

# Introduction

The present handbook focuses on recent developments in pseudo-Riemannian geometry and supersymmetry. In this introduction we give a short overview of the material contained in the various parts of the volume.

## Part A. Special geometry and supersymmetry

A classical field theory is usually specified by a Lagrangian  $L$ . The scalar fields of the theory are functions  $\phi^1, \dots, \phi^n$  on space-time  $\mathbb{M}$ . They can be interpreted as the components of a map  $\phi: \mathbb{M} \rightarrow M$  from space-time into a target manifold  $M$  with respect to some system of local coordinates  $x^1, \dots, x^n$  on  $M$ . The kinetic term  $-\frac{1}{2} \sum g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j$  for the scalars in the Lagrangian  $L$  defines a pseudo-Riemannian or even Riemannian metric  $g = \sum g_{ij} dx^i dx^j$  on  $M$ , provided that the symmetric matrix  $(g_{ij})$  of the scalar couplings is nondegenerate or even positive definite. Since the discovery of the first supersymmetric field theories, physicists have found that supersymmetry is often reflected in geometric properties of the target metric  $g$ . The specific restrictions imposed by supersymmetry depend on the dimension and signature of space-time, as well as on the field content of the theory. When the number of supercharges increases, the allowed target geometry is more and more restricted and becomes finally locally symmetric. The most interesting case is that of eight (real) supercharges. The corresponding geometry is called *special geometry*. In four-dimensional supergravity theories the matter fields are assembled in vector and hyper multiplets, which are coupled to the supergravity multiplet. The special geometry of supergravity coupled to vector multiplets is (projective) special Kähler geometry. That of supergravity coupled to hyper multiplets is quaternionic Kähler geometry. Both geometries are related by the so-called c-map, which associates a quaternionic Kähler manifold  $N$  of real dimension  $4n + 4$  (and negative scalar curvature) to any special Kähler manifold  $M$  of real dimension  $2n$ . Any quaternionic Kähler manifold  $N$  (of non-zero scalar curvature) is the base of an  $\mathbb{H}^*/\{\pm 1\}$ -bundle  $U(N) \rightarrow N$  called the Swann bundle of  $N$ . The Swann bundle is a hyper-Kähler cone. In particular, it carries a hyper-Kähler metric, the signature of which depends on the sign of the scalar curvature of the quaternionic Kähler base  $N$ . The hyper-Kähler metric can be locally derived from a (real valued) hyper-Kähler potential  $f$ , which effectively encodes also the geometry of  $N$ . Similarly, the geometry of the special Kähler manifold  $M$  is locally encoded in a holomorphic function  $F$  (called the holomorphic prepotential).

In Chapter 1 by Martin Roček, Cumrun Vafa and Stefan Vandoren, the hyper-Kähler potential  $f$  is determined in terms of the holomorphic prepotential  $F$ . The c-map is thus reduced to the correspondence  $F \mapsto f$ .

In Gregor Weingart's contribution, the bundle of differential forms on a quaternionic Kähler (or hyper-Kähler) manifold is decomposed into parallel subbundles. In

particular, the multiplicities of the corresponding irreducible  $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -representations (or  $\mathrm{Sp}(n)$ -representations in the hyper-Kähler case) are explicitly calculated.

Charles Boyer and Kris Galicki discuss Sasakian manifolds and their relation to special holonomy groups and supersymmetry in Chapter 3. Sasakian manifolds are intimately related to Kähler manifolds, which are fundamental objects in mathematics and theoretical physics. In fact, the metric cone over a Sasakian manifold is Kähler and the geometry transversal to the Sasakian vector field is also Kähler. Similarly, 3-Sasakian manifolds are intimately related to hyper-Kähler and quaternionic Kähler manifolds.

In the chapter by María A. Lledó, Oscar Maciá, Antoine Van Proeyen and Veeravalli S. Varadarajan the space-time signature remains Lorentzian but the signature of the special (pseudo-)Kähler target metric is arbitrary.

In Chapter 5, Thomas Mohaupt explains the role of special geometry in the theory of supersymmetric black holes. In particular, he shows how Euclidian supersymmetry in three dimensions can be used to study stationary black hole solutions in four dimensions. As one can see already from this example, the geometric structure of the target manifold of a supersymmetric theory can change significantly when the space-time signature changes from Lorentzian to Euclidian. Here not only the target metric changes from Riemannian to neutral, but from quaternionic Kähler to para-quaternionic Kähler. Such para-geometries are further discussed below.

## Part B. Generalized geometry

Mirror symmetry relates deformations of complex structures to deformations of symplectic structures (on the mirror manifold). Nigel Hitchin's notion of a *generalized complex structure* provides a superordinate conceptual framework in which complex and symplectic structures can be treated symmetrically.

