Preface

Since the fundamental results by Harish-Chandra and others, it is now well known that the algebra of invariant polynomials on the dual of a Lie algebra of a particular type (solvable [18], simple [24] or nilpotent) is isomorphic to the center of the corresponding universal enveloping algebra. This fact was generalized to an arbitrary finite-dimensional real Lie algebra by M. Duflo in 1977 [19]. His proof is based on Kirillov's orbits method that parametrizes infinitesimal characters of unitary irreducible representations of the corresponding Lie group in terms of co-adjoint orbits (see e.g. [28]). This isomorphism is called the *Duflo isomorphism*. It happens to be a composition of the well-known Poincaré–Birkhoff–Witt isomorphism (which is only an isomorphism at the level of vector spaces) with an automorphism of the space of polynomials (which descends to invariant polynomials), whose definition involves the power series $j(x) := \sinh(x/2)/(x/2)$.

In 1997 Kontsevich [29] proposed another proof, as a consequence of his construction of deformation quantization for general Poisson manifolds. Kontsevich's approach has the advantage of working also for Lie super-algebras and extending the Duflo isomorphism to a graded algebra isomorphism on the whole cohomology.

The inverse power series $j(x)^{-1} = (x/2)/\sinh(x/2)$ also appears in Kontsevich's claim that the Hochschild cohomology of a complex manifold is isomorphic as an algebra to the cohomology ring of holomorphic poly-vector fields on this manifold. We can summarize the analogy between the two situations in the following table:

Lie algebra	Complex geometry
symmetric algebra	sheaf of algebra of holomorphic poly-vector fields
universal enveloping algebra	sheaf of algebra of holomorphic poly-differential operators
taking invariants	taking global holomorphic sections
Chevalley–Eilenberg cohomology	sheaf cohomology

These lecture notes provide a self-contained proof of the Duflo isomorphism and its complex geometric analogue in a unified framework, and gives in particular a unifying explanation of the reason why the series $j(x)$ and its inverse appear. The proof is strongly based on Kontsevich's original idea, but actually differs from it (the two approaches are related by a Koszul type duality recently pointed out in [39] and proved in [8], this duality being itself a manifestation of Cattaneo–Felder constructions for the quantization of a Poisson manifold with two coisotropic submanifolds [12]).

Note that the series $j(x)$ also appears in the wheeling theorem by Bar-Natan, Le and Thurston [4] which shows that two spaces of graph homology are isomorphic as algebras (see also [31] for a completely combinatorial proof of the wheeling theorem, based on Alekseev and Meinrenken's proof [1], [2] of the Duflo isomorphism for quadratic Lie algebras). Furthermore this power series also shows up in various index theorems (e.g. Riemann–Roch theorems).

Throughout these notes we assume that k is a field with $\mathrm{char}(k) = 0$. Unless otherwise specified, algebras, modules, etc. are over k. Each chapter consists (more or less) of a single lecture.

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