Panoramic Overview

Compactness is a core concept in general topology, because it introduces finiteness in otherwise infinite geometric objects. When we combine compactness with group theory and its enormous background we can expect a theory rich in results, varied in direction, and fertile in applications. And we get it as is evidenced through a sizeable body of monographs and texts having come about in the second half of last century. Naturally, we like to cite our book on compact groups [102] that appeared in 1998 and that experienced a second revised and augmented edition in 2006. The standard examples are linear groups such as the orthogonal and unitary groups, or the additive groups of p-adic integers, and this confirms that the concept of a compact group is natural. The class of compact groups is closed in the class of all topological groups under the formation of arbitrary products and the passage to closed subgroups. This makes it a closed category in its own right, and that in itself is a fact from which many desirable properties of this category follow.

But there are before our eyes just as natural examples of groups that illustrate that there are many topological groups basic to analysis, geometry and algebra which are not compact; easily perceived examples are the additive groups of $\mathbb{R}^n$ or, more generally, finite-dimensional vector spaces over locally compact fields, and linear groups like the full linear groups $\mathrm{Gl}(n, \mathbb{R})$ and their closed subgroups. All of these groups, however, are locally compact. The most important locally compact groups are real Lie groups which are connected or have, at most, finitely many connected components. One definition of a Lie group is that it is a real analytic manifold with a group structure such that multiplication and inversion are analytic functions. A topological group which is isomorphic to a closed subgroup of the topological group $\mathrm{Gl}(n, \mathbb{R})$ is a Lie group, and we shall call such a Lie group a *linear Lie group*. Let us emphasize at this point that here, and in the following, when we speak about two topological groups as being isomorphic, we mean them to be isomorphic as topological groups; some writers like to stress this by saying that they are "isomorphic algebraically and topologically". We give a definition of a general Lie group in Appendix 1 to this book which allows a quicker access to the group theoretical aspects of Lie group theory than one involving analytical manifolds. In our book [102] on compact groups we devote a whole chapter to an introduction to linear Lie groups. It is shown in that book that all compact Lie groups are matrix groups, that is, linear Lie groups. All these groups possess identity neighborhoods which are homeomorphic to $\mathbb{R}^n$ for a suitable dimension n: they are locally euclidean.

In 1900 David Hilbert raised the question whether every locally euclidean group is a Lie group. It took half a century until this question was answered in the affirmative by the concerted joint efforts of Gleason [63], Montgomery and Zippin [144] published back to back in the Annals of Mathematics. The monograph [145] by Montgomery and Zippin appeared three years later and summarized the entire development including the important complements by YAMABE [206], [207] which followed one year later and to which we shall return presently. MONTGOMERY and ZIPPIN's book became a classic which has not been replaced to this day, in spite of an excellent secondary source authored by KAPLANSKY [129] sixteen years later.

Hilbert's Problems numbered 23 in all; they were formulated in order to indicate the directions which research in mathematics was to take in the 20th century. The problem concerning Lie groups is number 5, and its difficulty as well as the sheer quantity of research that it fertilized was very indicative of Hilbert's vision. So it is only natural that something even more influential came along with the affirmative solution of Hilbert's Fifth Problem, namely, YAMABE's Theorem. YAMABE's Theorem tells us that every connected locally compact group G is approximated by a connected Lie group in the sense that G contains arbitrarily small normal subgroups N such that G/N is a Lie group ([206], [207]). In fact YAMABE's Theorem applies to more than connected groups: it says that every locally compact group for which the group of connected components G/G_0 is compact is approximated by Lie groups in the sense just explained. The concept of being approximated by Lie groups is so important that it certainly deserves a definition of its own. For this purpose let us first recall that a topological group is *complete* if every Cauchy filter (or every Cauchy net) converges; this aptly generalizes the concept of completeness of a metric space which is complete if every Cauchy sequence converges. Every locally compact group is complete and so no mention of completeness need be made when one deals with locally compact groups.

Definition 1. (i) A topological group G is called a *pro-Lie group* if it is complete and if every identity neighborhood of G contains a normal subgroup N such that G/N is a Lie group. The category of all pro-Lie groups with continuous group morphisms between them is written $\mathsf{proLieGr}$. (3.39)

(ii) A topological group G is called *almost connected* if the factor group G/G_0 of G modulo the connected component G_0 of the identity is compact.

Let us then reformulate YAMABE's Theorem in this terminology:

Every almost connected locally compact group is a pro-Lie group.

It is a generally adopted notation that for a category $\mathcal{A}$ and objects A and B in $\mathcal{A}$, the set of all morphisms $A \to B$ is denoted by $\mathcal{A}(A, B)$. For instance, if TopGr denotes the category of all topological groups and continuous group homomorphisms between them, then $\mathsf{TopGr}(G, H)$ denotes the set of all continuous homomorphisms from the topological group G to the topological group H; if G and H are pro-Lie groups, then we have $\mathsf{proLieGr}(G, H) = \mathsf{TopGr}(G, H)$ by definition. This means that the category $\mathsf{proLieGr}$ is a *full subcategory* of the category TopGr of all topological groups.

Algebraists, in particular ring theorists, are rather familiar with a concept similar to that of pro-Lie groups, namely, profinite groups. A group G is *profinite* if it is a complete topological group such that every identity neighborhood of G contains a normal open subgroup N such that G/N is finite. Profinite groups are compact, and they are pro-Lie groups. Profinite groups generalize finite groups in the exact same way as pro-Lie groups generalize Lie groups.

Only three years before the solution of Hilbert's Fifth Problem was found by GLEASON, MONTGOMERY and ZIPPIN, a seminal paper by IWASAWA had appeared in the Annals of Mathematics [120]. In that paper he exposed fundamental properties of locally compact pro-Lie groups. So YAMABE's result made all of this available for the study of the structure and the representation theory of almost connected locally compact groups. This was the culmination of half a century of research on topological groups following HILBERT's vision in 1900. But at the same time, and certainly not less significant from the present vantage point, the work by IWASAWA, GLEASON, MONTGOMERY, ZIPPIN and YAMABE provided motivation and incentive for another half a century's worth of research on locally compact groups during the second half of the twentieth century. What went into this entire century of research naturally was the full body of highly developed structure and representation theory of finite-dimensional Lie groups and finite-dimensional Lie algebras.

Let us briefly say what we mean by the Lie theory of a topological group on a very general level; after all, the words *Lie theory* appear in the title of this book. To each topological group G one can easily associate a topological space $\mathfrak{L}(G)$, namely, the space $\mathrm{Hom}(\mathbb{R}, G)$ of all continuous group homomorphisms from the additive topological group $\mathbb{R}$ of real numbers to the topological group G, endowed with the topology of uniform convergence on compact sets. We also have a continuous function $\exp\colon \mathfrak{L}(G) \to G$ given by $\exp X = X(1)$ and a "scalar multiplication" $(r, X) \mapsto r \cdot X : \mathbb{R} \times \mathfrak{L}(G) \to \mathfrak{L}(G)$ given by $(r \cdot X)(s) = X(sr)$. Whether these concepts are useful depends in large measure on the degree to which additional properties are satisfied. In Chapter 2 we shall elaborate on the following definition.

Definition 2 (2.6ff.). A topological group G is said to *have a Lie algebra*, if $\mathfrak{L}(G)$ has a continuous addition and bracket multiplication making it into a topological Lie algebra in such a fashion that

$$(X + Y)(r) = \lim_{n\to\infty} \left(X\left(\frac{r}{n}\right) Y\left(\frac{r}{n}\right) \right)^n$$

and

$$[X, Y](r^2) = \lim_{n\to\infty} \left(X\left(\frac{r}{n}\right) Y\left(\frac{r}{n}\right) X\left(\frac{r}{n}\right)^{-1} Y\left(\frac{r}{n}\right)^{-1} \right)^{n^2}.$$

If G has a Lie algebra, then $\mathfrak{L}(G)$ is called *the Lie algebra of* G and $\exp\colon \mathfrak{L}(G) \to G$ is called its *exponential function*.

Clearly a topological group G has a Lie algebra if and only if the connected component G_0 of the identity has a Lie algebra G_0 and

$$\mathfrak{L}(G) = \mathrm{Hom}(\mathbb{R}, G) = \mathrm{Hom}(\mathbb{R}, G_0) = \mathfrak{L}(G_0).$$

The image of the exponential function is contained in G_0. If we believe that $\mathfrak{L}(G)$ and the exponential function encapsulate the Lie theory of G, then it is true that the identity component G_0 already captures the Lie theory of G.

We show in this book that

every pro-Lie group G has a Lie algebra $\mathfrak{L}(G)$ and the image $\exp\mathfrak{L}(G)$ *of the exponential function algebraically generates a subgroup which is dense in the connected component G_0 of the identity.*

We shall have much more to say about the topological Lie algebra $\mathfrak{L}(G)$ that can arise in this fashion. But for the moment we observe this: In every totally disconnected locally compact group, the open (hence closed) subgroups form a basis of the neighborhood filter of the identity element. If G is a locally compact group, then G/G_0 is a locally compact totally disconnected group, and so there is an open subgroup U of G containing G_0 such that U/G_0 is compact. Then U is almost connected and thus, by YAMABE's Theorem, is a pro-Lie group. Therefore every locally compact group has an open subgroup which is a pro-Lie group and which captures the Lie theory of G. Apart from individual studies such as [134], [64], [103], [104], [106], the Lie theory of locally compact groups has never been *systematically* considered or exploited, although a start was made in [102] for the purpose of a structure theory of *compact* groups. One of the thrusts of this book is to change this situation with determination.

In addition to the successful resolution of Hilbert's Fifth Problem there is yet a second prime reason for the success of the structure and representation theory of locally compact groups: The 1932 proof by A. HAAR of the existence and uniqueness of left invariant integration on a locally compact group G. Its full power for abstract harmonic analysis was recognized by A. WEIL in his influential monograph [198] of 1941.

Haar measure is the key to the representation theory of compact and locally compact groups on Hilbert space, and the wide field of abstract harmonic analysis with ever so many ramifications (including e.g. abstract probability theory on locally compact groups). A theorem due to A. WEIL shows that, conversely, a complete topological group with a left- (or right-) invariant σ-finite measure is locally compact (see e.g. [76], [198]). Thus the category of locally compact groups is that which is exactly suited for real analysis resting on the existence of an invariant integral based on σ-additive measures. One cannot expect to extend this aspect of locally compact groups to larger classes without abandoning σ-additivity. (BOURBAKI indicates in Chapter 9 of his "Intégration" [23], pp. 50–55, 70ff., how such an extension may be handled; however we shall not consider this aspect in this book.)

In quiet moments of introspection one might even admire the small miracle inherent in the fact that measure theory carries as far as locally compact groups go. The proper domain for an invariant measure theory again appears to be the category of compact groups, where one has a unique invariant two sided invariant measure P with respect to which G is measurable with measure $P(G) = 1$. That is, P is a veritable probability measure that allows averaging over G as a remarkably simple but effective device ([102]). Yet there it is, Haar measure of locally compact groups, infinite but eminently useful making locally compact groups the analysts' delight.

However, from each of a group theoretical, of a Lie theoretical, and of a category theoretical point of view, the class of a locally compact groups has serious defects which go rather deep.

Indeed, if we consider a family of Lie groups G_j, $j \in J$ for an index set J, then its product $\prod_{j\in J} G_j$ is a perfectly good Hausdorff topological group with a lucid structure, but it fails to be locally compact whenever infinitely many of the G_j fail to be compact.

Furthermore, while every locally compact group G does have a Lie algebra $\mathfrak{L}(G)$, the additive group of the Lie algebra is never locally compact unless it is finite-dimensional. Indeed even the additive topological group of the Lie algebra of a compact abelian group need not be locally compact; for example the product $G \stackrel{\text{def}}{=} \mathbb{T}^J$ of circle groups $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ has a Lie algebra $\mathfrak{L}(G)$ isomorphic to $\mathbb{R}^J$ and thus fails to be locally compact as soon as J is infinite, while the group $\mathbb{T}^J$ is comfortably within the realm of compact groups.

Each Lie group G has a tangent bundle which is again a Lie group, namely, the semidirect product $\mathfrak{L}(G) \rtimes_{\mathrm{Ad}} G$ with G acting on its Lie algebra by adjoint action induced by inner automorphisms. Does a locally compact group have a tangent bundle? The answer is yes, it does, in fact every pro-Lie group has one (as we shall show in this book), but it is almost never a locally compact group except when the group itself is finite-dimensional.

Thus the category of locally compact groups appears to have two major drawbacks:

– The topological abelian group underlying the Lie algebra $\mathfrak{L}(G)$ and the tangent bundle of a locally compact group fail to be locally compact unless $\mathfrak{L}(G)$ is finite-dimensional. In other words, the very Lie theory that makes the structure theory of locally compact groups interesting leads us outside the class.

– The category of locally compact groups is not closed under the forming of products, even of copies of $\mathbb{R}$; it is not closed under projective limits of projective systems of finite-dimensional Lie groups, let alone under arbitrary limits. In other words, the category of locally compact groups is badly incomplete.

This book presents an argument for a shift in the vantage point of looking at locally compact groups. We plead for a structure theory of topological groups that places the focus squarely and systematically on pro-Lie groups.

Recall that we denote the category of all (Hausdorff) topological groups and continuous group homomorphisms by TopGr. It turns out that the full subcategory $\mathsf{proLieGr}$ of TopGr consisting of *all* projective limits of finite-dimensional Lie groups avoids both of these difficulties. This would perhaps not yet be a sufficient reason for advocating this category if it were not for two facts:

– Firstly, while not every locally compact group is a projective limit of Lie groups, every locally compact group has an open subgroup which is a projective limit of Lie groups, so that, in particular, every connected locally compact group is a pro-Lie group; also all compact groups and all locally compact abelian groups are pro-Lie groups.

– Secondly, the category $\mathsf{proLieGr}$ is astonishingly well-behaved. Not only is it a complete category, it is closed under passing to closed subgroups and to those quotients which are complete, and it has a demonstrably good Lie theory.

It is therefore indeed surprising that this class of groups has been little investigated in a systematic fashion.

A serious attempt at such an investigation is made in this book where it is submitted that not only a general structure theory of locally compact groups can be based on a good understanding of the category proLieGr of pro-Lie groups, but that the category of pro-Lie groups is well worth a thorough study on its own account. In this book we will prove general structure theorems on pro-Lie groups which will include the best known general structure theorems on locally compact groups. Since the main strategy of the book is to provide a structure theory via Lie theory, en route we shall have to develop a full grown structure theory of those topological Lie algebras which occur as Lie algebras of pro-Lie groups. We shall call these *pro-Lie algebras*, because each of them is a complete topological Lie algebra such that every 0-neighborhood contains a closed ideal modulo which it is finite-dimensional (3.6).

Part 1. The Base Theory of Pro-Lie Groups

For a description of some basic results on the theory of projective limits of Lie groups some technical background information appears inevitable even for an overview, long before we delve into the actual study of our topic.

Core Definitions and Facts on Pro-Lie Groups

Definition 3. A *projective system D* of topological groups is a family of topological groups $(C_j)_{j\in J}$ indexed by a directed set J and a family of morphisms $\{f_{jk}\colon C_k \to C_j \mid (j,k) \in J \times J,\ j \leq k\}$, such that f_{jj} is always the identity morphism and $i \leq j \leq k$ in J implies $f_{ik} = f_{ij} \circ f_{jk}$. Then the *projective limit of the system* $\lim_{j\in J} C_j$ is the subgroup of $\prod_{j\in J} C_j$ consisting of all J-tuples $(x_j)_{j\in J}$ for which the equation $x_j = f_{jk}(x_k)$ holds for all $j, k \in J$ such that $j \leq k$.

Every cartesian product of topological groups may be considered as a projective limit. Indeed, if $(G_\alpha)_{\alpha\in A}$ is an arbitrary family of topological groups indexed by an infinite set A, one obtains a projective system by considering J to be the set of finite subsets of A directed by inclusion, by setting $C_j = \prod_{a\in j} G_a$ for $j \in J$, and by letting $f_{jk}\colon C_k \to C_j$ for $j \leq k$ in J be the projection onto the partial product. The projective limit of this system is isomorphic to $\prod_{a\in A} G_a$.

There are a few sample facts one should recall about the basic properties of projective limits (see e.g. [25], [64], [107], or this book 1.27 and 1.33):

Let $G = \lim_{j\in J} G_j$ be a projective limit of a projective system

$$\mathcal{P} = \{f_{jk}\colon G_k \to G_j \mid (j,k) \in J \times J,\ j \leq k\}$$

of topological groups with limit morphisms $f_j\colon G \to G_j$, and let $\mathcal{U}_j$ denote the filter of identity neighborhoods of G_j, $\mathcal{U}$ the filter of identity neighborhoods of G, and $\mathcal{N}$ the set $\{\ker f_j \mid j \in J\}$. Then $\mathcal{U}$ has a basis of identity neighborhoods $\{f_k^{-1}(U) \mid k \in J, U \in \mathcal{U}_k\}$ and $\mathcal{N}$ is a filter basis of closed normal subgroups converging to 1. If all bonding maps $f_{jk}\colon G_k \to G_j$ are quotient morphisms and all limit maps f_j are surjective, then the limit maps $f_j\colon G \to G_j$ are quotient morphisms. The limit G is complete if all G_j are complete.

Definition 4 (3.25). For a topological group G let $\mathcal{N}(G)$ denote the set of closed normal subgroups N such that all quotient groups G/N are finite-dimensional real Lie groups. Then $G \in \mathcal{N}(G)$ and G is said to be a *proto-Lie group* if every identity neighborhood contains a member of $\mathcal{N}(G)$.

By our earlier Definition 1, if in addition, G is a complete topological group, then G is a pro-Lie group.

While not every topological group can be embedded as a subgroup into a complete topological group, this is the case for proto-Lie groups, indeed

every proto-Lie group has a completion which is a pro-Lie group. (See 4.1.)

Every product of a family of finite-dimensional Lie groups $\prod_{j\in J} G_j$ is a pro-Lie group. In particular, $\mathbb{R}^J$ is a pro-Lie group for any set J which is locally compact if and only if the set J is finite. The product $\mathbb{Z}^{\mathbb{N}}$, accordingly, is a pro-Lie group. It is well known that the space $\mathbb{Z}^{\mathbb{N}}$ is homeomorphic to the space of irrational real numbers in the natural topology. We may formulate this by saying that

the space of irrational numbers supports the structure of a pro-Lie group.

It is a remarkable fact (which we discuss in Chapter 4) that the free abelian group $\mathbb{Z}^{(\mathbb{N})}$ in countably many generators carries the structure of a nondiscrete pro-Lie group. The underlying topological space cannot be a Baire space and so certainly cannot be Polish (second countable completely metrizable), nor locally compact; indeed a countable homogeneous Baire space is necessarily discrete.

These examples help us to realize from the beginning, that our general intuition of the topology of pro-Lie groups cannot be based on experience gathered from locally compact groups.

If $\{G_j : j \in J\}$ is a family of finite-dimensional real Lie groups then the subgroup

$$\left\{(g_j)_{j\in J} \in \prod_{j\in J} G_j : \{j \in J : g_j \neq 1\} \text{ is finite}\right\}$$

of the direct product $\prod_{j\in J} G_j$ is a proto-Lie group which is not a pro-Lie group if J is infinite and the G_j are nonsingleton.

We reiterate that a topological group G is called *almost connected* if the factor group G/G_0 modulo the connected component G_0 of the identity is compact. Everything that is proved for almost connected topological groups therefore is true for all connected groups and for all compact groups. One of the very weighty reasons why this concept is relevant for the theory of topological groups is the existence of YAMABE's crucial result:

Every almost connected locally compact group is a pro-Lie group.

The group $\mathrm{PSl}(2, \mathbb{Q}_p)$ of projective transformations of the p-adic projective line is locally compact, but has no nontrivial normal subgroups and is therefore a locally compact group which is not a pro-Lie group in our sense, while it is, of course, a p-adic Lie group.

Every pro-Lie group G gives rise to a projective system

$$\{p_{NM} \colon G/M \to G/N : M \supseteq N \text{ in } \mathcal{N}(G)\}$$

whose projective limit it is (up to isomorphism). The converse is a difficult issue, but it is true.

Theorem 5 (3.34, 3.35 (The Closed Subgroup Theorem)). *Every projective limit of pro-Lie groups is a pro-Lie group. Every closed subgroup of a pro-Lie group is a pro-Lie group. A topological group is a pro-Lie group if and only if it is isomorphic to a closed subgroup of a product of Lie groups.*

In fact in simple category theoretical parlance the following theorem holds.

Theorem 6 (3.3, 3.36). *The category* proLieGr *of pro-Lie groups is closed in* TopGr *under the formation of all limits and is therefore complete. It is the smallest full subcategory of* TopGr *that contains all finite-dimensional Lie groups and is closed under the formation of all limits.*

This shows that the category proLieGr does not have some of the shortcomings of the category of locally compact groups which is obviously incomplete. It remains yet to be seen how good the Lie theory of the category proLieGr is and we shall say good things about it shortly.

