the “putative cells are cells” in decorated Teichmüller theory. Subsequently, B. Bowditch and D. Epstein [22] gave a proof of the existence of the ideal simplicial decomposition of decorated Teichmüller space based on loci equidistant to specified horocycles, which coincides exactly with the convex hull construction here (as we show when we re-interpret the arithmetic problem geometrically).

Another singular and fundamental aspect of decorated Teichmüller theory is that there are global affine coordinates on the decorated Teichmüller space called “lambda lengths” with remarkable properties. These are the ambient coordinates in which we formulate and solve the arithmetic problems and prove the existence of the ideal simplicial decomposition, which amounts to proving the unique solvability for lambda lengths from appropriate simplicial coordinates. These lambda length coordinates, which are essentially inner products in Minkowski space, are absolutely central to our treatment, and we unapologetically take a decidedly nineteenth century viewpoint and perform essentially all basic calculations in suitably normalized lambda lengths. We parenthetically mention that Wolpert [173] has recently shown that lambda lengths are convex along earthquake paths.

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<sup>1</sup>One may thus start with a decorated hyperbolic structure, apply our convex hull construction to produce a fatgraph-with-numbers, where the numbers are given by simplicial coordinates, and then misinterpret these numbers as Strebel coordinates so as to produce a map from decorated Teichmüller space to itself. This map is *not* the identity, but we conjecture that this map has bounded distortion in an appropriate sense.

These coordinates are *not canonical* in that they depend upon the choice of a suitable fatgraph in the surface just as coordinates on a vector space depend upon a choice of basis. However, the coordinate transformations corresponding to different choices of fatgraph faithfully describe the action of the mapping class group of the surface in coordinates and are calculable in terms of “Ptolemy transformations”, which play a role by now in a number of fields of mathematics, for example, in the study of “cluster algebras” [46], [45] and [47], and “quantum Teichmüller theory” [32], [78], [31], and [33].

Not only that, the “Weil–Peterson Kähler two form” [172], [173] admits a simple and compact expression<sup>2</sup> in lambda lengths, which is also part-and-parcel of these other studies. Here, we shall compute and extend the basic WP Kähler two form and perform a few sample WP volume calculations partly as a paradigm for the general method of integration over moduli space. We shall also compute the Poincaré dual of the WP two form in Appendix B [133], which is primarily included because it illustrates further important general aspects of integration over Riemann’s moduli space.

Another of the author’s papers [136], on the Gauss product of binary integral quadratic forms, is included as Appendix A because just as this volume itself begins essentially *tabula rasa*, so too this paper was intended to be a self-contained introduction to topics in algebraic number theory from first principles and hence may be useful for a similar audience.

Furthermore, a joint paper [105] with Greg McShane is included as Appendix C because of the basic computations it describes on the asymptotics of lambda lengths during degeneration of the underlying surface.

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We have taken this opportunity to correct a few small calculational and other errors (which are explicitly noted in the text) and to present sometimes simpler and sometimes more detailed proofs than in the original papers. It is fair to say that results were not necessarily discovered in “correct” order temporally, so here we try to give a more systematic derivation of this theory from first principles. Variants and relative versions of the foregoing theory are discussed for “partially decorated surfaces” (where only certain of the punctures are decorated), for “bordered surfaces” (where the punctures are in effect required to lie in the boundary of the surface and all of them are decorated),

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<sup>2</sup>These two attributes of simple calculability, both for the action of the mapping class group and for the underlying symplectic geometry of the WP metric, distinguish lambda lengths among all known parametrizations of (decorated) Teichmüller spaces. So-called “Fricke coordinates” (i.e., entries of matrices in a Fuchsian group) transform explicitly under the action of the mapping class group with the WP two form unknown, and Fenchel–Nielsen coordinates, cf. Theorem 1.18 in Chapter 1, transform horribly under the mapping class group with the WP two form simply and beautifully expressed by Wolpert, cf. [172], [173]. In fact, our treatment of the WP two form is based on Wolpert’s formula Theorem 3.2 in Chapter 2, and if Fenchel–Nielsen coordinates are “length/twist” coordinates, then lambda lengths provide “length/length” coordinates on decorated Teichmüller space. Moreover, lambda lengths “tropicalize” to convenient coordinates on Thurston’s boundary as discussed in Section 5.4 of Chapter 5.

and the general case (where both interior and boundary punctures are allowed and only certain of them are decorated).

Most of the sections beyond the first two in Chapter 2 and all of Chapter 3 are independent, and all of them are optional. In fact after reading Chapter 1, a bee-line for the lambda length parameterization for punctured surfaces is directly to read the second section of Chapter 2, and a subsequent bee-line for the ideal simplicial decomposition of the decorated Teichmüller space of a punctured surface is to read the first four sections of Chapter 4.

It may be useful to comment further here on various sections. In Chapter 2, Section 4 covers the parallel theory of *undecorated* “surfaces with holes” where boundary components can be deformed to punctures. This is useful for quantization including the Poisson structure inherited from the Weil–Petersson Kähler form, which is also discussed. Chapter 3 extends lambda lengths and associated structures from the setting of surfaces to the topological group of homeomorphisms of the circle suitably manifest as the space of all “tesselations of the Poincaré disk” and studies an associated infinite-dimensional Lie algebra in Section 4. Section 3 treats a universal profinite object in Teichmüller theory, the “punctured hyperbolic solenoid”.

The main applications of the theory to mapping class groups and moduli spaces are given in Chapter 5, and the final Chapter 6 covers further applications, where we have sometimes included particularly interesting or illustrative excerpts from more recent papers. There are clear extensions of aspects of the theory to possibly non-orientable two-dimensional orbifolds (for example, lambda lengths extend immediately to coordinates in the non-orientable case), but these have not yet been fully articulated.

We have not strived for completeness in the bibliography, rather, we have cited papers and books whose bibliographies may be consulted for more complete references. Let us apologize here and now if the concomitant omissions from our listed references might cause offense. Let us also apologize for the quirk of notation that the *surface*  $F(G)$  associated to a *fatgraph*  $G$  is sometimes taken to be a “skinny” surface with boundary and sometimes to be the punctured surface that arises by capping off each boundary component with a punctured disk, where the distinction will always be explicitly stated.

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