The chapter by Hitchin is an introduction to the rapidly developing subject of generalized geometry, which incorporates central concepts of supergravity and string theory. In particular, the  $B$ -field and the 3-form gauge field  $H$  occur naturally in the twisting of the generalized tangent bundle by a gerbe. Moreover, the three-form  $H$  plays also the role of the torsion of a metric connection on the base manifold.

Alexei Kotov and Thomas Strobl focus on the role of such 'generalized' geometries encoded in some algebroid structure as targets of supersymmetric sigma models. In particular, generalized Kähler manifolds occur as such targets.

Ulf Lindström, Martin Roček, Rikard von Unge and Maxim Zabzine show in Chapter 8 that generalized Kähler structures can be derived from a generalized Kähler potential.

### Part C. Geometries with torsion

Let  $G \subset O(n)$  be a closed subgroup. A  $G$ -structure on an  $n$ -dimensional manifold admits a torsion-free connection only if the holonomy group of the Levi-Civita connection is a subgroup of  $G$ . Therefore, for a given  $G$ -structure, there may be no torsion-free connection at all. One is led to allow connections with non-zero torsion and to look for conditions on the torsion which ensure the uniqueness of the connection. It turns out that complete skew-symmetry of the torsion provides such a condition for certain  $G$ -structures. Moreover, connections with totally skew-symmetric torsion occur naturally in string theory and supergravity, as explained in the broad survey about geometries with torsion by Ilka Agricola.

A beautiful example of a  $G$ -structure admitting a unique connection with totally skew-symmetric torsion is provided by the class of nearly Kähler manifolds, which are almost Hermitian manifolds such that the Levi-Civita covariant derivative of the almost complex structure is completely skew-symmetric. General almost Hermitian structures admitting a unique connection with totally skew-symmetric torsion are discussed by Paul-Andi Nagy, whereas Jean-Baptiste Butruille explains the classification of homogeneous nearly Kähler manifolds.

Nearly Kähler structures with indefinite metric are considered by Lars Schäfer and Fabian Schulte-Hengesbach in Chapter 12. In particular, they prove that  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  admits a unique left-invariant nearly pseudo-Kähler structure.

As mentioned in Part A of the introduction, the Swann bundle provides a fundamental correspondence  $M \mapsto U(M)$ , which associates a hyper-Kähler cone  $U(M)$  with any quaternionic Kähler manifold  $M$ . The inverse construction, which associates (at least locally) a quaternionic Kähler manifold  $M(U)$  with any hyper-Kähler cone  $U$ , is known as the superconformal quotient in the physics literature. It relates a superconformal field theory with scalar manifold  $U$  to a Poincaré supergravity theory with target  $M(U)$ . It turns out that geometric and field theoretic constructions are often much simpler when described in terms of hyper-Kähler geometry. In Chapter 13 Andrew Swann explains how these results extend to the framework of geometries with torsion. The underlying superconformal algebra is now the one-parameter family of simple Lie superalgebras  $D(2, 1; \alpha)$ , which occurs, for instance, in the work of Michelson and Strominger on superconformal quantum mechanics.

### Part D. Para-geometries

A complex structure on a (smooth) manifold  $M$  can be defined as an endomorphism field  $J \in \Gamma(\text{End}(TM))$  such that  $J^2 = -\text{Id}$  and such that the eigendistributions  $T^{1,0}M, T^{0,1}M \subset TM \otimes \mathbb{C}$  are involutive. Similarly, a *para-complex* structure on a manifold is an endomorphism field  $J$  such that  $J^2 = \text{Id}$  with involutive eigendistributions  $T^+M, T^-M \subset TM$  of the same dimension. In virtue of the Frobenius theorem, a para-complex structure is simply a local product structure with factors of equal dimension. Nevertheless, it is helpful to make use of the analogy between complex

and para-complex manifolds. There is a useful para-holomorphic calculus in which the role of the field of complex numbers  $\mathbb{C} = \mathbb{R}[i]$ ,  $i^2 = -1$ , is played by the ring of para-complex numbers  $\mathbb{R}[e]$ ,  $e^2 = 1$ . Many interesting structures in Riemannian geometry have natural “para-analogues”. In particular, there is a notion of para-Kähler (or bi-Lagrangian), special para-Kähler, para-hyper-Kähler (or hypersymplectic) and para-quaternionic Kähler manifold. These manifolds carry pseudo-Riemannian metrics of split signature. Remarkably, these structures occur as special geometries of supersymmetric field theories, when the Lorentzian space-time metric is replaced by a positive definite metric, see the chapter by Thomas Mohaupt.

In Chapter 14, Stefan Ivanov, Ivan Minchev and Simeon Zamkovoy discuss twistor spaces of general almost para-quaternionic manifolds.

Matthias Krahe establishes a Darboux theorem for para-holomorphic symplectic and contact structures. This fundamental result can be applied, for instance, in the twistor theory of para-quaternionic Kähler manifolds. His contribution develops the twistor theory of para-pluriharmonic maps into symmetric spaces.