However, one must, at an early stage, admit that the category of pro-Lie groups has certain problems which are invisible as long as one stays inside the subcategory of locally compact pro-Lie groups. Indeed, every quotient group of a locally compact group is locally compact (which is a consequence of the fairly elementary observation that in any topological group, the product HK of a closed subset H and a compact subset K is closed, and the application of this fact to the case that H is a closed (normal) subgroup and K a compact identity neighborhood of G). It is one of the less elementary facts of Lie group theory that a quotient of a Lie group is a Lie group. Indeed the quotient of a linear Lie group need not be a linear Lie group, but is a Lie group nevertheless. The simplest example is the group of upper triangular matrices

$$G \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

and the discrete central subgroup

$$Z \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\};$$

here G is clearly a linear Lie group but the factor group G/Z, which is even locally isomorphic to G is not a linear Lie group. This was proved by GARRET BIRKHOFF in 1936 [9] by astute but elementary linear algebra. (See also [102], p. 169ff.) It is a much more debilitating fact for the study of pro-Lie groups that

quotient groups of pro-Lie groups need not be pro-Lie groups. (Corollary 4.11)

Still,

every quotient group of a pro-Lie group is a proto-Lie group and has a completion which is a pro-Lie group. (4.1)

So the defect here arises from a phenomenon that is well observed and studied, that quotients of complete topological groups may fail to be complete. (See [176].) We shall explain in Chapter 4 that the additive group of the topological vector space $\mathbb{R}^{[0,1]}$ has a nondiscrete closed subgroup K algebraically isomorphic to the free abelian group $\mathbb{Z}^{(\mathbb{N})}$ in countably many generators such that $\mathbb{R}^{[0,1]}/K$ is an abelian proto-Lie group which is dense in a compact connected and locally connected group (Corollary 4.11). We use this example in various places in the book to construct counterexamples. In this sense, this example is very helpful to build up our intuition on certain aspects of pro-Lie group theory that are invisible as long as we stay in the locally compact domain. Curiously, the counterexample itself arises from the theory of compact abelian groups, and it was discovered not so long ago ([106]).

The defect of proLieGr of not being closed under the passing to quotients is, as we have said, debilitating, because passing to quotient groups is an extremely helpful device of reduction to simple situations in many proofs; therefore it is a handicap not having this tool available at all times inside proLieGr.

Fortunately, we shall see that, even regarding quotients, the category proLieGr has its redeeming features.

Theorem 7 (The Quotient Theorem; 4.28). *Let G be an almost connected pro-Lie group and N a closed normal subgroup. Then G/N is a pro-Lie group provided at least one of the following conditions are satisfied by N:*

(i) *N is almost connected.*
(ii) *N is the kernel of a morphism from G onto some pro-Lie group.*
(iii) *N is locally compact or Polish.*

Part (iii) of this theorem arises from general topological group theory, and we refer to sources like the book [176] of DIEROLF and ROELKE for such pieces of information. Parts (i) and (ii) belong to the proper substance of this book, and neither of the two is a trivial matter (See Theorem 4.28 and Corollary 9.58.) In fact, Part (ii) is a consequence of another core result concerning pro-Lie groups, namely, the Open Mapping Theorem that is well known to functional analysts as applying to a variety of operators between suitable topological vector spaces, and that is equally well known to people working with locally compact or Polish topological groups. If conditions are right, then the surjectivity of a continuous group homomorphism $f: G \to H$ from a topological group onto another implies already that f is an open function, or, in equivalent terms that the canonical decomposition

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle\text{quot}}\downarrow & & \uparrow{\scriptstyle\text{id}_H} \\ G/\ker f & \xrightarrow[F]{} & H \end{array}$$

produces an isomorphism of topological groups $F\colon G/\ker f \to H$. If we let G be the additive group of real numbers $\mathbb{R}_d$ with the discrete topology, H the same group $\mathbb{R}$ but considered with its natural topology, then the identity function $f\colon G \to H$ is a bijective morphism between locally compact metric groups that is not open. We mentioned earlier that we shall expose a nondiscrete pro-Lie structure on the countable free abelian group H with infinitely many generators. So the identity morphism f from the discrete countable (hence locally compact Polish) group $G = \mathbb{Z}^{(\mathbb{N})}$ to H is a continuous morphism between pro-Lie groups which is not open. These examples show that the following theorem is not likely to be either obvious or trivial:

Theorem 8 (Open Mapping Theorem for Pro-Lie Groups; 9.60). *Let G be an almost connected pro-Lie group and $f\colon G \to H$ a continuous group homomorphism onto a pro-Lie group. Then f is an open mapping.*

That is, under these circumstances, f is equivalent to a quotient homomorphism.

One of the major impediments in the group theory of topological groups is the unavailability of the Second Isomorphism Theorem. The so called First Isomorphism Theorem says that if G is a topological group and $M \subseteq N$ are normal subgroups of G then the morphism $gM \mapsto gN : G/M \to G/N$ factors through an isomorphism of topological groups $(G/M)/(N/M) \to G/N$. This is a very robust theorem belonging to universal algebra. The environment of the so-called Second Isomorphism Theorem is as follows: Assume that G is a topological group with a closed normal subgroup N and a closed subgroup H such that $G = HN = NH$. Then the surjective morphism $h \mapsto hN\colon H \to G/N$ factors through a bijective continuous group homomorphism $H/(H \cap N) \to HN/N$. This may fail to be open even if $H \cap N = \{1\}$. In [108] there is an example of a topological abelian group G and two (isomorphic) closed subgroups H and N such that G is algebraically the direct sum of H and N and G/H and G/N are (isomorphic) compact groups, while G blatantly fails to be compact. However, if G is a pro-Lie group, then a closed subgroup H is a pro-Lie group by the Closed Subgroup Theorem. If N is an almost connected closed normal subgroup of G and G is almost connected, then G/N is a pro-Lie group by the Quotient Theorem. Therefore, from the Open Mapping Theorem we get the next theorem.

Theorem 9 (The Second Isomorphism Theorem for Pro-Lie Groups; 9.62). *Let N be an almost connected normal subgroup and H an almost connected subgroup of a topological group G and assume that H, N, and HN are pro-Lie groups. Then $N/(H \cap N)$ and HN/N are naturally isomorphic.*

The Coarse Lie Theory of Pro-Lie Groups

Let us consider a topological Lie algebra $\mathfrak{g}$ and on it the filter basis of closed ideals $\mathfrak{j}$ such that $\dim \mathfrak{g}/\mathfrak{j} < \infty$; we shall denote it by $\mathcal{I}(\mathfrak{g})$.

Definition 10 (3.6). A topological Lie algebra $\mathfrak{g}$ is called a *pro-Lie algebra* (short for *profinite-dimensional Lie algebra*) if $\mathcal{I}(\mathfrak{g})$ converges to 0 and if $\mathfrak{g}$ is a complete topological vector space.

Under these circumstances, $\mathfrak{g} \cong \lim_{\mathfrak{j}\in\mathcal{I}(\mathfrak{g})} \mathfrak{g}/\mathfrak{j}$, and the underlying vector space is a weakly complete topological vector space, that is, it is the algebraic dual of a real vector space with the weak $*$-topology. We give a systematic treatment of the duality of vector spaces and weakly complete topological vector spaces in Appendix 2 of this book. The category of pro-Lie algebras and continuous Lie algebra morphisms is denoted proLieAlg.

Proposition 11 (3.3, 3.36). *The category* proLieAlg *of pro-Lie algebras is closed in the category of topological Lie algebras and continuous Lie algebra morphisms under the formation of all limits and is therefore complete. It is the smallest category that contains all finite-dimensional Lie algebras and is closed under the formation of all limits.*

See also [104].

Our demonstration that Lie theory is applicable to pro-Lie groups begins with our showing results like the following:

Theorem 12 (3.12, 2.25). *Every pro-Lie group G has a pro-Lie algebra $\mathfrak{g}$ as Lie-algebra, and the assignment $\mathfrak{L}$ which associates with a pro-Lie group G its pro-Lie algebra is a limit preserving functor.*

These matters will be shown in Chapters 2 and 3. In fact, a portion of this set-up allows a considerable improvement which we summarize in the next section.

The Category Theoretical Version of Lie's Third Theorem

Theorem 13 (Lie's Third Theorem for Pro-Lie groups; 6.5, 6.6, 8.15). *The Lie algebra functor $\mathfrak{L}$:* proLieGr $\to$ proLieAlg *has a left adjoint Γ. It associates with every pro-Lie algebra $\mathfrak{g}$ a unique simply connected pro-Lie group $\Gamma(\mathfrak{g})$ and a natural isomorphism $\eta_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{L}(\Gamma(\mathfrak{g}))$ such that for every morphism $\varphi\colon \mathfrak{g} \to \mathfrak{L}(G)$ there is a unique morphism $\varphi'\colon \Gamma(\mathfrak{g}) \to G$ such that $\varphi = \mathfrak{L}(\varphi') \circ \eta_{\mathfrak{g}}$.*

A good portion of this theorem we shall prove in Chapter 6, but we find it necessary to introduce a preliminary concept of simple connectivity. Indeed we shall call a pro-Lie group *prosimply connected* if every member of $\mathcal{N}(G)$ contains a member N of $\mathcal{N}(G)$ such that G/N is a simply connected Lie group. This turns out to be, for a while, a very useful concept of simple connectivity for pro-Lie groups in all respects, and it reduces correctly to simple connectivity in the case of finite-dimensional Lie groups. Once we have developed enough structure theory we will be able in Chapter 8 to show that a pro-Lie group is prosimply connected if and only if it is simply connected. (See Theorem 8.15.)

Indeed, for each pro-Lie algebra $\mathfrak{g}$, the group $\Gamma(\mathfrak{g})$ is a projective limit of a projective system of simply connected Lie groups. The fact that $\mathfrak{L}$ is a right adjoint confirms its property of preserving all limits.

There is more to Theorem 13 than meets the eye, and we should alert the reader to these circumstances because they shed new light on the situation even when everything is restricted to the classical situation of finite-dimensional Lie groups. The adjointness of the two functors $\mathfrak{L}$ and Γ may be expressed in terms of universal properties as follows.

There is a natural isomorphism $\eta_{\mathfrak{g}}\colon \mathfrak{g} \to \mathfrak{L}(\Gamma(\mathfrak{g}))$ such that for any morphism $f\colon \mathfrak{g} \to \mathfrak{L}(H)$ of topological Lie algebras there is a unique morphism $f'\colon \Gamma(\mathfrak{g}) \to H$ such that $f = \mathfrak{L}(f') \circ \eta_{\mathfrak{g}}$. In diagrams:

$$
\begin{array}{ccc|c}
\multicolumn{3}{c}{\textsf{proLieAlg}} & \textsf{proLieGr} \\
\hline
\mathfrak{g} & \xrightarrow{\eta_{\mathfrak{g}}} & \mathfrak{L}(\Gamma(\mathfrak{g})) & \Gamma(\mathfrak{g}) \\
{\scriptstyle \forall f}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{L}(f')} & \big\downarrow{\scriptstyle \exists! f'} \\
\mathfrak{L}(H) & \xrightarrow[\mathrm{id}_{\mathfrak{L}(H)}]{} & \mathfrak{L}(H) & H.
\end{array}
$$

In fact, the natural isomorphism really allows us to identify $\mathfrak{g}$ with the Lie algebra of $\Gamma(\mathfrak{g})$. Sophus Lie's Third Fundamental Theorem (in his own terminology) says that for every finite-dimensional Lie algebra there is a Lie group having as Lie algebra the given one. So this theorem persists for pro-Lie groups.

The natural morphism η is what category theoreticians call the *front adjunction* or the *unit* of the adjunction. But any adjoint situation between two functors also has a *back adjunction* or *counit* with an appropriate version of the universal property. In the case of the present adjoint situation between $\mathfrak{L}$ and Γ, the back adjunction set-up is as follows.

There is a natural morphism $\pi_G\colon \Gamma(\mathfrak{L}(G)) \to G$ of pro-Lie groups with the following universal property: Given a pro-Lie group G and any morphism $f\colon \Gamma(\mathfrak{h}) \to G$ for some pro-Lie algebra $\mathfrak{h}$, there is a unique morphism $f'\colon \mathfrak{h} \to \mathfrak{L}(G)$ of pro-Lie algebras such that $f = \pi_G \circ \Gamma(f')$.

$$
\begin{array}{c|ccc}
\textsf{proLieAlg} & \multicolumn{3}{c}{\textsf{proLieGr}} \\
\hline
\mathfrak{L}(G) & \Gamma(\mathfrak{L}(G)) & \xrightarrow{\pi_G} & G \\
{\scriptstyle \exists! f'}\big\uparrow & \big\uparrow{\scriptstyle \Gamma(f')} & & \big\uparrow{\scriptstyle \forall f} \\
\mathfrak{h} & \Gamma(\mathfrak{h}) & \xrightarrow[\mathrm{id}_{\Gamma(\mathfrak{h})}]{} & \Gamma(\mathfrak{h})
\end{array}
$$

We shall abbreviate $\Gamma(\mathfrak{L}(G))$ by $\widetilde{G}$, and call the morphism $\pi_G\colon \widetilde{G} \to G$ the *universal morphism* of G. If G happens to be a pro-Lie group which has a universal covering group in the topological sense (in particular, if G is a finite dimensional Lie group), then $\pi_G\colon \widetilde{G} \to G$ is the universal covering morphism (8.21). In general the universal morphism is neither surjective nor a local isomorphism. This is best realized at an early stage by considering any connected compact abelian group G together with its exponential function $\exp_G\colon \mathfrak{L}(G) \to G$, $\mathfrak{L}(G) = \mathrm{Hom}(\mathbb{R}, G) \cong \mathrm{Hom}(\widehat{G}, R)$, where $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the discrete character group of G. These things are explained in great detail in [102], Chapters 7 and 8. In this case $\widetilde{G}$ may be equated with the additive group of $\mathfrak{L}(G)$ and π_G with $\exp_G\colon \mathfrak{L}(G) \to G$. The world of compact abelian groups of course is full of examples for which the exponential function fails to be surjective, beginning with the one-dimensional examples that are different from the circle group, that is, the solenoids, the character groups of which are the noncyclic infinite subgroups of $\mathbb{Q}$.

Let us consider within the complete category proLieGr the full subcategory proSimpConLieGr of all *simply connected pro-Lie groups*. Then we have the following corollary.

Corollary 14 (6.6(vi)). *The restrictions and corestrictions of the functors $\mathfrak{L}$ and Γ implement an equivalence of categories*

$$\mathsf{proLieAlg} \quad \underset{\Gamma}{\overset{\mathfrak{L}}{\rightleftarrows}} \quad \mathsf{proSimpConLieGr}.$$

Therefore, the category of pro-Lie algebras has a faithful copy inside the category of all pro-Lie groups, namely, the full subcategory of all simply connected pro-Lie groups. In this light, the universal morphism $\pi_G\colon \widetilde{G} \to G$ is a group theoretical substitute for the exponential function $\exp_G\colon \mathfrak{g} \to G$; indeed for abelian pro-Lie groups the two functions agree for all practical intents and purposes.

These matters are discussed in Chapter 6 but thereafter will pervade the whole book.

Considering the problems we have encountered with quotients in the category of pro-Lie groups, it is remarkable that the functor $\mathfrak{L}$ behaves well with regard to quotient morphisms. Indeed we see next that $\mathfrak{L}$ not only preserves all limits, but some colimits as well.

Conservation Laws for $\mathfrak{L}$ and Γ

Theorem 15 (4.20). *The functor $\mathfrak{L}$ preserves quotients. Specifically, assume that G is a pro-Lie group and N a closed normal subgroup and denote by $q\colon G \to G/N$ the quotient morphism. Then G/N is a proto-Lie group whose Lie algebra $\mathfrak{L}(G/N)$ is a pro-Lie algebra and the induced morphism of pro-Lie algebras $\mathfrak{L}(q)\colon \mathfrak{L}(G) \to \mathfrak{L}(G/N)$ is a quotient morphism. The exact sequence*

$$0 \to \mathfrak{L}(N) \to \mathfrak{L}(G) \to \mathfrak{L}(G/N) \to 0$$

induces an isomorphism $X + \mathfrak{L}(N) \mapsto \mathfrak{L}(f)(X) : \mathfrak{L}(G)/\mathfrak{L}(N) \to \mathfrak{L}(G/N)$.

The core of Theorem 15 is proved by showing that for every quotient morphism $f: G \to H$ of topological groups, where G is a pro-Lie group, every one parameter subgroup $Y: \mathbb{R} \to H$ lifts to one of G, that is, there is a one parameter subgroup σ of G such that $Y = f \circ \sigma$. (See 4.19, 4.20.) This requires the Axiom of Choice. It should be emphasized that, according to Theorem 15, a quotient group of a pro-Lie group always has a complete Lie algebra even if it is itself incomplete. Therefore, a proto-Lie group with an incomplete Lie algebra such as $\mathbb{R}^{(\mathbb{N})}$ cannot be a quotient of a pro-Lie group.

Corollary 16 (4.21). *Let G be a pro-Lie group. Then $\{\mathfrak{L}(N) \mid N \in \mathcal{N}(G)\}$ converges to zero and every closed ideal $\mathfrak{i}$ of $\mathfrak{L}(G)$ such that $\mathfrak{L}(G)/\mathfrak{i}$ is finite-dimensional contains an $\mathfrak{L}(N)$ for some $N \in \mathcal{N}(G)$.*

Furthermore, $\mathfrak{L}(G)$ is the projective limit $\lim_{N\in\mathcal{N}(G)} \mathfrak{L}(G)/\mathfrak{L}(N)$ of a projective system of bonding morphisms and limit maps all of which are quotient morphisms, and there is a commutative diagram

$$\begin{array}{ccccc}
\mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(\gamma_G)} & \mathfrak{L}(\lim_{N\in\mathcal{N}(G)} \frac{G}{N}) & \cong & \lim_{n\in\mathcal{N}(G)} \frac{\mathfrak{L}(G)}{\mathfrak{L}(N)} \\
{\scriptstyle \exp_G}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{L}(\lim_{N\in\mathcal{N}(G)} \exp_{G/N})} & & \\
G & \xrightarrow[\gamma_G]{} & \lim_{N\in\mathcal{N}(G)} G/N. & &
\end{array}$$

Theorem 15 expresses a version of exactness of $\mathfrak{L}$. But there is also an exactness theorem for Γ, the left adjoint of $\mathfrak{L}$.

Theorem 17 (6.7, 6.8, 6.9). *If $\mathfrak{h}$ is a closed ideal of a pro-Lie algebra $\mathfrak{g}$, then the exact sequence*

$$0 \to \mathfrak{h} \xrightarrow{i} \mathfrak{g} \xrightarrow{q} \mathfrak{g}/\mathfrak{h} \to 0$$

induces an exact sequence

$$1 \to \Gamma(\mathfrak{h}) \xrightarrow{\Gamma(j)} \Gamma(\mathfrak{g}) \xrightarrow{\Gamma(q)} \Gamma(\mathfrak{g}/\mathfrak{h}) \to 1,$$

in which $\Gamma(j)$ is an algebraic and topological embedding and $\Gamma(q)$ is a quotient morphism.

There are some other noteworthy consequences of Theorem 15.

Proposition 18 (4.22 (iv)). *Any quotient morphism $f: G \to H$ of pro-Lie groups onto a finite-dimensional Lie group is a locally trivial fibration.*

Proposition 19 (4.22 (i)). *For a pro-Lie group G, the subgroup $\langle \exp \mathfrak{g} \rangle$ generated by the image of the exponential function, is dense in G_0, that is, $\overline{\langle \exp \mathfrak{g} \rangle} = G_0$. In particular, a connected nonsingleton pro-Lie group has nontrivial one parameter subgroups.*

This may be viewed as an Existence Theorem for one parameter subgroups in pro-Lie groups, indeed of an abundance of them – unless of course, the group in question is totally disconnected. So, as an illuminating consequence we get the following characterisation of a pro-Lie group to be totally disconnected.

Corollary 20 (4.23). *For a pro-Lie group G the following statements are equivalent:*

(a) *G is totally disconnected.*
(b) $\mathfrak{L}(G) = \{0\}$.
(c) *The filter basis of open normal subgroups of G converges to* 1.

In this book we shall call topological groups satisfying these equivalent conditions *prodiscrete* groups. So every prodiscrete group is, in particular, a pro-Lie group. As we already mentioned and will observe again later, there are locally compact totally disconnected groups which are not prodiscrete. The group $\mathbb{Z}^{\mathbb{N}}$ in the product topology is prodiscrete but not locally compact. It is, as we remarked earlier, homeomorphic to the space of irrational numbers.

Semidirect products of two topological groups (and semidirect sums of topological Lie algebras) permeate the whole book, beginning from Chapter 1 where we remind the reader of its definition in Exercise E1.5 through Chapter 11, that is specifically devoted to the splitting of pro-Lie groups, that is to results that assert that, under suitable circumstances, a given pro-Lie group may be represented as a semidirect product. If $\pi : H \to \mathrm{Aut}(N)$ is a representation of a topological group in the group of automorphisms of a topological group N such that the function $(h, n) \mapsto h \cdot n \stackrel{\text{def}}{=} \pi(n)(h) : H \times N \to N$ is continuous, then the *semidirect product* $N \rtimes_\pi H$ of N by H is the topological product $N \times H$ endowed with the multiplication $(m, h)(n, k) = (m(h \cdot n), hk)$. That $N \rtimes_\pi H$ is a topological group is straightforwardly verified. Very simple examples show that semidirect products of pro-Lie groups need not be pro-Lie groups (see Examples 4.29). We shall demonstrate in this book that every pro-Lie group acts under what will be called the *adjoint action* or *adjoint representation* $\mathrm{Ad}\colon G \to \mathrm{Gl}(\mathfrak{L}(G))$ on $\mathfrak{L}(G)$ (see 2.27ff.). So we can form the semidirect product

$$\mathfrak{L}(G) \rtimes_{\mathrm{Ad}} G, \quad (X, g)(Y, h) = (X + \mathrm{Ad}(g)Y, gh),$$

and obtain this result.

Proposition 21 (4.29 (iii)). *For each pro-Lie group G, the semidirect product $T(G) \stackrel{\text{def}}{=} \mathfrak{L}(G) \rtimes_{\mathrm{Ad}} G$ is a well-defined pro-Lie group.*

We call $T(G)$ the *tangent bundle* of G.

Thus pro-Lie groups have tangent bundles that are pro-Lie groups. In particular, all (almost) connected locally compact groups have tangent bundles within the category of pro-Lie groups. However, for a locally compact group G its tangent bundle $T(G)$ is locally compact only if G is finite-dimensional.

We have seen that the category of pro-Lie groups

– contains all finite-dimensional real Lie groups,

– is closed in the category of topological groups under the formation of all limits and the passing to closed subgroups,

– has a substantial Lie algebra functor that possesses a very reasonable left adjoint,

– is closed under the passing from a group to the additive group of its Lie algebra and under the passing from a group to its tangent bundle.

In other words, we have seen that the category of pro-Lie groups has none of the defects which plague the category of locally compact groups while it still contains all almost connected locally compact groups. That is, it still houses comfortably all those locally compact groups that support all the Lie theory there is for locally compact groups. But can we exhibit, one might ask, enough fine structure theory of pro-Lie algebras and pro-Lie groups so that at least the known structure theory of locally compact groups is recovered?

Like with all categories of groups, the first test that a group theory has to face is how well it elucidates the structure of its abelian representatives.

Abelian Pro-Lie groups

Apart from a territory far removed from the domain of connected or even almost connected commutative pro-Lie groups, the situation is very satisfactory and is, as a first coarse approximation to the general structure theory of almost connected pro-Lie groups, rather representative and a good guide for one's intuition.

A *weakly complete vector space* is a real topological vector space V for which there is a real vector space E such that $V \cong E^*$, where E^* is the algebraic dual $\mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R}) \subseteq \mathbb{R}^E$ endowed with the weak *-topology, that is, the topology of pointwise convergence induced from $\mathbb{R}^E$ given the product topology. (See Appendix 2, notably Theorem A2.8.) If the cardinal $\dim E$ is the linear dimension of E, that is, the cardinality of one, hence every basis of E, then $E \cong \mathbb{R}^{(\dim E)}$ and thus $V \cong \mathbb{R}^{\dim E}$. Therefore, an equivalent definition of a weakly complete topological vector space is the postulate that there be a set J such that $V \cong \mathbb{R}^J$ (see Corollary A2.9). If $\mathcal{NS}(V)$ denotes the filter basis of all closed vector subspaces F of a locally convex Hausdorff topological vector space V such that $\dim V/F < \infty$, then

V is a weakly complete vector space if and only if the natural morphism $\lambda_V : V \to \lim_{F\in\mathcal{NS}(V)} V/F$, $\lambda_V(v) = (v + F)_{F\in\mathcal{NS}(V)}$ is an isomorphism of topological vector spaces.

If an abelian topological group is isomorphic to the additive group of a weakly complete topological vector space, that is, to $\mathbb{R}^J$ for some set J, then we shall call it a *weakly complete vector group*.

The abelian pro-Lie groups we know best are the compact abelian groups and the weakly complete vector groups. So it is very pleasing that we can state the following result.

Lemma 22 (Vector Group Splitting Lemma for Connected Abelian Pro-Lie Groups; 5.12). *Any abelian almost connected pro-Lie group is isomorphic to the direct product of a weakly complete vector group and a compact abelian group.*

This result is succinct and very lucid. It illustrates that abelian pro-Lie groups, at least if they are almost connected are built up from weakly complete vector groups and compact abelian groups in a certainly simple fashion.

In reality, we have better and more accurate information. For the more accurate information we have to pay a price: the formulations get more complicated. First we have a clean cut intermediate result showing that weakly complete topological vector spaces and tori are injectives in the category of abelian pro-Lie groups.

Theorem 23 (5.19). *Assume that G is an abelian pro-Lie group with a closed subgroup G_1 and assume that there are sets I and J such that $G_1 \cong \mathbb{R}^I \times \mathbb{T}^J$. Then G_1 is a homomorphic retract of G, that is, G_1 is a direct summand algebraically and topologically. So $G \cong G_1 \times G/G_1$.*

This allows us to argue that every abelian pro-Lie G group has a weakly complete vector subgroup V such that G is isomorphic to the direct product $V \times (G/V)$ where the factor G/V has no nontrivial vector subgroup. We call any such subgroup V a *vector group complement*. For a topological group G we let $\operatorname{comp}(G)$ denote the set of all elements which are contained in a compact subgroup.

Theorem 24 (Vector Group Splitting Theorem for Abelian Pro-Lie Groups; 5.20). *Let G be an abelian pro-Lie group and V a vector group complement. Then there is a closed subgroup H such that*

(i) $(v, h) \mapsto v + h \colon V \times H \to G$ *is an isomorphism of topological groups.*
(ii) H_0 *is compact and equals* $\operatorname{comp} G_0$ *and* $\operatorname{comp}(H) = \operatorname{comp}(G)$*; in particular,* $\operatorname{comp}(G) \subseteq H$.
(iii) $H/H_0 \cong G/G_0$*, and this group is prodiscrete.*
(iv) $G/\operatorname{comp}(G) \cong V \times S$ *for some prodiscrete abelian group S without nontrivial compact subgroups.*
(v) *G has a characteristic closed subgroup* $G_1 = G_0 + \operatorname{comp}(G)$ *which is isomorphic to* $V \times \operatorname{comp}(H)$ *such that G/G_1 is prodiscrete without nontrivial compact subgroups.*
(vi) *The exponential function* $\exp_G$ *of* $G = V \oplus H$ *decomposes as*

$$\exp_G = \exp_V \oplus \exp_H$$

where $\exp_V \colon \mathfrak{L}(V) \to V$ *is an isomorphism of weakly complete vector groups and* $\exp_H = \exp_{\operatorname{comp}(G_0)} \colon \mathfrak{L}((\operatorname{comp}(G_0)) \to \operatorname{comp}(G_0)$ *is the exponential function of the unique largest compact connected subgroup; here* $\mathfrak{L}(\operatorname{comp}(G_0)) = \operatorname{comp}(\mathfrak{L})(G)$ *is the set of relatively compact one parameter subgroups of G.*
(vii) *The arc component G_a of G is* $V \oplus H_a = V \oplus \operatorname{comp}(G_0)_a = \operatorname{im} \mathfrak{L}(G)$. *Moreover, if $\mathfrak{h}$ is a closed vector subspace of $\mathfrak{L}(G)$ such that* $\exp \mathfrak{h} = G_a$*, then* $\mathfrak{h} = \mathfrak{L}(G)$.

This theorem actually is the basis of a rather explicit structure theory of abelian pro-Lie groups. We recall that all locally compact abelian groups belong to this class. There are still some portions of an abelian pro-Lie group G which we do not fully control:

– The factor group $G/G_1 \cong H/\operatorname{comp} G$ is prodiscrete and has no compact subgroups but is otherwise uncharted.
– $\operatorname{comp}(G) = \operatorname{comp}(H)$ is a pro-Lie group that is a directed union of compact subgroups. We do not know much more in the absence of local compactness.

If we impose certain natural additional hypothesis that are traditionally invoked in topological group theory, the situation is at once much better. A topological group is called *compactly generated* if it is algebraically generated by a compact subset.

Theorem 25 (The Compact Generation Theorem for Abelian Pro-Lie Groups; 5.32).

(i) *For a compactly generated abelian pro-Lie group G the characteristic closed subgroup* $\operatorname{comp}(G)$ *is compact and the characteristic closed subgroup G_1 is locally compact.*

(ii) *In particular, every vector group complement V is isomorphic to a euclidean group $\mathbb{R}^m$ for some $m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$.*

(iii) *The factor group G/G_1 is a compactly generated prodiscrete group without compact subgroups. If G/G_1 is Polish, then G is locally compact and*

$$G \cong \mathbb{R}^m \times \operatorname{comp}(G) \times \mathbb{Z}^n.$$

(iv) *If G is a pro-Lie group containing a finitely generated abelian dense subgroup, then* $\operatorname{comp}(G)$ *is compact and $G \cong \operatorname{comp}(G) \times \mathbb{Z}^m$. In particular, G is locally compact.*

(v) *A finitely generated abelian pro-Lie group is discrete.*

The full subcategory of locally compact abelian groups in the category of abelian pro-Lie groups has a celebrated structure theory that is primarily due to the highly effective and elegant duality theory going back to L. S. Pontryagin [169] and E. R. van Kampen [125] in the early thirties of the 20th century. For any topological abelian group G we let $\widehat{G} = \operatorname{Hom}(G, \mathbb{T})$ denote its dual with the compact open topology. (See e.g. [102, Chapter 7].) There is a natural morphism of abelian groups $\eta_G\colon G \to \widehat{\widehat{G}}$ given by $\eta_G(g)(\chi) = \chi(g)$ which may or may not be continuous; information regarding this issue is to be found for instance in [102, pp. 298ff], notably in Theorem 7.7 on p. 300. We shall call a topological abelian group *semireflexive* if $\eta_G\colon G \to \widehat{\widehat{G}}$ is bijective and *reflexive* if η_G is an isomorphism of topological groups; in the latter case G is also said *to have duality* (see [102, p. 305]). In the direction of a duality theory of abelian pro-Lie groups we offer the following results.

Proposition 26 (5.35). *Let G be an abelian pro-Lie group and let V be a vector group complement. Then G is reflexive, respectively, semireflexive iff G/V is reflexive, respectively, semireflexive. The character group $\widehat{G}$ is isomorphic to a product $E \times A$ where E is the additive group of a real vector space with its finest locally convex topology and A is the character group of an abelian pro-Lie group whose identity component is compact.*

Theorem 27 (5.36). *Every almost connected abelian pro-Lie group is reflexive, and its character group is a direct sum of the additive topological group of a real vector space* *endowed with the finest locally convex topology and a discrete abelian group. Pontryagin duality establishes a contravariant functorial bijection between the categories of almost connected abelian pro-Lie groups and the full subcategory of the category of topological abelian groups containing all direct sums of vector groups with the finest locally convex topology and discrete abelian groups.*

Part 2. The Algebra of Pro-Lie Algebras

The success of the Lie theory of classical Lie groups as well as in our case the Lie theory of pro-Lie groups depends on the effectiveness of the mechanism that allows us to translate problems of the topological group structure on the group level to algebraic problems on the Lie algebra level and back. Experience demonstrates that problems are more easily attacked in a purely algebraic environment. In the present case we know, however, that the Lie algebra of a pro-Lie group is a topological algebra itself. So we hope to repeat the classical success story only to the extent to which the topological algebra and the representation theory of pro-Lie groups themselves reduce to pure algebra – more or less. We shall see that this is largely the case for pro-Lie algebras due to the fact that the underlying topological vector spaces are weakly complete vector spaces and that these have a perfect duality theory that allows us to translate their topological linear algebra to pure linear algebra upon passing to the vector space duals. (See Appendix 2.)

The Module Theory of Pro-Lie Algebras

We saw that for every pro-Lie group G there exists a simply connected pro-Lie group $\widetilde{G}$ and a natural morphism with dense image $\pi_G\colon \widetilde{G} \to G$. Thus the structure of simply connected pro-Lie groups has no small influence on the structure of pro-Lie groups in general. We further saw that the structure of simply connected pro-Lie groups, in a well-understood sense, is completely determined by the structure of their Lie algebra. The lesson learned from Lie Theory of finite-dimensional Lie groups is that one must first study the structure of Lie algebras carefully and then apply the information gathered in this fashion to the group theory of Lie groups. It is no different with pro-Lie groups even though the connection between pro-Lie algebras and pro-Lie groups is more tenuous than in the finite-dimensional case.

We develop the representation theory and structure theory of pro-Lie groups simultaneously. Elementary module theory is usually preceded by a rush of simple definitions which still turn out to be very effective. We record some to the extent they are necessary for the reader to follow this overview.

Let L be a Lie algebra and E a vector space. Then E is an *L-module* if there is a bilinear map

$$(x, v) \mapsto x \cdot v : L \times E \to E \quad \text{satisfying} \quad [x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$$

for all $x, y \in L$ and $v \in E$. A function $f\colon E_1 \to E_2$ between L-modules is said to be a *morphism of L-modules* if it is linear and satisfies

$$(\forall x \in L,\ v \in E_1) \quad f(x \cdot v) = x \cdot f(v).$$

A submodule F of an L-module E is a vector subspace such that $L \cdot F \subseteq F$.

An L-module E is said to be *simple* if $\{0\}$ and $E \neq \{0\}$ are its only submodules.

An L-module E is called *semisimple* if every submodule is a direct module summand.

If L is a topological Lie algebra, then a topological vector space V is said to be a *topological L-module* if $(x, v) \mapsto x \cdot v\colon L \times V \to V$ is continuous in each variable separately.

If the topological vector space V is weakly complete, and if the filter basis of closed submodules W such that $\dim V/W < \infty$ converges to 0, then V is said to be a *profinite-dimensional L-module*. The profinite-dimensional modules have a perfect duality; indeed if E is the topological dual of a profinite-dimensional L- module, then E is an L-module with respect to the module operation defined by $\langle x \cdot \omega, v\rangle = -\langle \omega, x \cdot v\rangle$ for $x \in L$, $v \in V$ and $\omega \in E$.

Duality permits us to transfer concepts from algebraic module theory to topological module theory. For instance, let V be a profinite-dimensional topological vector space and an L-module. Then the module is said to be *reductive* if its dual module is semisimple.

Duality then permits us to prove theorems like the following:

Theorem 28 (7.18). (a) *Let V be a profinite-dimensional L-module for a Lie algebra L. Then the following statements are equivalent:*

(i) *V is reductive.*
(ii) *Every finite-dimensional quotient module of V is reductive.*
(iii) *V is the projective limit of finite-dimensional reductive module quotients.*
(iv) *V is isomorphic to a product of finite-dimensional simple modules.*

(b) *Every profinite-dimensional L-module has a unique smallest submodule V^{ss} such that V/V^{ss} is reductive.*

The theory and duality of L-modules are discussed in great detail, among many other things, in Chapter 7.

Now these module theoretical concepts apply to the structure theory of pro-Lie algebras. The key is the following remark. If $\mathfrak{g}$ is a pro-Lie algebra, then the underlying weakly complete topological vector space $|\mathfrak{g}|$ is a topological L-module with respect to the module operation defined by $x \cdot v = [x, v]$ for $x \in \mathfrak{g}$ and $v \in |\mathfrak{g}|$. This module is called the *adjoint module* $\mathfrak{g}_{\mathrm{ad}}$. A pro-Lie algebra $\mathfrak{g}$ is called *reductive* if its adjoint module $\mathfrak{g}_{\mathrm{ad}}$ is a reductive $\mathfrak{g}$-module. It is called *semisimple* if it is reductive and its center $\mathfrak{z}(\mathfrak{g})$ is zero.

While the duality theory of profinite-dimensional L-modules works perfectly, the theory of pro-Lie algebras has no duality theory in the sense that a pro-Lie algebra $\mathfrak{g}$ could have a Lie algebra as a dual object. However, its adjoint $\mathfrak{g}$-module $\mathfrak{g}_{\mathrm{ad}}$ has a dual $\mathfrak{g}$-module, also called its *coadjoint module* $\mathfrak{g}_{\mathrm{coad}}$. This module duality, however, attaches to each pro-Lie algebra $\mathfrak{g}$ an almost purely algebraic object, the coadjoint module $\mathfrak{g}_{\mathrm{coad}}$, and that is extremely helpful for the structure theory of pro-Lie algebras as the following results will show.

Theorem 29 (The Structure Theorem of Reductive and Semisimple Pro-Lie Algebras).
(a) *For a pro-Lie algebra* $\mathfrak{g}$ *the following conditions are equivalent.*

(i) $\mathfrak{g}$ *is reductive.*
(ii) $\mathfrak{g}$ *is the product of a family of finite-dimensional simple or one-dimensional ideals of* $\mathfrak{g}$.

(b) *Let* $\mathfrak{g}$ *be a reductive pro-Lie algebra. Then the commutator algebra* $[\mathfrak{g}, \mathfrak{g}]$ *is closed and is a product of finite simple real Lie algebras. Further* $\mathfrak{g} \cong \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ *algebraically and topologically, and* $\mathfrak{z}(\mathfrak{g}) \cong \mathbb{R}^I$ *for some set* I.
(c) *A pro-Lie algebra is semisimple iff it is a product of finite simple real Lie algebras.*
(d) *Every pro-Lie algebra has a unique smallest ideal* $\mathfrak{n}_{\mathrm{cored}}(\mathfrak{g})$ *such that* $\mathfrak{g}/\mathfrak{n}_{\mathrm{cored}}(\mathfrak{g})$ *is reductive.*

Consequently, for a pro-Lie algebra, the following statements are equivalent.

(I) $\mathfrak{g}$ is semisimple.
(II) $\mathfrak{g}$ is the product of a family of finite-dimensional simple ideals of $\mathfrak{g}$. (7.27, 7.29)

In the light of the fact that pro-Lie algebras $\mathfrak{g}$ arise as the Lie algebras of pro-Lie groups G, the very appealing duality theory of profinite-dimensional $\mathfrak{g}$-modules is a surprisingly effective tool for making the structure theory of pro-Lie groups algebraic.

Pro-Lie Algebras and Solvability

Recalling in the structure theory of finite-dimensional Lie algebras that there is always a unique largest solvable ideal, called the radical, we cannot hope to be able to bypass the question of solvability in any structure theory of pro-Lie algebras that is deserving of this name. The fact that the underlying vector spaces of pro-Lie algebras are infinite-dimensional as soon as the theory begins to be new and interesting is an ominous warning that solvability is going to be a delicate matter likely to involve set theory including well-ordering and ordinals.

Firstly, on a purely algebraic basis, in any Lie algebra we must define a transfinite commutator series and use this transfinite series to define a general concept of solvability. This proceeds as follows.

Let $\mathfrak{g}$ be a Lie algebra. Set $\mathfrak{g}^{(0)} = \mathfrak{g}$ and define sequences of ideals $\mathfrak{g}^{(\alpha)}$ indexed by the ordinals α, $\mathrm{card}\,\alpha \leq \mathrm{card}\,\mathfrak{g}$ via transfinite induction.

Assume that $\mathfrak{g}^{(\alpha)}$ is defined for $\alpha < \beta$.

(i) If β is a limit ordinal, set $\mathfrak{g}^{(\beta)} = \bigcap_{\alpha<\beta} \mathfrak{g}^{(\alpha)}$.
(ii) If $\beta = \alpha + 1$, set $\mathfrak{g}^{(\beta)} = [\mathfrak{g}^{(\alpha)}, \mathfrak{g}^{(\alpha)}]$.

For cardinality reasons, there is a smallest ordinal γ such that $\mathfrak{g}^{(\gamma+1)} = \mathfrak{g}^{(\gamma)}$. Set $\mathfrak{g}^{(\infty)} = \mathfrak{g}^{(\gamma)}$.

Let ω denote the first infinite ordinal. Then $\mathfrak{g}$ is said to be *transfinitely solvable* if $\mathfrak{g}^{(\infty)} = \{0\}$.

If $\mathfrak{g}$ is transfinitely solvable and $\gamma \leq \omega$, then $\mathfrak{g}$ is called *countably solvable*.

If γ is finite and $\mathfrak{g}^{(\gamma)} = \{0\}$, then $\mathfrak{g}$ is called *solvable*. Thus we have implications solvable $\Rightarrow$ countably solvable $\Rightarrow$ transfinitely solvable. Any simple Lie algebra such as $\mathfrak{sl}(2, \mathbb{R})$ (that is, the Lie algebra of 2×2-matrices with trace 0) yields an example with $\gamma = 0$ and $\mathfrak{g}^{(\infty)} = \mathfrak{g} \neq \{0\}$.

But we are dealing with *topological* Lie algebras. The natural objects here are the members of the *closed* commutator series, giving another three reasonable concepts of solvability right away.

Indeed, let $\mathfrak{g}$ be a topological Lie subalgebra of a topological Lie algebra $\mathfrak{h}$. (For instance, $\mathfrak{h} = \mathfrak{g}$.) Set $\mathfrak{g}^{((0))} = \bar{\mathfrak{g}}$ and define sequences of ideals $\mathfrak{g}^{((\alpha))}$ indexed by the ordinals α, $\operatorname{card}\alpha \leq \operatorname{card}\mathfrak{g}$ via transfinite induction.

Assume that $\mathfrak{g}^{((\alpha))}$ is defined for $\alpha < \beta$.

(i) If β is a limit ordinal, set $\mathfrak{g}^{((\beta))} = \bigcap_{\alpha<\beta} \mathfrak{g}^{((\alpha))}$.
(ii) If $\beta = \alpha + 1$, set $\mathfrak{g}^{((\beta))} = \overline{[\mathfrak{g}^{((\alpha))}, \mathfrak{g}^{((\alpha))}]}$.