Dmitri V. Alekseevsky, Constantino Medori and Adriano Tomassini classify maximally homogeneous para-CR manifolds of semisimple type.

## Part E. Holonomy theory

The *holonomy group* of a pseudo-Riemannian manifold  $M$  of signature  $(p, q)$  at a point  $x \in M$  is the subgroup  $\text{Hol}_x \subset \text{O}(T_x M) \cong \text{O}(p, q)$  generated by parallel transports along loops based at  $x$ . For connected manifolds this yields a subgroup  $\text{Hol} \subset \text{O}(p, q)$  well defined up to conjugation in the pseudo-orthogonal group  $\text{O}(p, q)$ . Holonomy groups were introduced by Élie Cartan in the twenties for the study of Riemannian symmetric spaces and became a powerful tool in Riemannian geometry with Berger’s classification of holonomy groups of complete simply connected Riemannian manifolds in the fifties.

Anton Galaev and Thomas Leistner review recent developments in the holonomy theory of pseudo-Riemannian manifolds. These include their classification of Lorentzian holonomy groups and Anton Galaev’s classification of holonomy groups which are subgroups of  $\text{U}(1, n)$ . The general classification problem for holonomy groups of pseudo-Riemannian manifolds of arbitrary signature remains unsolved. It includes the classification of pseudo-Riemannian symmetric spaces of arbitrary signature, which is already too complicated a problem to expect a simple solution.

Chapter 18 by Antonio J. Di Scala, Thomas Leistner and Thomas Neukirchner contains proofs of some facts about irreducible representations of Lie groups and applications of these results in holonomy theory.

In the chapter by Konrad Waldorf, the notion of holonomy of a line bundle (endowed with a connection) around a loop is extended to the holonomy of a gerbe along a closed oriented surface, which corresponds to the interaction of a string with a three-form gauge field.

## Part F. Symmetric spaces and spaces of constant curvature

A pseudo-Riemannian manifold  $M$  is called a *symmetric space* if every point  $x \in M$  is an isolated fixed point of an involutive isometry. This includes the complete simply connected pseudo-Riemannian manifolds of constant curvature.

Ines Kath reviews the state of the art in the classification of pseudo-Riemannian symmetric spaces. Like for the classification of pseudo-Riemannian holonomy groups, little is known beyond metrics of index 2. She explains various approaches and partial results, for instance under the assumption of additional geometric structures.

Dmitry Alekseevsky discusses in Chapter 21 the classification problem for pseudo-Kähler and para-Kähler symmetric spaces. In particular, he describes some classes of Ricci-flat examples.

Oliver Baues develops the theory of flat pseudo-Riemannian manifolds in the general context of flat affine structures and prehomogeneous affine representations. Flat Riemannian manifolds are well understood by Bieberbach's theorems, but there are still many long standing open problems concerning flat pseudo-Riemannian manifolds of arbitrary signature. For instance, it is not known whether every compact flat pseudo-Riemannian manifold of signature  $(p, q)$  (with  $p \geq q \geq 2$ ) is a quotient of the pseudo-Euclidian space  $\mathbb{R}^{p,q}$  and also cocompact properly discontinuous groups of pseudo-Euclidian motions are scarcely understood.

## Part G. Conformal geometry

A *conformal structure* of signature  $(p, q)$  on a smooth manifold  $M$  is a ray subbundle  $L \subset S^2T^*M$  such that any local section of  $L$  defines a pseudo-Riemannian metric of signature  $(p, q)$ . In particular, any pseudo-Riemannian metric  $g$  on  $M$  defines a conformal structure  $L = \mathbb{R}^+g$ . Conformal geometry is concerned with properties which do not depend on the choice of a section  $g \in \Gamma(L)$ . The holonomy group of a pseudo-Riemannian manifold  $(M, g)$ , for instance, is not a conformal invariant.

Helga Baum's contribution is a survey on the holonomy theory of Cartan connections. This theory applies, in particular, to conformal geometry. As explained in her exposition, the conformal holonomy group contains important information about a pseudo-Riemannian manifold. The knowledge of the holonomy group allows one to decide, for example, whether the pseudo-Riemannian manifold admits conformal Killing spinors or an Einstein metric in the conformal class. She describes Lorentzian manifolds with conformal holonomy in  $SU(1, n)$ .

Chapter 24 by Yoshinubo Kamishima is also concerned with conformal and related geometric structures. It provides a unified treatment of conformal, CR and quaternionic CR-structures. In the positive definite case, the corresponding model spaces are the boundaries at infinity of the hyperbolic spaces over the real, complex and quaternionic numbers, respectively.

**Part H. Other topics of recent interest**

The chapter by Christian Bär summarises the analytic theory of linear wave equations on globally hyperbolic Lorentzian manifolds, as developed in his book with Nicolas Ginoux and Frank Pfäffle. In the final chapter, Dan Freed explains the relation between D-branes in string theory and K-theory.