For cardinality reasons, there is a smallest ordinal $\bar{\gamma}$ such that $\mathfrak{g}^{((\bar{\gamma}+1))} = \mathfrak{g}^{((\bar{\gamma}))}$. Set $\mathfrak{g}^{((\infty))} = \mathfrak{g}^{((\bar{\gamma}))}$.

Let ω denote the first infinite ordinal. Then $\mathfrak{g}$ is said to be *transfinitely topologically solvable*, if $\mathfrak{g}^{(\infty)} = \{0\}$. If $\mathfrak{g}$ is transfinitely topologically solvable and $\gamma \leq \omega$, then $\mathfrak{g}$ is called *countably topologically solvable*.

If $\bar{\gamma}$ is finite and $\mathfrak{g}^{((\bar{\gamma}))} = \{0\}$, then $\mathfrak{g}$ is called *topologically solvable*.

However, the Lie algebras we have to consider here are not only topological Lie algebras, they are in fact pro-Lie algebras, that is projective limits of finite-dimensional ones. That suggests yet another concept of solvability, namely, a pro-Lie algebra $\mathfrak{g}$ is called *prosolvable* if every finite-dimensional quotient algebra of $\mathfrak{g}$ is solvable.

It is known and proved just as in the case of topological groups in general that a topological Lie algebra is solvable if and only if it is topologically solvable. Thus there is a glimmer of hope that some of these seven reasonable concepts of solvability coincide.

Theorem 30 (The Equivalence Theorem for Solvability of Pro-Lie Algebras; 7.53). *Let $\mathfrak{g}$ be a pro-Lie algebra. Then the following assertions are equivalent:*

(i) $\mathfrak{g}$ *is transfinitely solvable.*
(ii) $\mathfrak{g}$ *is transfinitely topologically solvable.*
(iii) $\mathfrak{g}$ *is countably solvable.*

(iv) $\mathfrak{g}$ *is countably topologically solvable.*
(v) $\mathfrak{g}$ *is prosolvable.*
(vi) $\mathfrak{g}$ *does not contain a finite-dimensional simple Lie algebra.*

What a relief! We have to deal with only two concepts: The classical concept of solvability (which does not play a great theoretical role in our context) and one concept of "infinite" solvability which, in the later parts of the book, will be called prosolvability.

Actually this theorem is astonishing. Simple examples show that there are prosolvable algebras that are not solvable such as an infinite product of a family of solvable algebras with an unbounded family of solvable lengths. But it is not a priori clear that there cannot exist a prosolvable pro-Lie algebra with transfinite commutator series of arbitrary length in terms of ordinals.

We shall show that a pro-Lie algebra $\mathfrak{g}$ has a unique largest prosolvable ideal which is called its *radical* or *solvable radical* and is denoted by $\mathfrak{r}(\mathfrak{g})$.

Pro-Lie Algebras and Nilpotency

It is of course no surprise, that we play a similar game with the nilpotency of pro-Lie algebras arriving, somewhere down the line, at the following result.

Theorem 31 (The Equivalence Theorem for Nilpotency of Pro-Lie Algebras; 7.57). *Let $\mathfrak{g}$ be a pro-Lie algebra. Then the following assertions are equivalent:*

(i) $\mathfrak{g}$ *is transfinitely nilpotent.*
(ii) $\mathfrak{g}$ *is transfinitely topologically nilpotent.*
(iii) $\mathfrak{g}$ *is countably nilpotent.*
(iv) $\mathfrak{g}$ *is countably topologically nilpotent.*
(v) $\mathfrak{g}$ *is pronilpotent.*
(vi) *For every pair x, y of elements in $\mathfrak{g}$ the vector space endomorphism* $\operatorname{ad} x$ *satisfies* $\lim_n (\operatorname{ad} x)^n y = 0$.

A comparison of the preceding two results produces a certain difference in the structure of condition (vi) in the two cases. This in indicative of the fact, that the treatment of the two cases is not entirely parallel.

In the case of nilpotency we shall show that a pro-Lie algebra $\mathfrak{g}$ has a unique largest pronilpotent ideal which is called its *nilradical* and is denoted by $\mathfrak{n}(\mathfrak{g})$.

We remarked earlier that every pro-Lie algebra has a unique smallest ideal $\mathfrak{n}_{\text{cored}}(\mathfrak{g})$ such that $\mathfrak{g}/\mathfrak{n}_{\text{cored}}(\mathfrak{g})$ is reductive. We shall see (7.66 and 7.67) that $\mathfrak{n}_{\text{cored}}(\mathfrak{g})$ is pronilpotent and that

$$\mathfrak{n}_{\text{cored}}(\mathfrak{g}) = \overline{[\mathfrak{g}, \mathfrak{g}]} \cap \mathfrak{r}(g) = \overline{[\mathfrak{g}, \mathfrak{r}(\mathfrak{g})]}.$$

Clearly, we have a chain of radicals to which we might add the center $\mathfrak{z}(\mathfrak{g}) \stackrel{\text{def}}{=} \{x \in \mathfrak{g} : (\forall y \in \mathfrak{g})\ [x, y] = 0\}$:

$$\mathfrak{z}(g) \subseteq \mathfrak{n}_{\text{cored}}(\mathfrak{g}) \subseteq \mathfrak{n}(\mathfrak{g}) \subseteq \mathfrak{r}(\mathfrak{g})$$

and they may very well all be different.

While some results of the classical Lie algebra theory are rather difficult to recover for the case of pro-Lie algebras such as for instance the theory of Cartan subalgebras, there are others that are obtainable with comparatively little effort by some well chosen basic definitions and the classical results. An example of these is the Theorem of Ado.

Theorem 32 (Theorem of Ado for Pro-Lie Algebras; 7.105). *Every pro-Lie algebra* $\mathfrak{g}$ *has a faithful profinite-dimensional module* M.

In addition, this module has the property that for every cofinite-dimensional submodule N and the associated finite-dimensional representation $\pi_N : \mathfrak{g} \to \mathfrak{gl}(M/N)$ *for every element x from the nilradical, the endomorphism* $\pi_N(x)$ *of the finite-dimensional vector space* M/N *is nilpotent.*

The Levi–Mal'cev Theorem for Pro-Lie Algebras

One of the core results of the theory of finite-dimensional Lie algebras over fields of characteristic 0 is the semidirect splitting of the radical, a result that is usually labelled the Levi–Mal'cev Theorem. It is one of the remarkable facts of the theory of pro-Lie algebras that this theorem generalizes intact even though this does not happen by a simple generalization of the finite-dimensional proof. Nor is the proof based on a simple passage to the limit. We recall that in order merely to define the concepts of the various radicals, one had to understand first what solvability and nilpotency meant in the infinite-dimensional case.

The Levi–Mal'cev Theorem for a finite-dimensional Lie algebra $\mathfrak{g}$ has two essential parts: The first is an existence statement saying that there is a subalgebra $\mathfrak{s}$ such that the vector space $\mathfrak{g}$ is the direct sum of the radical $\mathfrak{r}(\mathfrak{g})$ and $\mathfrak{s}$. The second part tells us in which sense two subalgebras $\mathfrak{s}_1$ and $\mathfrak{s}_2$ satisfying these conditions agree: There is an element x of the coreductive radical for which, due to its provenance, the vector space endomorphism $\operatorname{ad} x$, defined by $(\operatorname{ad} x)(y) = [x, y]$, is nilpotent and yields $e^{\operatorname{ad} x}\mathfrak{s}_2 = \mathfrak{s}_1$. Since $\operatorname{ad} x$ is nilpotent, $e^{\operatorname{ad} x}$ is a polynomial and is therefore defined over any field of characteristic 0 and without any recourse to convergence and topology.

In the case of pro-Lie algebras, there is topology involved in the semidirect sum decomposition and extra effort has to go into the question whether indeed $\mathfrak{s}$ is closed and whether the vector space direct sum is also a topological one.

If $x \in \mathfrak{n}_{\text{cored}}(\mathfrak{g})$, then $e^{\operatorname{ad} x} = \operatorname{id}_{\mathfrak{g}} + \operatorname{ad} x + \frac{1}{2!} \cdot (\operatorname{ad} x)^2 + \cdots$ is a well-defined automorphism of $\mathfrak{g}$ called the *special automorphism* (implemented by x); in order to show that it is well defined, issues of convergence with respect to the topology of $\mathfrak{g}$ have to be clarified, and this topology is rarely first countable.

Let us now summarize what we shall show in the line of Levi–Mal'cev type results on pro-Lie algebras.

Theorem 33 (Levi–Mal'cev Theorem for Pro-Lie Algebras: Existence and Conjugacy; 7.52, 7.77).

(i) *A pro-Lie algebra* $\mathfrak{g}$ *is the semidirect sum* $\mathfrak{r}(\mathfrak{g}) \oplus \mathfrak{s}$ *of the radical and a closed semisimple subalgebra* $\mathfrak{s}$*.*
(ii) *The radical* $\mathfrak{r}(\mathfrak{g})$ *is prosolvable, and any Levi summand is the cartesian product of a family of finite-dimensional simple Lie algebras.*
(iii) *If* $\mathfrak{h}$ *is a closed subalgebra of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) + \mathfrak{h}$ *then* $\mathfrak{h}$ *contains a Levi summand* $\mathfrak{s}$ *of* $\mathfrak{g}$*.*
(iv) *Two Levi summands of a pro-Lie algebra are conjugate under a special automorphism.*
(v) *A pro-Lie algebra* $\mathfrak{g}$ *has only one Levi summand* $\mathfrak{s}$*, if and only if* $\mathfrak{g}$ *is the direct sum, algebraically and topologically,* $\mathfrak{r}(\mathfrak{g}) \oplus \mathfrak{s}$*.*
(vi) *If* $\mathfrak{m}$ *is a semisimple closed subalgebra of a pro-Lie algebra* $\mathfrak{g}$ *and* $\mathfrak{s}$ *is a Levi summand of* $\mathfrak{g}$*, then a conjugate of* $\mathfrak{m}$ *under an inner automorphism of* $\mathfrak{g}$ *is contained in* $\mathfrak{s}$*, and* $\mathfrak{m}$ *is contained in some Levi summand.*
(vii) *A semisimple closed ideal is contained in every Levi summand.*

Simply Connected Pro-Lie Groups Revisited

Of course we wish to apply the Levi–Mal'cev Theorem for pro-Lie algebras to finding out information about the structure theory of pro-Lie groups. This is really where the well-known structure theory of compact connected groups arises (see [102, Chapter 9]), and in that special case, the group structure rather well reflects the algebra structure. The noncompact situation is much more complicated. However, in the class of simply connected pro-Lie groups, the group structure perfectly reflects the Lie algebra structure. This was anticipated in our statement 14 above. Recalling that, on the level of pro-Lie algebras we have an algebraic and topological semidirect sum $\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) \oplus \mathfrak{s}$ with a semisimple Levi–Mal'cev summand $\mathfrak{s}$ and recalling that the structure of semisimple pro-Lie algebras is very lucid by Theorem 29 above, we have to clarify first the structure of simply connected prosolvable groups. Let us first remark, that we shall be able to show that on a pronilpotent pro-Lie algebra $\mathfrak{n}$ the Campbell–Hausdorff multiplication $(x, y) \mapsto x * y = x + y + \frac{1}{2} \cdot [x, y] + \cdots$ is well defined, since the infinite series that defines it in the ring of formal power series in two noncommuting variables can be shown to be summable for all pairs $(x, y) \in \mathfrak{n} \times \mathfrak{n}$. With respect to this multiplication, $(\mathfrak{n}, *)$ is a pro-Lie group.

Theorem 34 (Theorem on the Topological Structure of Simply Connected Pro-Lie Groups with Prosolvable Lie Algebras; 8.13). *Let* G *be a prosimply connected pro-Lie group whose Lie algebra* $\mathfrak{g} = \mathfrak{L}(G)$ *is prosolvable, that is, which is its own radical.*

Let $\mathfrak{n}$ *denote its Lie radical or its reductive radical, as the case may be. Then the following statements hold:*

(i) $\Gamma(\mathfrak{n}) \cong (\mathfrak{n}, *)$ *may be considered as a closed normal subgroup* N *of* G *such that* G/N *is an abelian pro-Lie group whose exponential function is a homeomorphism, and* $\mathfrak{L}(G/N)$ *is naturally isomorphic to* $\mathfrak{g}/\mathfrak{n}$*. Indeed,*

(ii) $\exp_{G/N}\colon \mathfrak{g}/\mathfrak{n} \to G/N$ *is an isomorphism of weakly complete vector groups.*

(iii) *The quotient morphism* $q\colon G \to G/N$ *admits a continuous cross section* $\sigma\colon G/N \to G$ *such that* $\sigma(N) = 1$

(iv) *There is an* N*-equivariant homeomorphism* $\varphi\colon G \to N \times (G/N)$ *such that* $\varphi(n) = (n, N)$ *for all* $n \in N$*, and* $\mathrm{pr}_{G/N} \circ \varphi = q$*.*

(v) G *is homeomorphic to* $\mathbb{R}^J$ *for some set* J*.*

(vi) G *is simply connected in any sense for which the additive group of a weakly complete topological vector space is simply connected.*

This is indeed, for simply connected pro-Lie groups with a prosolvable pro-Lie algebra, a fairly satisfactory state of affairs. The class of examples that illustrates how prosolvable pro-Lie groups arise by extending the nilradical by an abelian group is as follows.

Lemma 35 (The Center-Free Embedding Lemma; 9.41). *Let* K *be any pro-Lie group possessing enough finite-dimensional fixed point-free representations to separate the points. This is the case for all compact groups* K *and all locally compact abelian groups and all almost connected abelian pro-Lie groups. Then there is a center-free pro-Lie group* G *with a normal subgroup* V *such that* $G/V \cong K$*.*

The construction is surprisingly straightforward: Let $\{V_j : j \in J\}$ be a family of fixed point free finite-dimensional K-modules providing enough representations $\pi_j\colon K \to \mathrm{Gl}(V_j)$ of K to separate the points. Then we set $V = \prod_{j\in J} V_j$ and define $\pi\colon K \to \mathrm{Gl}(V)$ by $\pi(k)(v_j)_{j\in J} = (\pi_j(k)(v_j))_{j\in J}$. Then we define G to be the semidirect product $V \rtimes_\pi K$. If K is itself a weakly complete vector group, then G is a simply connected metabelian pro-Lie group (that is, solvable with commutative commutator subgroup) such that $\mathfrak{n} = V \times \{0\}$ is the nilradical and coreductive radical.

Another instructive example arises when we let K be any compact connected abelian group. Then each character $\chi \in \widehat{K}$ determines an irreducible representation $\pi_\chi\colon K \to \mathbb{C} \cong \mathbb{R}^2$. The construction of Lemma 35 provides us with a center-free metabelian pro-Lie group $G = \mathbb{C}^{\widehat{K}} \rtimes_\pi K$. The Lie algebra $\mathfrak{k}$ of K is isomorphic to $\mathrm{Hom}(\widehat{K}, \mathbb{R})$, and the image $\exp\mathfrak{k}$ of the exponential function is the dense proper analytic subgroup $A(\mathfrak{k}, K)$, which is exactly the arc component K_a of K and $G_a = \mathbb{C}^{\widehat{K}} \rtimes_{\pi|K_a} K_a$ is the unique dense subgroup $A(\mathfrak{g}, G)$ of G with Lie algebra $\mathfrak{g}$. We discuss “analytic” subgroups of pro-Lie groups extensively in Chapter 9 of this book.

The abstract quotient group $\pi_0(K) = K/K_a$ is isomorphic to $\mathrm{Ext}(\widehat{K}, \mathbb{Z})$. Thus whenever this group is nonzero, the analytic subgroup $A(\mathfrak{k}, K)$ is proper and dense. It then follows that $A(\mathfrak{g}, G)$ is a proper subgroup as well. All nontrivial Lie group homomorphic images G/N of G are of the form $\mathbb{R}^{2p} \times \mathbb{T}^q$ with $q > 0$ and thus have an infinite Poincaré group.

A good special case is the character group $K = \widehat{\mathbb{Q}_d}$ of the discrete additive group of rational numbers. Then by Pontryagin Duality the character group $\widehat{K}$ of K may be identified with $\mathbb{Q}_d$. Here for each $N \in \mathcal{N}(G)$ the factor group G/N is a circle group and so $P(G/N) \cong \mathbb{Z}$ while G/N_0 is of the form $\mathbb{R}^{2p} \times K$ and thus has a nontrivial center.

Within any category of locally compact groups the construction of such groups would be impossible as G is rarely locally compact in these examples.

The structure of simply connected pro-Lie groups is now rather completely described in the next theorem.

Theorem 36 (Structure Theorem for Simply Connected Pro-Lie Groups; 8.14). *Let G be a simply connected pro-Lie group with Lie algebra $\mathfrak{g}$. Then*

(i) *G is the semidirect product $R \rtimes_I S$ of a closed normal subgroup R whose Lie algebra $\mathfrak{L}(R)$ is the radical $\mathfrak{r}(\mathfrak{g})$ and a closed subgroup S whose Lie algebra $\mathfrak{s}$ is a Levi summand of $\mathfrak{g}$.*

(ii) *There is a family of simply connected simple Lie groups S_j, $j \in J$ such that $S \cong \prod_{j \in J} S_j$.*

(iii) *There is a closed normal subgroup N of G contained in R such that the pro-Lie algebra $\mathfrak{L}(N) = \mathfrak{n}_{\mathrm{cored}}(\mathfrak{g})$ is the coreductive radical of $\mathfrak{g}$ and that there is an N-equivariant isomorphism $\varphi\colon R \to N \times (R/N)$, where $N \cong (\mathfrak{n}_{\mathrm{cored}}(\mathfrak{g}), *)$ and where $R/N \cong \mathfrak{r}(\mathfrak{g})/\mathfrak{n}_{\mathrm{cored}}(\mathfrak{g})$ is a vector group.*

(iv) *R is homeomorphic to $\mathbb{R}^J$ for some set J.*

(v) *G is homeomorphic to a product of copies of $\mathbb{R}$ and of a family of simply connected real finite-dimensional simple Lie groups.*

(vi) *If C denotes the identity component of the center of G, then G is C-equivariantly homeomorphic to $C \times G/C$.*

The actual course of events will be this: we shall arrive at this structure theorem with the hypothesis of G being *prosimply* connected. This result will then allow us to demonstrate finally that the following are equivalent statements for a pro-Lie group G:

(i) *G is prosimply connected.*

(ii) *G is simply connected.*

(iii) *$\pi_G\colon \widetilde{G} \to G$ is an isomorphism.*

(iv) *$\pi_G\colon \widetilde{G} \to G$ is bijective.*

The equivalence of (iii) and (iv) is a consequence of the Open Mapping Theorem 8. In the course of the development of the theory, however, the equivalence of (iii) and (iv) is proved directly at this point in Chapter 8.

Part 3. The Fine Lie Theory of Pro-Lie Groups

We have seen that pro-Lie algebras have a remarkably good structure theory and that, in a first step, the structure of pro-Lie algebra translates in an almost one-to-one fashion to simply connected pro-Lie algebras. Let us reiterate again that in the category of locally compact groups this very satisfactory phenomenon cannot be seen because the simply connected manifestation $\widetilde{G} = \Gamma(\mathfrak{L}(G))$ of a locally compact group is rarely locally compact; indeed it is locally compact if and only if the radical $\mathfrak{r}(\mathfrak{L}(G))$ is finite-dimensional and all simple factors of the semisimple pro-Lie algebra $\mathfrak{L}(G)/\mathfrak{r}(\mathfrak{L}(G))$ are compact simple Lie algebras (that is, simple real Lie algebras with a negative definite Killing form) with the possible exception of finitely many simple factors.

Yet as soon as one renounces simple connectivity the problems of a global structure theory start and are, as a rule, more serious than in the theory of Lie groups. First we have to deal with the issue of what, in the situation of a pro-Lie group, constitutes an analytic subgroup and what the relation between (closed) subalgebras of the Lie algebra and connected subgroups might be.

The Lie Theory of Analytic Subgroups

In the theory of topological groups in general one gets accustomed to thinking of subgroups as being closed. If one has a closed normal subgroup N of a topological group, then G/N is a Hausdorff topological group, and in the absence of normality, the quotient space is still a good Hausdorff homogeneous space on which G acts transitively. However, as soon as Lie groups emerge in the picture, certain nonclosed subgroups simply *have to* be taken into account, namely, the so-called analytic subgroups. A subgroup H of a Lie group G is called analytic if it is an immersed submanifold. The classical example upon which people base their intuition is the torus $G = \mathbb{R}^2/\mathbb{Z}^2$ and the subgroups $H \stackrel{\text{def}}{=} (\mathbb{R}\cdot(1, r)+\mathbb{Z}^2)/\mathbb{Z}^2$ which are compact and therefore closed if and only if r is rational and are nonclosed and dense in G if and only if r is irrational. These, together with the subgroup $(\mathbb{R}\cdot(0, 1)+\mathbb{Z}^2)/\mathbb{Z}^2$ are all analytic subgroups other than the singleton one and G itself. Each one corresponds uniquely to a vector subspace (and subalgebra) of the Lie algebra $\mathbb{R}^2$ of G and all of them have to be taken into account to make this correspondence work well. The example illustrates well the fact that analytic subgroups depend in a chaotic fashion on the subalgebras. As one progresses more deeply into the theory of finite-dimensional Lie groups, one learns that the example is perhaps more typical than meets the eye at a first encounter. Of course, the *closed* connected subgroups of a finite-dimensional Lie group are indeed analytic, these are the "good" analytic subgroups, and the nonclosed ones which we illustrated above are the "bad" analytic subgroups – but they are needed.

Another thing one learns in finite-dimensional Lie group theory is that analytic subgroups can be characterized group theoretically in one of a variety of ways. We wish to generalize the concept of "analytic subgroup" to the environment of pro-Lie groups, and an elaborate analysis and manifold theory is not immediately available here – nor is it needed in the context of a structure theory of pro-Lie groups. Therefore we should select a flexible and useful version of the various characterisations of analytic subgroups and use it for a definition of an analytic subgroup of a pro-Lie group in the general case. The one we opted for says that a subgroup H of a Lie group G is analytic if and only there is some connected Lie group C and a morphism of topological groups $f: C \to G$ such that $H = f(C)$. In Chapter 9 we select this definition for pro-Lie groups:

Definition 37 (9.5). (i) Let G be a pro-Lie group and H a subgroup. Then H is said to be an *analytic subgroup* of G if there is a morphism $f: C \to G$ of topological groups from a connected pro-Lie group C into G such that $H = f(C)$ and $\mathfrak{L}(f)(\mathfrak{L}(C))$ is closed in $\mathfrak{L}(G)$.

(ii) A subgroup H of a pro-Lie group G is said to be *exponentially generated* if $\mathfrak{h} \stackrel{\text{def}}{=} \mathfrak{L}(H)$ is a closed Lie subalgebra of $\mathfrak{L}(G)$ and $H = \langle \exp \mathfrak{h} \rangle$.

In any infinite-dimensional Lie theory there is the added complication that subalgebras of the Lie algebra may or may not be closed, and on the Lie algebra level we insist that the Lie subalgebras we consider are closed. The definition above does not demand that $\operatorname{im} \mathfrak{L}(f) = \mathfrak{L}(H)$, however we shall show (9.6 (ii)) that this is the case. This means that

> *any analytic subgroup H of a pro-Lie group uniquely determines its own Lie algebra $\mathfrak{L}(H)$.*

Every connected closed subgroup of a pro-Lie group is analytic (9.7).

For each closed subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ of a pro-Lie group the inclusion $\mathfrak{h} \to \mathfrak{g}$ induces a morphism $\Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{g}) = \widetilde{G}$, and in view of the universal morphism $\pi_G : \widetilde{G} \to G$, the composition yields a morphism $i_{\mathfrak{h}} : \Gamma(\mathfrak{h}) \to G$. If we write $\mathfrak{L}(\Gamma(\mathfrak{h})) = \mathfrak{h}$, then $i_{\mathfrak{h}} : \Gamma(\mathfrak{h}) \to G$ is the unique morphism inducing the inclusion $\mathfrak{L}(i_{\mathfrak{h}}) : \mathfrak{h} \to \mathfrak{g}$. It is this morphism that has the special analytic subgroup $A(\mathfrak{h})$ or, more accurately, $A(\mathfrak{h}, G)$ of G as its image. Let us retain this notation in the formulation of the following proposition:

Proposition 38 (9.10, 9.11). *For each closed subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{L}(G)$ of a pro-Lie group G, there is at least one analytic subgroup H such that $\mathfrak{L}(H) = \mathfrak{h}$, namely, $H = A(\mathfrak{h}, G)$.*

If $\mathfrak{h}$ is an ideal, then $\Gamma(\mathfrak{h})$ may and will be identified with a closed normal subgroup $A(\mathfrak{h}, \Gamma(\mathfrak{g}))$ of $\Gamma(\mathfrak{g})$.

The analytic subgroup $A(\mathfrak{h})$ is exponentially generated and satisfies $\mathfrak{L}(A(\mathfrak{h})) = \mathfrak{h}$. In particular, the analytic subgroup $A(\mathfrak{h})$ is arcwise connected.

Among all analytic subgroups H satisfying $\mathfrak{L}(H) = \mathfrak{h}$, the subgroup $A(\mathfrak{h})$ is the smallest; it is contained in each H satisfying $\mathfrak{L}(H) = \mathfrak{h}$.

As a consequence of this proposition we obtain the following assertion that is one possible generalisation of the classical correspondence between subalgebras of the Lie algebra of a Lie group and its analytic subgroups.

Scholium (Scholium following 9.12). *Let G be a pro-Lie group and* $\mathfrak{g}$ *its Lie algebra. Denote the set of all analytic subgroups of G by* $\mathcal{A}(G)$ *and the set of all* minimal *analytic subgroups of G by* $\mathcal{A}_0(G)$. *Then the assignment* $H \mapsto \mathfrak{L}(H)\colon \mathcal{A}_0(G) \to \mathfrak{C}(\mathfrak{g})$ *is a bijection with inverse function* $\mathfrak{h} \mapsto A(\mathfrak{h})$ *and the function* $\mathfrak{h} \mapsto A(\mathfrak{h})\colon \mathfrak{C}(\mathfrak{g}) \to \mathcal{A}(G)$ *is a surjection.*

However, from most compact connected abelian non-Lie groups we can learn what happens with analytic subgroups. Indeed let G be a compact connected abelian group and G_a the arc component of the identity (see for instance [102, Chapter 8]). Then $G_a = \langle \exp_G \mathfrak{g}\rangle = A(\mathfrak{g}, G)$ is the minimal analytic subgroup with Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$, and G is the largest analytic subgroup with Lie algebra $\mathfrak{g}$. All subgroups H with $G_a \subseteq H \subseteq G$ satisfy $\mathfrak{L}(H) = \mathfrak{g}$ and some of these may very well be analytic: For instance (9.8 (iv))

> *the compact connected metric abelian group whose character group is the discrete group* $\mathbb{Q}^{(\mathbb{N})}$ *has a continuum cardinality of different analytic subgroups all of whose Lie algebras agree with* $\mathfrak{g}$.

In formulating a nomenclature for analytic subgroups of a pro-Lie group G, one is in a quandary.

On the one hand, it is pretty clear that a closed connected subgroup of a pro-Lie group should be called analytic. These have always, classically or otherwise, been the "good" analytic subgroups.

The relevant representatives of the nonclosed analytic subgroups are the minimal ones of the type $A(\mathfrak{h})$, and they are in a clean bijective correspondence with the closed Lie subalgebras of the Lie algebra $\mathfrak{L}(G)$. They are exactly the subgroups $\langle \exp_G \mathfrak{h}\rangle$ for the closed subalgebras; they are the ones that are amenable to all deeper developments of a Lie theory of pro-Lie groups. They are the "good bad" analytic subgroups. They are certainly needed in the Lie theory of pro-Lie groups. One might be tempted to reserve the term "analytic subgroup" for these subgroups exclusively – but then one would have excluded the "good analytic subgroups", the closed connected ones.

So we think that our terminology is a good compromise, still extending the classical concept, including the "good analytic subgroups" and the "good bad analytic subgroups" under one common roof. But we do have to allow for the presence of the myriad other analytic subgroups that have little, if any theoretical significance we can perceive. But there they are, and as long as they do not upset our scheme of things, they may stay.

Centralizers and Normalizers

Like in classical Lie theory, centralizers pose no problems.

Let G be a pro-Lie group and H any subset. Then the centralizer or commutant $Z(H, G) = \{g \in G : (\forall h \in H)\, gh = hg\}$ *is closed in G and is therefore a pro-Lie group.*

Let H be a subgroup of a connected pro-Lie group G and assume that $H \subseteq \overline{\langle \exp_G \mathfrak{h} \rangle}$, where $\mathfrak{h} = \mathfrak{L}(H)$. (This assumption is automatically satisfied if H is exponentially generated or analytic.) Then the following conclusions hold:

(i) *An automorphism α of G satisfies $\alpha(h) = h$ for all $h \in H$ iff $\mathfrak{L}(\alpha)X = X$ for all $X \in \mathfrak{h}$.*
(i′) *An element $g \in G$ is in $Z(H, G)$ iff $\mathrm{Ad}(g)X = X$ for all $X \in \mathfrak{h}$.*
(ii) $\mathfrak{L}(Z(H, G)) = \mathfrak{z}(\mathfrak{h}, \mathfrak{g})$.
(iii) $Z(H, G)_0 = \overline{\langle \exp_G \mathfrak{z}(\mathfrak{h}, \mathfrak{g}) \rangle}$.

These results satisfy most demands when it comes to centralizers, and we provide a few additional pieces of information in the book, but we do not need to go into them here. If $H = G$ then $Z(H, G)$ is the center $Z(G)$ of G; likewise $\mathfrak{z}(\mathfrak{h}, \mathfrak{g})$ is the center of $\mathfrak{g}$ on the Lie algebra level. So $\mathfrak{L}(Z(G)) = \mathfrak{z}(\mathfrak{g})$ and $Z(G)_0 = \overline{\langle \exp_G \mathfrak{z}(\mathfrak{g}) \rangle}$. In particular here is a proof that a connected pro-Lie group is abelian iff its Lie algebra is abelian.

The normalizer story is a bit more delicate. Let H be a subgroup of a group G. The *normalizer* of H in G is the set $N(H, G) = \{g \in G : gHg^{-1} = H\}$. If $\mathfrak{h}$ is a subalgebra of a Lie algebra $\mathfrak{g}$, then the *normalizer* of $\mathfrak{h}$ in $\mathfrak{g}$ is the set $\mathfrak{n}(\mathfrak{h}, \mathfrak{g}) = \{X \in \mathfrak{g} : [X, \mathfrak{h}] \subseteq \mathfrak{h}\}$. Sometimes $\mathfrak{n}(\mathfrak{h}, \mathfrak{g})$ is said to be the *idealizer* of the subalgebra $\mathfrak{h}$ in $\mathfrak{g}$.

We shall prove the following facts (9.20), that illustrate well the significance of the maximal and the minimal analytic subgroups having a fixed Lie algebra.

Let H be a subgroup of a pro-Lie group G and assume that H satisfies at least one of the following conditions:

(a) *H is a minimal analytic subgroup of G.*
(b) *H is a closed connected subgroup.*

Then the following conclusions hold:

(i) *An automorphism α of G satisfies $\alpha(H) = H$ iff $\mathfrak{L}(\alpha)(\mathfrak{h}) = \mathfrak{h}$.*
(i′) *Let g be an element of G. Then $gHg^{-1} = H$ iff $\mathrm{Ad}(g)\mathfrak{h} = \mathfrak{h}$.*
(ii) *The normalizer $N(H, G)$ is closed in G.*
(iii) $\mathfrak{L}(N(H, G)) = \mathfrak{n}(\mathfrak{h}, \mathfrak{g})$.

Item (ii) may be surprising, since H may be a nonclosed analytic subgroup. The closedness of the normalizer arises from the possibility of transporting the issue to the Lie algebra level where H has a closed Lie algebra. This is not very much different from the way it is in classical Lie theory for finite-dimensional Lie groups (see for instance [102], Proposition 5.54), but some extra care is required.

Commutator Subgroups

The Lie theory of commutator subgroups of analytic groups is hard already in finite dimensions, and many fairly elementary examples in the domain of pro-Lie groups show that certain difficulties we encounter here cannot be circumnavigated. We give a pronilpotent simply connected pro-Lie group whose commutator subgroup is not analytic. What we can prove is sampled by the following theorems, in which we denote the closed commutator subgroup (respectively, closed commutator subalgebra) by dots.

Theorem 39 (9.26). *For a connected pro-Lie group G, the closed commutator subgroup $\dot{G}$ is a closed analytic subgroup which agrees with the closure $\overline{A(\dot{\mathfrak{g}})}$ of the unique smallest analytic subgroup whose Lie algebra is the closed commutator subalgebra $\dot{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ of G.*

Theorem 40 (Theorem on Commutator Subgroups of Dense Analytic Subgroups; 9.32). *Let G be a connected pro-Lie group and $\mathfrak{h}$ a closed subalgebra of $\mathfrak{g} = \mathfrak{L}(G)$. Then*

(i) $\mathfrak{L}(\overline{A(\mathfrak{h})})^{\cdot} \subseteq \dot{\mathfrak{h}} \subseteq \mathfrak{h}$.
(ii) *In particular, $\mathfrak{L}(\overline{A(\mathfrak{h})})/\mathfrak{h}$ is abelian.*
(iii) *The abstract group $A(\mathfrak{L}(\overline{A(\mathfrak{h})}))/A(\mathfrak{h})$ is abelian.*

Corollary 41 (9.34). *Let G be a pro-Lie group and H a dense analytic subgroup with Lie algebra $\mathfrak{h}$, then for any $N \in \mathcal{N}(G)$, the algebraic commutator subgroup G' of G is contained in $A(\dot{\mathfrak{h}})N \subseteq HN$. As a consequence*

$$G' \subseteq \bigcap_{N \in \mathcal{N}(G)} A(\dot{\mathfrak{h}})N \subseteq \overline{A(\dot{\mathfrak{h}})}$$

and

$$A(\dot{\mathfrak{g}}) = A(\dot{\mathfrak{h}}) \subseteq \overline{A(\dot{\mathfrak{g}})} = \dot{G}.$$

Finite Dimensional Connected Pro-Lie Groups

We shall say, provisionally, but in perfect accord with a finer theory of topological dimension, that a pro-Lie group is *finite-dimensional*, if $\dim \mathfrak{L}(G) < \infty$. Armed with the arsenal of analytic subgroups we are able to deal with finite-dimensional pro–Lie groups and show, that they are rather close to finite-dimensional Lie groups in most respects. The class of almost connected finite-dimensional pro-Lie groups is seen to coincide with the class of almost connected locally compact groups. (For the compact case see for instance [102], Theorem 9.52.)

We shall denote the kernel of the universal morphism $\pi_G : \widetilde{G} \to G$ by $P(G)$ and call it the *Poincaré group* of G. It is natural that in a theory of pro-Lie groups, finite-dimensional pro-Lie groups play a significant role. This is primarily due to the fact that for every normal subgroup $N \in \mathcal{N}(G)$ and its identity component N_0 the pro-Lie group G/N_0 is finite-dimensional, having the same Lie algebra as the Lie group G/N. If G itself is finite-dimensional, then $N_0 = \{1\}$ for every sufficiently small $N \in \mathcal{N}(G)$. In our discussion of finite-dimensional pro-Lie groups, we pass through the following theorem, which in itself does not refer to the hypothesis of finite-dimensionality but is, despite its technical character at the root of a number of significant results. It deals with a connected pro-Lie group G, a normal subgroup N such that G/N is a Lie group, and with an intermediate member N_1 of $\mathcal{N}(G)$, $N_0 \subseteq N_1 \subseteq N$ and the Lie group $L \overset{\text{def}}{=} G/N_1$:

$$\begin{array}{ccccccc}
 & & P(G/N_0) & \longrightarrow & P(L) & \longrightarrow & P(G/N) \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
 & & \widetilde{L} & \xrightarrow{=} & \widetilde{L} & \xrightarrow{=} & \widetilde{L} \\
 & & {\scriptstyle\varphi}\big\downarrow & & {\scriptstyle\pi_L}\big\downarrow & & \big\downarrow{\scriptstyle\pi_{G/N}} \\
N_1/N_0 & \longrightarrow & G/N_0 & \xrightarrow{\rho} & L & \longrightarrow & G/N.
\end{array}$$

Theorem 42 (9.39). *Let G be a connected pro-Lie group and N an arbitrary member of the filter basis $\mathcal{N}(G)$. Then there is a characteristic subgroup $N_1 \in \mathcal{N}(G)$ that is open in N such that the connected Lie group $L \overset{\text{def}}{=} G/N_1$ and its universal covering $\pi_L : \widetilde{L} \to L$, together with the quotient map $\rho : G/N_0 \to L$ satisfy the following conditions:*

- ($*$) *There is a lifting morphism $\varphi : \widetilde{L} \to G/N_0$ such that $\pi_L = \rho \circ \varphi$,*
- ($**$) $N_1/N_0 = \ker(\rho_1)_N = \overline{\varphi(P(L))} = \operatorname{comp}(G/N_0)$ *is a compact metric totally disconnected central subgroup of G/N_0.*
  The group $P(L)/(P(L) \cap \ker\varphi)$ is finite iff N_0 has finite index in N_1 iff $N_0 \in \mathcal{N}(G)$. This is the case if $P(L)$ is finite and this is the case if $P(G/N) = \ker \pi_{G/N}$ is finite.
- ($***$) *Let $D \overset{\text{def}}{=} \{(\varphi(g)^{-1}, g) \in N_1/N_0 \times \widetilde{L} : g \in P(L)\} \cong P(L)$; then*

$$(nN_0, g)D \mapsto nN_0 \cdot \varphi(g) : \frac{N_1/N_0 \times \widetilde{L}}{D} \to G/N_0$$

  is a well-defined isomorphism of locally compact metric groups, and $N/N_1 \times \widetilde{L}$, $N/N_0 \times \widetilde{L}$ and G/N_0 are all locally isomorphic.
- ($\dagger$) *The subgroup $(\varphi(P(L)) \times \widetilde{G/N})/D$ is isomorphic to the minimal analytic subgroup $A(\mathfrak{L}(G/N_0), G/N_0)$ of G/N_0 with Lie algebra $\mathfrak{L}(G/N_0) \cong \mathfrak{L}(G/N)$.*

While this statement is fairly technical, it proves its value in the subsequent conclusions:

Proposition 43 (9.40). *Let G be a connected pro-Lie group with the following property:*

- ($\bullet$) *In the filter basis $\mathcal{N}(G)$ every member contains a member N such that N/N_0 is finite.*

Then $G = A(\mathfrak{g}, G)$.

Theorem 44 (9.44). *Let G be a finite-dimensional connected pro-Lie group with Lie algebra $\mathfrak{g}$. Then G is locally compact metric, and there is a compact metric totally disconnected member $\Delta \in \mathcal{N}(G)$ such that the Lie group $F \stackrel{\text{def}}{=} G/\Delta$ and the quotient morphism $\rho\colon G \to F$ satisfy the following conditions:*

(∗) *There is a morphism $\varphi\colon \widetilde{F} \to G$ such that $\pi_F = \rho \circ \varphi$.*
(∗∗) $\Delta = \overline{\varphi(P(F))}$.
(∗∗∗) *Let $D \stackrel{\text{def}}{=} \{(\varphi(g)^{-1}, g) : g \in P(F)\} \cong P(F)$; then*

$$(c, g)D \mapsto c\varphi(g) : \frac{\Delta \times \widetilde{F}}{D} \to G$$

is a well-defined isomorphism of locally compact metric groups, and $\Delta \times F$, $\Delta \times \widetilde{F}$ and G are all locally isomorphic.
(†) *The subgroup $\frac{\varphi(P(F)) \times \widetilde{F}}{D}$ is isomorphic to the minimal analytic subgroup $A(\mathfrak{L}(G), G)$ of G with Lie algebra $\mathfrak{g} \cong \mathfrak{L}(F)$.*

We keep in mind: Connected finite-dimensional pro-Lie groups are locally compact metric. Consequently, *almost connected finite-dimensional pro-Lie groups are locally compact*. For any connected finite-dimensional pro-Lie group G there are a totally disconnected compact abelian group Δ and a simply connected Lie group L such that we have a quotient morphism $\Delta \times L \to G$ whose kernel is a discrete central subgroup of $\Delta \times L$ projecting onto a dense subgroup of Δ.

We shall conclude a number of useful pieces of information from these developments. It is sometimes useful to know that the limit representation $G \cong \lim_{N \in \mathcal{N}(G)} G/N$ of a pro-Lie group in terms of its Lie group quotients yields a representation

$$G \cong \lim_{N \in \mathcal{N}(G)} G/N_0$$

of G in terms of finite-dimensional metric quotients of G modulo *connected* normal subgroups N_0.

There is a fairly significant conclusion coming out of this context:

Theorem 45 (Existence of the Largest Compact Normal Abelian Subgroup; 9.50). *Let G be a connected pro-Lie group.*

(i) *Then G has a unique largest compact central subgroup* $\mathrm{KZ}(G)$. *The factor group $G/\mathrm{KZ}(G)$ does not have nondegenerate compact central subgroups.*
(ii) *The center $Z(G)$ is a direct product of a weakly complete vector group V and a subgroup A of $Z(G)$ containing the characteristic subgroup* $\mathrm{KZ}(G)$*; moreover, the factor group $Z(G)/V\,\mathrm{KZ}(G) \cong A/\mathrm{KZ}(G)$ is prodiscrete and free of nonsingleton compact subgroups. The characteristic closed subgroup* $Z(G)_0 \operatorname{comp}(Z(G))$ *is the direct product of V and* $\mathrm{KZ}(G)$.

By way of illustration, this theorem says that a connected pro-Lie group cannot contain a central subgroup isomorphic to the additive group $\mathbb{Q}_p$ of a p-adic rational field or a discrete group isomorphic to a discrete Prüfer group $\mathbb{Z}(p^\infty) = \frac{1}{p^\infty}\mathbb{Z}/\mathbb{Z}$.

However, on the other hand, we give a construction that shows the following (9.51):

Given an abelian pro-Lie group A such that the union comp(A) *of its compact subgroups is compact then there is a connected* (*metabelian*) *pro-Lie group G such that A is* (*isomorphic to*) *the center of G.*

A second major result using conclusions from our discussion of finite-dimensional pro-Lie groups is the Open Mapping Theorem which we have already recorded in Theorem 8. But a lot of information that has accrued at this stage in the book enters its proof.

Part 4. Global Structure Theory of Connected Pro-Lie Groups

Since we have established a reasonable correspondence between subalgebras and analytic subgroups and handled the Lie theory of commutator subgroups with some success, we may hope to embark upon a global structure theory and deal with issues like solvability, nilpotency, reductivity, semisimplicity in the absence of simple connectivity.

Solvability and Nilpotency of Pro-Lie Groups

The question of solvability of infinite-dimensional Lie algebras that we discussed earlier is paralleled by the question of solvability of arbitrary groups.

Definition 46 (10.1). Let G be a group. Set $G^{(0)} = G$ and define sequences of subgroups $G^{(\alpha)}$ indexed by the ordinals α, $\operatorname{card}\alpha \leq \operatorname{card} G$ via transfinite induction.

Assume that $G^{(\alpha)}$ is defined for $\alpha < \beta$.

(i) If β is a limit ordinal, set $G^{(\beta)} = \bigcap_{\alpha<\beta} G^{(\alpha)}$.
(ii) If $\beta = \alpha + 1$, set $G^{(\beta)} = [G^{(\alpha)}, G^{(\alpha)}]$.

For cardinality reasons, there is a smallest ordinal γ such that $G^{(\gamma+1)} = G^{(\gamma)}$. Set $G^{(\infty)} = G^{(\gamma)}$.

Let ω denote the first infinite ordinal. Then G is said to be *transfinitely solvable*, if $G^{(\infty)} = \{0\}$. If G is transfinitely solvable and $\gamma \leq \omega$, then G is called *countably solvable*.

If γ is finite and $G^{(\gamma)} = \{0\}$, then G is called *solvable*.

We proceed to make a parallel definition for the infinite version of nilpotency.

Definition 47 (10.5). Let G be a group. Set $G^{[0]} = G$ and define sequences of normal subgroups $G^{[\alpha]}$ indexed by the ordinals α, $\operatorname{card}\alpha \leq \operatorname{card} G$ via transfinite induction.

Assume that $G^{[\alpha]}$ is defined for $\alpha < \beta$.

(i) If β is a limit ordinal, set $G^{[\beta]} = \bigcap_{\alpha<\beta} G^{[\alpha]}$.

(ii) If $\beta = \alpha + 1$, set $G^{[\beta]} = [G, G^{[\alpha]}]$.

For cardinality reasons, there is a smallest ordinal δ such that $G^{[\delta+1]} = G^{[\delta]}$. Set $G^{[\infty]} = G^{[\delta]}$. Then G is said to be *transfinitely nilpotent* if $G^{[\infty]} = \{0\}$. If G is transfinitely nilpotent and $\delta \leq \omega$, then G is called *countably nilpotent*.

If δ is finite and $G^{[\delta]} = \{0\}$, then G is called *nilpotent*.

Since $G^{(\alpha)} \subseteq G^{[\alpha]}$, *any transfinitely nilpotent Lie group is transfinitely solvable*.

As we are dealing with topological groups, we have topological versions of these concepts as well.

Definition 48 (10.8). Let G be a subgroup of a topological group H. (For instance, $H = G$.) Set $\mathfrak{g} = \mathfrak{L}(G)$ and $G^{((0))} = \overline{G}$; define sequences of normal subgroups $G^{((\alpha))}$ indexed by the ordinals α, $\operatorname{card}\alpha \leq \operatorname{card}\mathfrak{g}$ via transfinite induction.

Assume that $G^{((\alpha))}$ is defined for $\alpha < \beta$.

(i) If β is a limit ordinal, set $G^{((\beta))} = \bigcap_{\alpha<\beta} G^{((\alpha))}$.
(ii) If $\beta = \alpha + 1$, set $G^{((\beta))} = \overline{[G^{((\alpha))}, G^{((\alpha))}]}$.

For cardinality reasons, there is a smallest ordinal $\overline{\gamma}$ such that $G^{((\overline{\gamma}+1))} = G^{((\overline{\gamma}))}$. Set $G^{((\infty))} = G^{((\overline{\gamma}))}$.

Let ω denote the first infinite ordinal. Then G is said to be *transfinitely topologically solvable*, if $G^{((\infty))} = \{1\}$. If $\mathfrak{g}$ is transfinitely topologically solvable and $\gamma \leq \omega$, then G is called *countably topologically solvable*.

If $\overline{\gamma}$ is finite and $G^{((\overline{\gamma}))} = \{0\}$, then G is called *topologically solvable*.

And the nilpotent counterpart follows at once.

Definition 49 (10.9). Let G be a subgroup of a topological group. We set $G^{[[0]]} = \overline{G}$ and define sequences of closed normal subgroups $G^{[[\alpha]]}$ indexed by the ordinals α, $\operatorname{card}\alpha \leq \operatorname{card}\mathfrak{g}$ via transfinite induction.

Assume that $G^{[[\alpha]]}$ is defined for $\alpha < \beta$.

(i) If β is a limit ordinal, set $G^{[[\beta]]} = \bigcap_{\alpha<\beta} G^{[[\alpha]]}$.
(ii) If $\beta = \alpha + 1$, set $G^{[[\beta]]} = \overline{[G, G^{[[\alpha]]}]}$.

For cardinality reasons, there is a smallest ordinal $\overline{\delta}$ such that $G^{[[\overline{\delta}+1]]} = G^{[[\overline{\delta}]]}$. Set $G^{[[\infty]]} = G^{[[\overline{\delta}]]}$. Then G is said to be *transfinitely topologically nilpotent*, if $G^{[\infty]} = \{0\}$.

If G is transfinitely topologically nilpotent and $\delta \leq \omega$, then G is called *countably topologically nilpotent*.

If $\overline{\delta}$ is finite and $G^{[[\overline{\delta}]]} = \{0\}$, then G is called *topologically nilpotent*.

All of this may look a bit tedious, but as we are dealing with infinite groups and with topological groups there does not appear any way to bypass these definitions. However, since we are dealing here with pro-Lie groups more definitions are to follow inevitably.

Definition 50 (10.12). A pro-Lie group G is called *prosolvable* if every (finite-dimensional) quotient Lie group G/N, $N \in \mathcal{N}(G)$ is solvable. It is called *pronilpotent* if every (finite-dimensional) quotient Lie group G/N, $N \in \mathcal{N}(G)$, is nilpotent.

We have transfinite theories of solvability and nilpotency for pro-Lie algebras on the one hand and for pro-Lie groups on the other. The theory of analytic subgroups and their correspondence to closed subalgebras is now launched on this side-by-side situation with good success (see 10.14ff)

The results are sizeable.

Theorem 51 (The Equivalence Theorem for Solvability of Connected Pro-Lie Groups; 10.18). *Let G be a connected pro-Lie group and $\mathfrak{g}$ its Lie algebra $\mathfrak{L}(G)$. Then the following assertions are equivalent:*

(i) *G is transfinitely solvable.*
(ii) *G is countably solvable.*
(iii) *G is transfinitely topologically solvable.*
(iv) *G is countably topologically solvable.*
(v) *G is prosolvable.*
(vi) *G does not contain a finite-dimensional analytic simple subgroup.*
(vii) *$\mathfrak{g}$ is prosolvable.*
(viii) *$\mathfrak{g}$ does not contain a finite-dimensional simple Lie algebra.*

Again it is a remarkable feature of connected pro-Lie groups that all reasonable concepts of infinite solvability coalesce and that a genuine transfinite solvability does in fact not occur.

The situation with nilpotency is, alas, not equally perfect as far as our knowledge is concerned.

Theorem 52 (The Equivalence Theorem for Nilpotency of Connected pro-Lie Groups; 10.36). *Let G be a connected pro-Lie group and $\mathfrak{g}$ its Lie algebra $\mathfrak{L}(G)$. Then the following assertions are equivalent:*

(i) *G is transfinitely topologically nilpotent.*
(ii) *G is countably topologically nilpotent.*
(iii) *G is pronilpotent.*
(iv) *$\mathfrak{g}$ is pronilpotent.*

These conditions imply the following ones:

(v) *G is transfinitely nilpotent.*
(vi) *G is countably nilpotent.*

Is a transfinitely nilpotent connected pro-Lie group pronilpotent? We do not know. A transfinitely nilpotent group has to be prosolvable since it is transfinitely solvable and then Theorem 51 applies. The impediment for a proof is the failure of transfinite nilpotency to be preserved by passing to quotients. Free topological groups are free groups in the algebraic sense and thus are countably nilpotent; but *every* topological group is a quotient of a free topological group and thus of a transfinitely countably nilpotent topological group. In the meantime, we are content to have what Theorem 52 gives us.

The relationship between the topological commutator series and the topological descending central series on the Lie algebra and on the group level are expressed in a somewhat delicate fashion involving the minimal analytic subgroups associated with a closed subalgebra as follows:

Theorem 53 (Theorem on the Commutator Series of Pro-Lie Groups; 10.20). *Let G be a connected pro-Lie group. Then*

$$G^{((\alpha))} = \overline{A(\mathfrak{g}^{((\alpha))})} \quad \textit{for all ordinals } \alpha.$$

Theorem 54 (Theorem on the Descending Central Series of Pro-Lie Groups; 10.38). *Let G be a connected pro-Lie group. Then*

$$G^{[[\alpha]]} = \overline{A(\mathfrak{g}^{[[\alpha]]})} \quad \textit{for all ordinals } \alpha.$$

It is quite natural that we should introduce the counterparts of the radical $\mathfrak{r}(\mathfrak{g})$, the nilradical $\mathfrak{n}(\mathfrak{r})$, and the coreductive radical $\mathfrak{n}_{\text{cored}}(\mathfrak{g})$ of a pro-Lie algebra $\mathfrak{g}$ for a connected pro-Lie group G. The definitions are a bit delicate, because some obvious attempts at a definition are not feasible. Here is the way we proceed:

Definition 55 (10.23, 10.40). Let G be a pro-Lie group and $\mathfrak{g} = \mathfrak{L}(G)$ its Lie algebra. Then the closed subgroup $\overline{\langle \exp_G \mathfrak{r}(\mathfrak{g})\rangle}$ will be denoted by $R(G)$. This group is called the *radical* of G or, if more clarity is required, the *solvable radical of the group G*.

The closed subgroups $\overline{\langle \exp_G \mathfrak{n}(\mathfrak{g})\rangle}$ and $\overline{\langle \exp_G \mathfrak{n}_{\text{cored}}(\mathfrak{g})\rangle}$ will be denoted by $N(G)$, respectively $N_{\text{cored}}(G)$. These groups are called the *nilradical*, respectively, *coreductive radical* of G.

Recall that $Z(G)$ denotes the center of G. Since $Z(G)_0 = \overline{\langle \exp_G \mathfrak{z}(\mathfrak{g})\rangle}$ and $\mathfrak{z}(g) \subseteq \mathfrak{n}(\mathfrak{g}) \subseteq \mathfrak{r}(\mathfrak{g})$ we notice that $Z(G)_0 \subseteq N(G) \subseteq R(G)$.

We say that a connected pro-Lie group G is *semisimple*, if $R(G) = \{1\}$, and *reductive*, if $R(G) = Z(G)_0$. The radical $R(G)$ is a prosolvable connected closed characteristic subgroup; the nilradical and the coreductive radical are pronilpotent connected closed characteristic subgroups. More precisely (10.25, 10.28):

Theorem 56 (10.25). *If G is a pro-Lie group, then the radical $R(G)$ is the largest connected transfinitely topologically solvable normal subgroup and is a closed connected characteristic subgroup of G such that $\mathfrak{L}(R(G)) = \mathfrak{r}(\mathfrak{g})$. The factor group $G_0/R(G)$ is semisimple.*

If $f: G \to H$ is a quotient morphism of connected pro-Lie groups, then

$$\overline{f(R(G))} = R(H).$$

Recall that for a morphism of almost connected pro-Lie groups to be a quotient morphism, by the Open Mapping Theorem it suffices to be surjective.

Theorem 57 (10.42). *If G is a pro-Lie group, then the nilradical $N(G)$ is the largest connected transfinitely topologically nilpotent normal subgroup and is a closed connected characteristic subgroup of G such that $\mathfrak{L}(N(G)) = \mathfrak{n}(\mathfrak{g})$.*

Theorem 58 (10.43). *If G is a connected pro-Lie group, then the coreductive radical $N_{\text{cored}}(G)$ is the smallest connected closed normal subgroup N such that G/N is reductive. In particular, $G/N_{\text{cored}}(G)$ and $G/N(G)$ are reductive. It is a closed connected characteristic subgroup of G such that $\mathfrak{L}(N_{\text{cored}}(G)) = \mathfrak{n}_{\text{cored}}(\mathfrak{g})$.*

If $f : G \to H$ is a quotient morphism of connected pro-Lie groups, then

$$\overline{f(N_{\text{cored}}(G))} = N_{\text{cored}}(H).$$

In a pro-Lie group we have therefore a hierarchy of characteristic connected closed subgroups:

$$\begin{array}{c} G \\ | \\ G_0 \\ | \\ R(G) \\ | \\ N(G) \\ | \\ N_{\text{cored}}(G) \\ | \\ \{1\}. \end{array}$$

The factor group $G_0/R(G)$ is semisimple, the factor group $G_0/N_{\text{cored}}(G)$ is reductive. We remember also $Z(G)_0 \subseteq N(G)$.

So it is clearly time to say something about semisimple and reductive groups:

Theorem 59 (Characterisation of Semisimple and Reductive Connected Pro-Lie Groups; 10.29). *Let G be a connected pro-Lie group.*

(i) *G is semisimple iff $\mathfrak{g}$ is semisimple, and G is reductive iff $\mathfrak{g}$ is reductive.*
(ii) *G is semisimple iff $\widetilde{G}$ is a product $\prod_{j\in J} S_j$ of simply connected simple finite-dimensional Lie groups S_j, $j \in J$. Also G is reductive iff $\widetilde{G}$ is a product $\prod_{j\in J} S_j$ of pro-Lie groups S_j, $j \in J$ which are either simply connected simple finite-dimensional Lie groups or copies of $\mathbb{R}$.*
(iii) *Assume that P is a connected proto-Lie group, embedded into its completion G according to Theorem* 4.1 *and assume that $\mathfrak{g} = \mathfrak{L}(P) = \mathfrak{L}(G)$ such that $\mathfrak{g}$ is a semisimple pro-Lie algebra. This assumption is satisfied if $P = G$ is a semisimple pro-Lie group by* (i) *above. Then we have the following conclusions:*

   (a) *$\widetilde{G} = \widetilde{P} = \Gamma(\mathfrak{g}) \cong \prod_{j\in J} S_j$ where all S_j are simply connected simple Lie groups, and*

$$\pi_P : \widetilde{G} = \prod_{j\in J} S_j \to P$$

is a morphism with dense image whose kernel D is a closed subgroup of $\prod_{j\in J} Z(S_j)$ and thus is a totally disconnected central subgroup of $\widetilde{G}$.

(b) *There is a quotient morphism $f\colon P \to \prod_{j\in J} S_j/Z(S_j)$ with a totally disconnected kernel* $\ker f = Z(P)$. *The quotient $P/Z(P)$ is a center-free semisimple pro-Lie group. The completion G of P is $Z(G)P$, a semisimple connected pro-Lie group satisfying $P/Z(P) \cong G/Z(G)$.*

(c) (Sandwich Theorem) *The group P is 'sandwiched' between two products via two morphisms*

$$\prod_{j\in J} S_j \xrightarrow{\pi_G} P \xrightarrow{f} \prod_{j\in J} S_j/Z(S_j)$$

whose composition is just the quotient morphism obtained by passing to the quotient $S_j \to S_j/Z(S_j)$ in each factor.

(d) *Let G be a semisimple pro-Lie group and let $A(\mathfrak{g})$ be the minimal analytical subgroup with Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$. Then $G = Z(G)A(\mathfrak{g})$, and $A(\mathfrak{g})/Z(A(\mathfrak{g})) \cong G/Z(\mathfrak{g}) \cong \prod_{j\in J} G_j/Z(G_j)$. If $A(\mathfrak{g})$ is center-free then $A(\mathfrak{g})$ is complete and therefore equal to G.*

Theorem 60 (Theorem on the Closure of Semisimple Analytic Subgroups; 10.32). *Let G be a pro-Lie group and $\mathfrak{s}$ a closed semisimple subalgebra of $\mathfrak{g}$. Set $H \stackrel{\text{def}}{=} \overline{A(\mathfrak{s})}$. Then the following conclusions hold:*

(i) *H is topologically perfect, that is, $\overline{[H, H]} = H$.*
(ii) $[H, H] \subseteq A(\mathfrak{s})$.
(iii) *H is reductive such that $\overline{[\mathfrak{L}(H), \mathfrak{L}(H)]} = \mathfrak{s}$.*
(iv) *In particular,*

$$\mathfrak{L}(H) \cong \mathbb{R}^I \times \mathfrak{s}, \quad \mathfrak{s} \cong \prod_{j\in J} \mathfrak{s}_j$$

for some set I and J and a family of simple finite-dimensional Lie algebras $\mathfrak{s}_j$, $j \in J$. Therefore, $\widetilde{H} \cong \mathbb{R}^I \times \prod_{j\in J} S_j$, $S_j = \Gamma(\mathfrak{s}_j)$.
(v) *$H = Z(H)A(\mathfrak{s})$ and $H/Z(H)$ is a center-free pro-Lie group.*
(vi) *Let S be the image of $\prod_{j\in J} \Gamma(\mathfrak{s}_j)$ in $\widetilde{G}$; then*

  (a) *$A(\mathfrak{s}) = \pi_G(S)$, and*
  (b) *$Z(H) = \overline{Z(A(\mathfrak{s}))}$.*
  (c) *If $\mathfrak{L}(H) = \mathfrak{s}$, that is, if H is semisimple, then* $\mathrm{KZ}(H)A(\mathfrak{s}, H) = H$.

(vii) *The minimal analytic subgroup $A(\mathfrak{s})$ is closed in G if and only if its center $Z(A(\mathfrak{s}))$ is closed in G. This is the case for instance if $Z(A(\mathfrak{s}))$ is compact.*

Much of what has been said, but not everything, is included in the following summary:

Theorem 61 (Characterisation of Reductive Pro-Lie Groups; 10.48). *Let G be a connected pro-Lie group. Then the following statements are equivalent:*

(i) *G is reductive.*
(ii) *$\mathfrak{g}$ is reductive.*
(iii) *$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \dot{\mathfrak{g}}$ for a unique semisimple pro-Lie algebra $\dot{\mathfrak{g}}$, obtained as the closed commutator subalgebra of $\mathfrak{g}$.*
(iv) *$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{r}$ for a central closed subalgebra $\mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{g})$ and a closed reductive subalgebra $\mathfrak{r} \stackrel{\text{def}}{=} \mathfrak{L}(\overline{A(\dot{\mathfrak{g}})})$.*
(v) *$G = \overline{A\overline{S}} = \overline{AS}$ for a closed connected central subgroup A and a semisimple minimal analytic subgroup S.*
(vi) *$G = Z(G)S$ for a semisimple minimal analytic subgroup S, that is, $S = A(\mathfrak{s})$ for some semisimple subalgebra (indeed ideal) $\mathfrak{s}$.*
(vii) *$G = Z(G)S$ for the minimal analytic subgroup $S = A(\dot{\mathfrak{g}})$, and $\dot{\mathfrak{g}}$ is semisimple.*

Essential ingredients of theorems on semisimple and reductive groups G are the simple factors of $\dot{\mathfrak{g}} \cong \prod_{j\in J} \mathfrak{s}_j$. We say that $\mathfrak{s}_j$ is of *bounded type* if the simply connected Lie group $\Gamma(\mathfrak{s}_j)$ has a compact center. (See 10.50 for more details.) We say that a semisimple pro-Lie algebra $\mathfrak{s}$ is of *bounded type* if all of its simple factors are of bounded type. This amounts to saying that the center of $\Gamma(\mathfrak{s})$ is compact.

This concept is very important in determining the structure of the minimal analytic subgroup whose Lie algebra is a Levi summand.

Theorem 62 (10.52). *Let G be a pro-Lie group and let $\mathfrak{s}$ be a semisimple pro-Lie subalgebra of $\mathfrak{g} \stackrel{\text{def}}{=} \mathfrak{L}(G)$ defining a minimal analytic subgroup $A(\mathfrak{s})$.*

(a) *If $Z(A(\mathfrak{s}))$ is compact, then $A(\mathfrak{s})$ is closed.*
(b) *The following statements are equivalent:*

 (i) *$\mathfrak{s}$ is of bounded type.*
 (ii) *$\mathfrak{s} \subseteq \mathfrak{L}(G)$ for a pro-Lie group $\mathfrak{L}(G)$ then $A(\mathfrak{s}, G)$ is closed in G.*

In other words, $\mathfrak{s}$ is of bounded type if and only if $A(\mathfrak{s}, G))$ is closed in all pro-Lie groups G.

Splitting Theorems for Pro-Lie Groups

An important class of structure theorems for topological groups is formed by the so called splitting theorems. Assume that N is a normal subgroup of a topological group G. There is a representation $\iota\colon G \to \mathrm{Aut}(N)$ defined by $\iota(g)(n) = gng^{-1}$; we do not worry here about a topological group structure on the group $\mathrm{Aut}(N)$ of automorphisms of the topological group N or any continuity properties of ι; however what is relevant here is that the function $(g, n) \mapsto \iota(g)(n) = gng^{-1} : G \times N \to N$ is continuous, allowing us to define a semidirect product $G \rtimes_\iota N$, that is, the product space $N \times G$ with the multiplication $(m, g)(n, h) = (m\iota(g)n, gh)$. The function $\mu\colon N \rtimes_\iota G \to G$ defined by $\mu(n, g) = ng$ is a morphism of topological groups. If H is any subgroup of G, then $N \rtimes_\iota H$ is a subgroup of $N \rtimes_\iota G$, and the morphism μ restricts to a morphism $\mu_H : N \rtimes_\iota H \to G$ whose image is NH. We note that the kernel of μ_H is $\{(h^{-1}, h) : h \in N \cap H\}$ and that $h \mapsto (h^{-1}, h) : N \cap H \to \ker \mu_H$ is an isomorphism of topological groups. In particular $\mu_H : N \rtimes_\iota H \to G$ is bijective if and only if $N \cap H = \{1\}$ and $NH = G$. In the absence of any Open Mapping Theorem we cannot assert that μ_H is an isomorphism. Now assume that we are given G and N; then a *splitting theorem* provides sufficient conditions for the existence of a subgroup H of G such that $\mu_H : N \rtimes_\iota H \to G$ is an isomorphism. This is sometimes expressed by saying that N is a *semidirect factor* of G and that H is a *semidirect cofactor*. In fact, under these circumstances one also writes $G = N \rtimes H$ which some readers may consider a mild abuse of notation since the semidirect product sign $\rtimes$ is reserved for the "external" semidirect product.

We have already encountered some typical splitting theorems, for instance Theorem 24 and Theorem 36. Those two theorems had quite different proofs. This variety of methods will also be typical for the splitting theorems that we will discuss now and prove in Chapter 11.

The information we have accumulated on reductive pro-Lie groups allows us to establish a splitting theorem for reductive pro-Lie groups as follows:

Theorem 63 (Splitting Theorem for Reductive Pro-Lie Groups; 11.8). *Let $\mathfrak{s} = \dot{\mathfrak{g}}$ be the Levi summand of the pro-Lie algebra $\mathfrak{g}$ of a semisimple connected pro-Lie group G, and assume that $\mathfrak{s}$ is of bounded type. Then $\langle \exp_G \mathfrak{g} \rangle$, the unique minimal analytic subgroup with Lie algebra $\mathfrak{s}$, is the closed commutator subgroup $\dot{G}$ of G, and is a semidirect factor. That is, there is a closed connected abelian subgroup A of G acting on $\dot{G}$ under inner automorphisms such that $\iota(a)(n) = ana^{-1}$ and such that*

$$\mu_A : \dot{G} \rtimes_\iota A \to G, \quad \mu_A(n, a) = na$$

is an isomorphism of pro-Lie groups.

Of course, $A \cong G/\dot{G}$, so we could express this as saying that *a reductive pro-Lie group G whose Lie algebra has no simple factor of unbounded type is a semidirect product of its closed commutator subgroup $\dot{G}$ and its commutator factor group $G/\dot{G}$*. If $\mathfrak{s} = \mathfrak{sl}(2, \mathbb{R})$, then this assertion fails already for locally compact connected groups of dimension 3. On the other hand, every compact connected group satisfies the hypothesis of the theorem yielding the so-called Borel–Hofmann–Scheerer Splitting Theorem (see [102, Theorem 9.39]):

Corollary 64. *Every compact connected group G is the semidirect product of its commutator subgroup G' and a closed abelian subgroup isomorphic to G/G'.*

Here one also uses the fact that the algebraic commutator subgroup of a compact connected group is closed. (See [102, Theorem 9.2].)

One should draw the reader's attention to the fact that under the hypotheses of Theorem 63, by Theorems 61 and 62 there is a surjective morphism $\dot{G} \times Z_0 \to G$ for the identity component Z_0 of the center Z which, by the Open Mapping Theorem, is in fact a quotient morphism. But in general it has a prodiscrete kernel isomorphic to $Z_0 \cap \dot{G}$. Therefore $\dot{G}$ is not a direct factor.

The question how many cofactors we can have for $\dot{G}$ is answered by the following remark:

Assume that the hypotheses of Theorem 63 are satisfied. Let $\mathcal{C}(\dot{G})$ denote the set of cofactors of $\dot{G}$ in G. Then the function

$$\Phi\colon \operatorname{Hom}_D(Z(G)_0, \dot{G}') \to \mathcal{C}(\dot{G}), \quad \Phi(f) = \{f(z)^{-1}z \mid z \in Z\},$$

is a bijection.(11.9)

An entirely different splitting theorem with rather powerful consequences arises when the quotient group modulo a normal vector subgroup is compact.

Theorem 65 (The Vector Group Splitting Theorem for Compact Quotients; 11.15, 11.31). *Let G be a pro-Lie group with a normal weakly complete vector subgroup N such that G/N is compact. Then G has a compact subgroup K such that $G = N \rtimes K$. Moreover, two semidirect cofactors K_1 and K_2 for N are conjugate under an inner automorphism implemented by an element of N.*

For Lie groups and indeed for locally compact groups this result is fairly well known to mathematicians working in the area of locally compact groups. (See e.g. [108].) Yet we prove it here for the first time for pro-Lie groups in general.

Theorem 65 does generalize to prosolvable normal subgroups as follows:

Corollary 66 (Splitting Simply Connected Prosolvable Groups; 11.17, 11.32). *Let G be a pro-Lie group with a normal subgroup N such that N is simply connected prosolvable and G/N is compact. Then G has a compact subgroup K such that $G = N \rtimes K$. Moreover, two semidirect cofactors K_1 and K_2 for N are conjugate under an inner automorphisms implemented by an element of N.*

These results allow us to prove strong structure theorems for prosolvable groups which give us a reasonably good insight into the structure of prosolvable connected pro-Lie groups. First we need to attend to some business concerning pronilpotent pro-Lie groups; we recall that the closed commutator subgroup of a connected prosolvable group is pronilpotent.

Lemma 67 (Simple Connectivity of Pronilpotent Pro-Lie Groups; 11.27). *For a connected pronilpotent pro-Lie group G, the following statements are equivalent:*

(i) *G has no compact subgroups.*
(ii) *G has no compact normal subgroups.*
(iii) *G has no compact normal connected subgroups.*
(iv) *G is simply connected.*
(v) $\exp_G\colon (\mathfrak{g}, *) \to G$ *is an isomorphism of topological groups.*

We note that Conditions (i), (ii), (iii) are group theoretical conditions, Condition (iv) is a topological condition, and (v) is a Lie theoretical condition. The last one reconfirms what we said earlier: The Lie algebra here determines the group, and does so in a particularly explicit way.

Now we have all the ingredients to prove the following structure theorem which belongs to the type of "almost" splitting theorems.

Theorem 68 (Structure of Almost Connected Prosolvable Pro-Lie Groups; 11.28). *Let G be an almost connected pro-Lie group whose identity component is prosolvable. Let C be the unique largest compact central connected subgroup, clearly contained in the nilradical $N(G_0)$. Then there is a maximal compact subgroup K which is abelian containing C, and there is a connected closed normal subgroup V containing $N(G_0)$ such that V/C is simply connected and $G/C = V/C \rtimes K/C$. There is a compact abelian subgroup M of G such that $G = VM$ and $V \cap M$ is totally disconnected central.*

$$
\begin{array}{cl}
G & \\
\Vert & \Big\}\ \text{compact, splits mod } V\cap M \\
V & \\
| & \Big\}\ \cong (\mathfrak{L}(V/N(G_0)), +) \text{ splits topologically over } N(G_0) \\
N(G)_0 & \\
| & \Big\}\ \cong (\mathfrak{L}(N(G_0))/C, *) \\
C & = \text{maximal compact connected central} \\
\Vert & \Big\}\ \text{compact central} \\
\{1\} &
\end{array}
$$

Roughly speaking this theorem says that the compact subgroups come in two kinds: one is deep down in the center and is connected, while the other is at the top and "almost" splits as a semidirect cofactor for V which is simply connected modulo C.

From these results we can get a group theoretical characterisation of simple connectivity for prosolvable groups in the spirit of Lemma 67 for pronilpotent groups.

Corollary 69 (Simple Connectivity of Prosolvable Lie Groups; 11.29). *Let G be a connected prosolvable pro-Lie group. Then the following statements are equivalent:*

(i) *G does not contain any nontrivial compact subgroup.*
(ii) *G is simply connected.*
(iii) *G is homeomorphic to* $(\mathfrak{L}(N(G)), *) \times (\mathfrak{L}(G/N(G)), +)$.

If these conditions are satisfied, then $G = \langle \exp_G \mathfrak{g} \rangle$.

Part 5. The Role of Compactness on the Pro-Lie Algebra Level

We have seen in various results on the structure of pro-Lie groups that compact subgroups play an important role. In Theorem 45 we say that each connected pro-Lie group contains a unique largest compact central subgroup. In describing the global structure of pronilpotent and prosolvable pro-Lie groups (see notably Theorem 68) we observed the crucial role of compact subgroups. In Chapter 12 of this book we investigate this role further in a systematic way. In particular, we are looking for maximal compact connected subgroups and aim to show their conjugacy, where possible. Our strategy, however, must be Lie theoretical. That is, we should be able to detect compact connected subgroups by identifying their Lie algebras. This requires that we look back at the module theory of pro-Lie algebras and try to transform the topological property of compactness on the group level into algebraic properties on the algebra level.

Lie Algebra Modules and Compactness

Definition 70 (7.8). Let L be a Lie algebra, E a vector space and V a topological vector space such that E and V are L-modules.

(i) V is called a *profinite-dimensional L-module* if it is complete as a topological vector space and the filter basis $\mathcal{M}$ of closed submodules $M \subseteq V$ such that $\dim V/M < \infty$ converges to 0.

(ii) E is called a *locally finite-dimensional L-module* if for each finite subset S of E there is a finite-dimensional submodule F of E containing S.

Definition 71 (12.1). (i) Let L be a Lie algebra and let V be an L-module. Then V is called a *pre-Hilbert L-module* if V is a real vector space with an inner product $(\bullet \mid \bullet)$, that is, a symmetric positive bilinear form, such that

$$(\forall x \in L,\ v, w \in V) \quad (x \cdot v|w) = -(v|x \cdot w).$$

It is called a *Hilbert L-module* if, in addition, V is a complete topological vector space with respect to the norm $\|v\|_2 = \sqrt{(v|v)}$.

(ii) V is called a *compact L-module* if V can be given an inner product relative to which it is a Hilbert L-module, and $\dim V < \infty$.

(iii) An L-module V is called *procompact* if V is profinite-dimensional and if all finite-dimensional quotient modules are compact L-modules.

(iv) An L-module V is called an *algebraically locally compact L-module* if V is locally finite-dimensional and if all finite-dimensional submodules are compact L-modules.

The terminology of a “compact” L-module formulated in (ii) derives from the adjoint module of a compact Lie group; for details see for instance [102, pp. 188ff., notably Proposition 6.2].

We define an algebraic property of a locally finite-dimensional module as “locally compact” even though it has nothing to do with the topological property of local compactness, but this is not any more deleterious than calling certain finite-dimensional Lie algebras compact, and this practice is well established.

It is not hard to see that some of the concepts we introduced appear in dual pairs. Indeed for a profinite-dimensional L-module V over a Lie algebra L, the following conditions are equivalent:

(i) V is a procompact L-module.
(ii) The topological dual V' of V is an algebraically locally compact L-module.

Therefore we understand the structure of procompact L- modules if we understand the largely algebraic concept of an algebraically locally compact L-module. In this regard we shall prove the following theorem with the aid of the Axiom of Choice:

Theorem 72 (The Structure of Algebraically Locally Compact Modules; 12.4).
(i) *An algebraically locally compact L-module is a pre-Hilbert L-module which is an orthogonal direct sum of compact submodules.*
(ii) *Any L-submodule of an algebraically locally compact L-module is algebraically locally compact and is an orthogonally direct summand.*
(iii) *Any L-module homomorphic image of an algebraically locally compact L-module is algebraically locally compact.*
(iv) *For each algebraically locally compact L-module E there is a Hilbert L-module $\widetilde{E}$ in which E is dense in the Hilbert space norm.*

As a consequence, every algebraically locally compact L-module E is a semisimple L-module. By way of duality, we now can instantly formulate the following results on procompact L-modules.

Theorem 73 (The Structure of Procompact Modules; 12.6). (i) *A procompact L-module is a direct product of compact simple L-modules and is a semisimple L-module.*
(ii) *A continuous homomorphic image of a procompact L-module is procompact.*
(iii) *A closed submodule of a procompact L-module is procompact.*

The concept of an L-module for a Lie algebra L is parallelled by that of a G-module where G is a group. This is what one learns in the elementary linear algebra of group representations. If V is a topological vector space and G a topological group, then V is called a *jointly continuous topological G-module* if the module action $(g, v) \mapsto g \cdot v : G \times V \to V$ is continuous. We mentioned that the adjoint module of a compact Lie group G is a compact $\mathfrak{g}$-module, and indeed the adjoint module of a compact group G is a procompact $\mathfrak{g}$ modules as we shall observe presently. It is both noteworthy in its own right and useful in various applications we shall make that, conversely, procompact modules give rise to compact groups.

Theorem 74 (The Compact Group Associated with a Procompact Module; 12.8). *Let V be a procompact L-module. Then there is a compact connected group $G_V \subseteq \mathrm{Aut}(V)$ such that V is a jointly topological G_V-module and there is a homomorphism of Lie algebras $\lambda\colon L \to \mathfrak{L}(G_V)$ such that*

(i) $G_V = \overline{\langle \exp_{G_V} \lambda(L) \rangle}$,
(ii) $(\forall x \in L,\ v \in V)\ x \cdot v = \lim_{h \to 0, h \neq 0} \frac{1}{h}((\exp_{G_V} \lambda(h \cdot x)) \cdot v - v)$, *and*

(iii) $(\forall x \in \mathfrak{g}, w \in V)\ (\exp_{G_V} \lambda(x))(w) = w + x_V(w) + \frac{1}{2!} \cdot x_V^2(w) + \cdots \in V.$
(iv) *A closed vector subspace W of V is an L-module if and only if it is a G_V-module.*

One of the applications which we derive from this result says that if $L = L_1 + L_2$ is a Lie algebra with two subalgebras L_1 and L_2 satisfying $[L_1, L_2] \subseteq L_2$, and if V is a profinite-dimensional L-module such that V is a procompact L_j-module for each of $j = 1$ and $j = 2$, then V is a procompact L-module.

Procompact Lie Algebras and Compactly Embedded Lie Subalgebras of Pro-Lie Algebras

If $\mathfrak{g}$ is a pro-Lie algebra, then the adjoint module $\mathfrak{g}_{\rm ad}$ is a profinite-dimensional $\mathfrak{g}$-module, and the coadjoint module $\mathfrak{g}_{\rm coad}$ is a locally finite-dimensional $\mathfrak{g}$-module.

If $\mathfrak{k}$ is any Lie subalgebra of $\mathfrak{g}$, then the restriction of the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}_{\rm ad}$ to $\mathfrak{k}$ makes $\mathfrak{g}_{\rm ad}$ a profinite-dimensional $\mathfrak{k}$-module and the restriction of the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}_{\rm coad}$ to $\mathfrak{k}$ makes $\mathfrak{g}_{\rm coad}$ into a locally finite-dimensional $\mathfrak{k}$-module which is dual to the $\mathfrak{k}$-module $\mathfrak{g}_{\rm ad}$. In Definition 71 (iii) we defined the notion of a procompact L-module for a Lie algebra L. This allows us now to say when a subalgebra of a pro-Lie algebra is "compactly embedded."

Definition 75. Let $\mathfrak{g}$ be a pro-Lie algebra and $\mathfrak{k}$ a Lie subalgebra. Then $\mathfrak{k}$ is said to be *compactly embedded* into $\mathfrak{g}$ if the adjoint module $\mathfrak{g}_{\rm ad}$ is a procompact $\mathfrak{k}$-module.

If $\mathfrak{g}$ is compactly embedded into itself, then $\mathfrak{g}$ is said to be a *procompact pro-Lie algebra*.

Obviously, a closed compactly embedded subalgebra is procompact in its own right. One notices that every commutative pro-Lie algebra, that is, every weakly complete vector space is a procompact Lie algebra, but a noncentral one-dimensional subalgebra of the three-dimensional Heisenberg algebra is a compact (hence procompact) Lie algebra which is not compactly embedded into the Heisenberg algebra. In the Heisenberg algebra, every 2-dimensional vector subspace containing the 1-dimensional center and commutator subalgebra is a maximal abelian subalgebra and is also an ideal. This shows that maximal abelian subalgebras and ideals are not unique. It also shows that the analytic subgroup belonging to a maximal abelian subalgebra may not be compact in a Lie group having the Heisenberg algebra as Lie algebra.

Here is what it means to be a procompact pro-Lie algebra:

Theorem 76 (The Structure Theorem of Procompact Lie Algebras; 12.12).
(A) *Let $\mathfrak{g}$ be a pro-Lie algebra. Then the following statements are equivalent:*

(i) *$\mathfrak{g}$ is procompact.*
(ii) *The coadjoint $\mathfrak{g}$-module $\mathfrak{g}_{\rm coad}$ is an algebraically locally compact $\mathfrak{g}$-module.*
(iii) *The coadjoint $\mathfrak{g}$-module $\mathfrak{g}_{\rm coad}$ is a direct sum of simple compact $\mathfrak{g}$-modules.*
(iv) *$\mathfrak{g}$ is a direct product of simple compact Lie algebras or copies of $\mathbb{R}$.*

(v) $\mathfrak{g}$ *is a direct product of its center* $\mathfrak{z}(\mathfrak{g})$ *and its commutator algebra* $\mathfrak{g}'$, *and* $\mathfrak{g}'$ *is a product of simple compact Lie algebras.*

In particular, for a procompact Lie-algebra $\mathfrak{g}$, *the radical* $\mathfrak{r}(\mathfrak{g})$ *agrees with its center* $\mathfrak{z}(\mathfrak{g})$.

(B) *A closed subalgebra of a procompact pro-Lie algebra is procompact.*

(C) *The image of a procompact pro-Lie algebra under a continuous morphism of Lie algebras is procompact.*

(D) *A product of any family of procompact pro-Lie algebras is procompact.*

(E) *A closed procompact semisimple subalgebra* $\mathfrak{k}$ *of a pro-Lie algebra* $\mathfrak{g}$ *is compactly embedded in* $\mathfrak{g}$.

(F) $\mathfrak{g}/\mathfrak{r}(\mathfrak{g})$ *is procompact iff* $\mathfrak{g}/\mathfrak{n}_{\mathrm{cored}}(\mathfrak{g})$ *is procompact.*

(G) *If* V *is a profinite-dimensional* $\mathfrak{k}$*-module for a semisimple procompact pro-Lie algebra* $\mathfrak{k}$, *then* V *is a procompact* $\mathfrak{k}$*-module.*

(H) *If* $\mathfrak{k}$ *is a procompact semisimple closed subalgebra of a pro-Lie algebra* $\mathfrak{g}$, *then it is compactly embedded in* $\mathfrak{g}$.

(I) *Assume that a subalgebra* $\mathfrak{g}$ *of a pro-Lie algebra* $\mathfrak{h}$ *is the sum of two subalgebras* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *such that, firstly,* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *are compactly embedded in* $\mathfrak{h}$, *and, secondly,* $[\mathfrak{g}_1, \mathfrak{g}_2] \subseteq \mathfrak{g}_2$. *Then* $\mathfrak{g}$ *is compactly embedded in* $\mathfrak{h}$.

Compactly embedded subalgebras are preserved under homomorphisms in the following sense: If $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a surjective morphism of pro-Lie algebras and $\mathfrak{k}$ is a compactly embedded subalgebra of $\mathfrak{g}_1$, then $\varphi(\mathfrak{k})$ is compactly embedded in $\mathfrak{g}_2$.

Compactly embedded subalgebras behave well in many ways:

(i) If $\mathfrak{k}$ is a compactly embedded subalgebra of $\mathfrak{g}$, then $\overline{\mathfrak{z}(\mathfrak{g}) + \mathfrak{k}}$ is compactly embedded.

(ii) The center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ is compactly embedded. The closure of a compactly embedded subalgebra is compactly embedded.

(iii) A compactly embedded subalgebra $\mathfrak{k}$ contained in a pronilpotent ideal $\mathfrak{n}$ is contained in the center $\mathfrak{z}(\mathfrak{g})$. In particular, a pronilpotent compactly embedded ideal is central.

(iv) Any procompact subalgebra $\mathfrak{k}$ of a prosolvable pro-Lie algebra $\mathfrak{g}$ is abelian. In particular, a compactly embedded subalgebra of a prosolvable pro-Lie algebra is abelian.

Maximal Compactly Embedded Subalgebras of Pro-Lie Algebras

With the Axiom of Choice one establishes without major difficulty the existence of maximal compactly embedded subalgebras:

Theorem 77 (Maximal Compactly Embedded Subalgebras: Existence; 12.15). *Let* $\mathfrak{g}$ *be a pro-Lie algebra. Then*

(i) *every compactly embedded subalgebra of* $\mathfrak{g}$ *is contained in a maximal compactly embedded subalgebra, and*
(ii) *every compactly embedded abelian subalgebra of* $\mathfrak{g}$ *is contained in a maximal compactly embedded abelian subalgebra.*

So existence is an easy matter, but a proof of the fact that maximal compactly embedded subalgebras are conjugate is quite a challenge. It was primarily for a proof of this fact that we developed the theory of Cartan subalgebras since they turn out to be the key here.

Cartan Subalgebras of Pro-Lie Algebras

Even in defining Cartan subalgebras in the context of pro-Lie algebras we need some auxiliary concepts. The essence is this

Definition 78 (7.84). Let $\mathfrak{g}$ be a pro-Lie algebra and $\mathfrak{h}$ a subalgebra. Then we set

$$\begin{aligned}\mathfrak{g}^0(\mathfrak{h}) &= \{x \in \mathfrak{g} : (\forall h \in \mathfrak{h}, \mathfrak{j} \in \mathcal{I}(\mathfrak{g}))(\exists n = n(x,h,\mathfrak{j}) \in \mathbb{N})\ (\operatorname{ad} h)^n(x) \in \mathfrak{j}\} \\ &= \bigcap_{h\in\mathfrak{h},\mathfrak{j}\in\mathcal{I}(\mathfrak{g})} \bigcup_{n\in\mathbb{N}} ((\operatorname{ad} h)^n)^{-1}(\mathfrak{j}) \\ &= \bigcap_{h\in\mathfrak{h},\mathfrak{j}\in\mathcal{I}(\mathfrak{g})} ((\operatorname{ad} h)^{\dim \mathfrak{g}/\mathfrak{j}})^{-1}(\mathfrak{j}),\end{aligned}$$

and

$$\begin{aligned}\mathfrak{g}_0(\mathfrak{h}) &= \{x \in \mathfrak{g} : (\forall h \in \mathfrak{h}, \mathfrak{j} \in \mathcal{I}(\mathfrak{g}))\ (\operatorname{ad} h)(x) \in \mathfrak{j}\} \\ &= \{x \in \mathfrak{g} : (\forall h \in \mathfrak{h})\ [h,x] = 0\} = \mathfrak{z}(\mathfrak{h},\mathfrak{g}),\end{aligned}$$

the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.

These definitions still reflect their finite-dimensional counterpart, but the adjustment to the pro-Lie algebra environment causes things to be more complicated. Accordingly, the following theorem is not exactly easy to prove.

Theorem 79 (7.87). *For a closed pronilpotent subalgebra* $\mathfrak{h}$ *of a pro-Lie algebra* $\mathfrak{g}$*, the following conditions are equivalent:*

(i) $\mathfrak{g}^0(\mathfrak{h}) = \mathfrak{h}$.
(ii) $\mathfrak{h}$ *is its own normalizer.*

Once we have this theorem we can at least proceed with the definition of a Cartan subalgebra of a pro-Lie algebra.

Definition 80 (7.88). A subalgebra $\mathfrak{h}$ of a pro-Lie algebra $\mathfrak{g}$ is said to be a *Cartan subalgebra* if it is a closed pronilpotent subalgebra satisfying the equivalent conditions of Theorem 79. That is, a Cartan subalgebra of a pro-Lie algebra is a closed pronilpotent subalgebra that agrees with its own normalizer.

For finite-dimensional Lie algebras, the existence of Cartan subalgebras is proved by establishing that each regular element is contained in a unique Cartan subalgebra where the regular elements are determined by finite-dimensional linear algebra and form an open dense subset. We cannot follow this path in our environment. But with the aid of the Axiom of Choice we prove:

Theorem 81 (Existence of Cartan Subalgebras; 7.93). *Let $\mathfrak{g}$ be a pro-Lie algebra and $\mathfrak{i}$ a cofinite-dimensional closed ideal. Then for each subalgebra $\mathfrak{h}_\mathfrak{i}$ of $\mathfrak{g}$ containing $\mathfrak{i}$ such that $\mathfrak{h}_\mathfrak{i}/\mathfrak{i}$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{i}$ there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h}+\mathfrak{i}=\mathfrak{h}_\mathfrak{i}$.*

In fact we shall see that the union of all Cartan subalgebras in a pro-Lie algebra is dense. It is also a very useful fact to know that if $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a surjective morphism of pro-Lie algebras and if $\mathfrak{h}_2$ is a Cartan subalgebra of $\mathfrak{g}_2$, then there exists a Cartan subalgebra $\mathfrak{h}_1$ of $\mathfrak{g}_1$ such that $f(\mathfrak{h}_1)=\mathfrak{h}_2$. We shall finally show the following result, generalizing a well-known fact on finite-dimensional Lie algebras but rather delicate to prove in the context of pro-Lie algebras:

Theorem 82 (Conjugacy of Cartan Subalgebras of Prosolvable Pro-Lie Algebras; 7.101). *Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be two Cartan subalgebras of a prosolvable pro-Lie algebra $\mathfrak{g}$. Then there is an $x \in \mathfrak{n}_{\mathrm{cored}}(\mathfrak{g})$ such that $e^{\mathrm{ad}\,x}\mathfrak{h}_1=\mathfrak{h}_2$.*

Now we are able to exploit these facts for a proof of the conjugacy of maximal compactly embedded subalgebras. First one verifies that a lemma which is prominently proved in Bourbaki [19, Chapter 7], persists for pro-Lie algebras. Indeed:

Lemma 83 (12.17). *Let $\mathfrak{g}$ be a pro-Lie algebra and $\mathfrak{a}$ a subalgebra satisfying the following two conditions:*

(i) *$\mathfrak{a}$ is abelian.*
(ii) *$\mathfrak{g}$ is a semisimple $\mathfrak{a}$-module under the adjoint action.*

Then $\mathfrak{a}$ is contained in a Cartan subalgebra of $\mathfrak{g}$; in fact the nonempty set $\mathfrak{C}(\mathfrak{a},\mathfrak{g})$ of Cartan subalgebras of $\mathfrak{g}$ containing $\mathfrak{a}$ is the set of Cartan subalgebras of $\mathfrak{z}(\mathfrak{a},\mathfrak{g})$, the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$.

Definition 84. An *inner automorphism* of a pro-Lie algebra is a finite composition of automorphisms of the form $e^{\mathrm{ad}\,x}$ with $x \in \mathfrak{g}$. An element $a \in \mathfrak{g}$ is said to be conjugate to b in $\mathfrak{g}$ if there is an inner automorphism φ such that $\varphi(a)=b$.

Now we proceed by proving that

(i) if in a pro-Lie algebra $\mathfrak{g}$ all Cartan subalgebras are conjugate, then all maximal compactly embedded abelian subalgebras are conjugate,

and we conclude from this information and Theorem 82 that

(ii) in a prosolvable pro-Lie algebra all maximal compactly embedded subalgebras are conjugate under inner automorphisms.

This is the first step in a longer chain of arguments which end up in the following conjugacy theorem.

Theorem 85 (Maximal Compactly Embedded Subalgebras: Conjugacy; 12.27). *Let* $\mathfrak{g}$ *be a pro-Lie algebra.*

(i) *If* $\mathfrak{k}_1$ *and* $\mathfrak{k}_2$ *are two maximal compactly embedded subalgebras of* $\mathfrak{g}$*, then there is an inner automorphism* φ *of* $\mathfrak{g}$ *such that* $\varphi(\mathfrak{k}_1) = \mathfrak{k}_2$*.*

(ii) *If* $\mathfrak{t}_1$ *and* $\mathfrak{t}_2$ *are two maximal compactly embedded abelian subalgebras of* $\mathfrak{g}$*, then there is an inner automorphism* φ *of* $\mathfrak{g}$ *such that* $\varphi(\mathfrak{t}_1) = \mathfrak{t}_2$*.*

(iii) *If* $\mathfrak{t}$ *is a maximal compactly embedded abelian subalgebra of* $\mathfrak{g}$ *then there is a compactly embedded subalgebra* $\mathfrak{k}$ *of* $\mathfrak{g}$ *containing* $\mathfrak{t}$ *and* $\mathfrak{k}$ *is unique modulo the coreductive radical* $\mathfrak{n}_{\mathrm{cored}}(\mathfrak{g})$*, that is, if* $\mathfrak{k}_1$ *is a maximal compactly embedded subalgebra containing* $\mathfrak{t}$*, then there is an* $x \in \mathfrak{n}_{\mathrm{cored}}(\mathfrak{g})$ *such that* $e^{\mathrm{ad}\,x}\mathfrak{k}_1 = \mathfrak{k}$*. In particular, if* $\mathfrak{g}$ *is reductive, then* $\mathfrak{k}$ *is unique. Moreover, there is an inner automorphism* φ *of* $\mathfrak{g}$ *such that* $\varphi(\mathfrak{k}_1) = \mathfrak{k}$ *and* $\varphi(\mathfrak{t}) = t$*.*

From the conjugacy theorem we can derive some rather immediate consequences.

Corollary 86 (12.31). *Let* Ω *be the set of pairs* $(\mathfrak{a}, \mathfrak{k})$ *where* $\mathfrak{k}$ *is a maximal compactly embedded subalgebra of a pro-Lie algebra* $\mathfrak{g}$ *and* $\mathfrak{a}$ *is a maximal compactly embedded abelian subalgebra of* $\mathfrak{g}$ *contained in* $\mathfrak{k}$*. Let the group* $\mathrm{Inn}(\mathfrak{g})$ *of all inner automorphisms act on* Ω *via* $\varphi \cdot (\mathfrak{a}, \mathfrak{k}) = (\varphi(\mathfrak{a}), \varphi(\mathfrak{k}))$*. Then the action is transitive.*

So in principle, if $\mathrm{Inn}(\mathfrak{g})_{(\mathfrak{a},\mathfrak{k})}$ is the isotropy subgroup of this action fixing the pair $(\mathfrak{a}, \mathfrak{k})$, then the set Ω is bijectively equivalent to the quotient space $\mathrm{Inn}(\mathfrak{g})/\,\mathrm{Inn}(\mathfrak{g})_{(\mathfrak{a},\mathfrak{k})}$. By Corollary 8.18 in the book, which exploits Corollary 14 above, it then follows that Ω is bijectively equivalent to a quotient space of $\Gamma(\mathfrak{g})$.

Theorem 85 permits us to intersect all maximal compactly embedded subalgebras and to conclude, that in this fashion we obtain a unique maximal compactly embedded ideal:

Corollary 87 (The Largest Compactly Embedded Ideal of a Pro-Lie Algebra; 12.34). *Let* $\mathfrak{g}$ *be a pro-Lie algebra and* $\mathfrak{r} = \mathfrak{r}(\mathfrak{g})$ *its radical. Then*

(i) *there is a unique largest compactly embedded ideal* $\mathfrak{m}(\mathfrak{g})$*;*

(ii) *there is a unique largest compactly embedded abelian ideal, namely, the center* $\mathfrak{z}(\mathfrak{m}(\mathfrak{g})) = \mathfrak{z}(\mathfrak{g})$*;*

(iii) $\mathfrak{m}(\mathfrak{g}) \cap \mathfrak{r}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{m}(\mathfrak{g}))$ *and* $\mathfrak{m}(\mathfrak{g})' = \mathfrak{m}(\mathfrak{g}) \cap \mathfrak{s}$ *for each Levi summand* $\mathfrak{s}$ *of* $\mathfrak{g}$*.*

Further $\mathfrak{m}(\mathfrak{g})$*,* $\mathfrak{z}(\mathfrak{g})$*, and* $\mathfrak{m}(\mathfrak{g})'$ *are invariant under all automorphisms of* $\mathfrak{g}$*.*

As far as the quotient $\mathfrak{g}/\mathfrak{m}(\mathfrak{g})$ is concerned, we have to be circumspect; indeed if $\mathfrak{g}$ is the 3-dimensional Heisenberg algebra, then $\mathfrak{m}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}/\mathfrak{m}(\mathfrak{g})$ is abelian, hence procompact. Nevertheless, this appears to be the only point of caution, because we have:

Corollary 88 (12.35). *The quotient algebra* $\mathfrak{g}/\mathfrak{m}(\mathfrak{g})$ *has no nondegenerate compactly embedded semisimple ideal.*

Part 6. The Role of Compact Subgroups of Pro-Lie Groups

The entire purpose of studying the nature of compactly embedded subalgebras of pro-Lie algebras was to investigate compact subgroups of pro-Lie groups via their Lie theory. A test of how good our chances are that this will work is the following characterisation theorem of procompact pro-Lie algebras:

Theorem 89 (Group Theoretical Characterisation of Procompact Pro-Lie Algebras; 12.36). *For a pro-Lie algebra* $\mathfrak{g}$ *the following statements are equivalent.*

(i) $\mathfrak{g}$ *is a procompact Lie algebra.*
(ii) *There is a simply connected topological group* G *of the form* $\mathbb{R}^I \times S$ *for some set* I *and a compact simply connected compact group* S *such that* $\mathfrak{g} \cong \mathfrak{L}(G)$.
(iii) *There is a compact connected group* G *such that* $\mathfrak{L}(G) \cong \mathfrak{g}$.
(iv) *There is a unique projective compact connected group* G *such that* $\mathfrak{L}(G) \cong \mathfrak{g}$.
(v) $\Gamma(\mathfrak{g}) = \prod_{i\in I} S_i$ *for a family of Lie groups* S_i *each of which is either isomorphic to* $\mathbb{R}$ *or else is a compact simply connected Lie group.*

In these circumstances, if $\mathfrak{L}(\Gamma(\mathfrak{g}))$ *is identified with* $\mathfrak{g}$ *as is possible by Theorem* 6.4, *then* $\exp_{\Gamma\mathfrak{g})} \mathfrak{g} = \Gamma(\mathfrak{g})$. *If* H *is a pro-Lie group with Lie algebra* $\mathfrak{h}$ *containing* $\mathfrak{g}$, *then* $\exp_H \mathfrak{g}$ *is an analytic subgroup of* H, *indeed the minimal analytic subgroup with Lie algebra* $\mathfrak{g}$.

For the concept of a projective compact group, which occurred in a unique fashion in (iv) we must refer the reader to [102, Definition 9.75ff.] and Theorem 8.78ff. As a consequence and further test for the effectiveness of pro-Lie theory we derive the core structure theorem of compact connected groups from this result (12.37), which of course is discussed in source books on compact groups:

Corollary 90 (The Structure Theorem of Semisimple Compact Connected Groups and the Levi–Mal'cev Structure Theorem for Connected Compact Groups; 12.37). *Each compact connected group is a quotient modulo a central totally disconnected compact subgroup of the product* $Z_0(G) \times \prod_{j\in J} S_j$, *where the* S_j *are simply connected simple Lie groups and where* $Z_0(G)$ *is the identity component of the center of* G.

We are now beginning to look for maximal compact subgroups of a pro-Lie group, if there are any. The additive group of p-adic rationals (see Example 1.20 (A)(i)) is a nondiscrete locally compact but noncompact abelian group which is a union of an ascending chain of compact (open) subgroups; thus there is no maximal compact subgroup in such a pro-Lie group. Of course, there are simple discrete abelian examples: The groups $\mathbb{Z}(p^\infty) = \left(\bigcup_{n=1}^{\infty} \frac{1}{p^n}\mathbb{Z}\right)/\mathbb{Z}$ and $\bigoplus_{n=1}^{\infty} \mathbb{Z}/m_n\mathbb{Z}$ (for a family of positive integers m_n) are countably infinite torsion groups which are the union of ascending towers of finite groups. It is therefore not a priori clear whether, for instance, connected pro-Lie groups have maximal compact subgroups at all.

We attack this question by linking the group theory of connected pro-Lie groups with the Lie algebra theory of pro-Lie algebras. In one direction we have:

Proposition 91 (12.41). *Let G be a pro-Lie group and H a compact subgroup with Lie algebra $\mathfrak{h} \stackrel{\text{def}}{=} \mathfrak{L}(H)$. Then $\overline{e^{\text{ad}_{\mathfrak{g}} \mathfrak{h}}} = \text{Ad}(H) \subseteq \text{Aut}(\mathfrak{g})$ is a compact subgroup and $\mathfrak{h}$ is compactly embedded.*

In the other direction we must recall what we said in Theorem 74 about the compact group G_V associated with a procompact module V and find:

Proposition 92 (12.42). *Let G be a pro-Lie group and $\mathfrak{h}$ a compactly embedded subalgebra of $\mathfrak{g} = \mathfrak{L}(G)$. Assume that H is an analytic subgroup of G with Lie algebra $\mathfrak{h}$, and denote by V the weakly complete vector space $\mathfrak{g}$ considered as the procompact $\mathfrak{h}$-module under the adjoint action. Then $\text{Ad}(\overline{H}) \subseteq G_V \subseteq \overline{\text{Ad}(\mathfrak{g})} \subseteq \text{Aut}(\mathfrak{g})$, and $\overline{\text{Ad}(H)} = \overline{\text{Ad}(\overline{H})}$. In particular, $\overline{\text{Ad}(H)}$ is a compact group and agrees with $\overline{e^{\text{ad}\,\mathfrak{h}}}$.*

When we discussed analytic subgroups we saw that for a closed subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of a pro-Lie group G we could have many analytic subgroups H with $\mathfrak{L}(H) = \mathfrak{h}$. There was always a minimal one, there may fail to be a maximal one, let alone a closed one. Yet if $\mathfrak{h}$ is compactly embedded, the situation is better in this regard:

Corollary 93. *Let $\mathfrak{h}$ be a maximal compactly embedded subalgebra of the Lie algebra $\mathfrak{g}$ of a pro-Lie group G. Then for any analytic subgroup H of G with $\mathfrak{L}(H) = \mathfrak{h}$ one has $\mathfrak{L}(\overline{H}) = \mathfrak{h}$. In other words, among the analytic subgroups with Lie algebra $\mathfrak{h}$ there is a closed one which is the unique largest one.*

If nothing else is being said we consider on $\text{Aut}(\mathfrak{g})$ the topology of pointwise convergence, that is, the one induced from $\mathfrak{g}^{\mathfrak{g}}$. If we now summarize the essential features of this discussion we notice that on the group side, compactness occurs within the adjoint group. Indeed the center of any pro-Lie algebra is always compactly embedded, but the identity component of the center in the group may easily fail to be compact:

Theorem 94 (12.45). *Let $\mathfrak{h}$ be a closed subalgebra of the Lie algebra $\mathfrak{g}$ of a pro-Lie group G. Then the following statements are equivalent:*

(i) $\mathfrak{h}$ *is compactly embedded into* $\mathfrak{g}$.
(ii) $\overline{\text{Ad}(\exp \mathfrak{h})} = \overline{e^{\text{ad}\,\mathfrak{h}}}$ *is a compact subgroup of* $\text{Aut}(\mathfrak{g})$.
(iii) $\overline{\text{Ad}(\langle \exp \mathfrak{h} \rangle)} = \overline{\langle e^{\text{ad}\,\mathfrak{h}} \rangle}$ *is a compact subset of* $\text{Aut}(\mathfrak{g})$.

If one were to create a name for those groups which have procompact Lie algebras or those whose adjoint groups are compact one might come up with the following nomenclature:

Definition 95 (12.46). (i) A connected pro-Lie group G is called *potentially compact*, if its Lie algebra $\mathfrak{g}$ is a procompact pro-Lie algebra.

(ii) A subgroup H of a pro-Lie group G is called *compactly embedded*, if $\overline{\mathrm{Ad}(H)}$ is compact in $\mathrm{Aut}(\mathfrak{g})$ (with respect to the topology of pointwise convergence).

With this terminology we get the following results:

Corollary 96 (12.47). *Let H be an analytic subgroup of a pro-Lie group G and $\mathfrak{h}$ the Lie algebra of H inside the Lie algebra $\mathfrak{g}$ of G. Then the following conditions are equivalent:*

(i) *H is compactly embedded in G.*
(ii) *$\mathfrak{h}$ is compactly embedded in $\mathfrak{g}$.*

Theorem 97 (Characterisation of Potentially Compact Connected Pro-Lie Groups; 12.48). *Let G be a connected pro-Lie group. Then the following conditions are equivalent.*

(i) *G is potentially compact.*
(ii) *G contains a closed weakly complete central vector subgroup V and a maximal compact subgroup C which is characteristic, such that the function $\mu\colon V \times C \to G$, $\mu(v, c) = vc$ is an isomorphism of topological groups.*
(iii) *There is a morphism $f\colon G \to K$ into a compact group with dense image such that $\mathfrak{L}(f)\colon \mathfrak{L}(G) \to \mathfrak{L}(K)$ is an isomorphism.*

If these conditions are satisfied, then $G = Z_0(G)G'$, where $Z_0(G)$, the identity component of the center is isomorphic to $V \times \mathrm{comp}(Z_0(G))$, and where the algebraic commutator group G' is a semisimple compact connected characteristic subgroup.

That terminology we chose also allows us to formulate very briefly what corresponds, on the group level, to maximal compact embedded subalgebras, respectively, maximal compactly embedded abelian subalgebras on the Lie algebra level.

Proposition 98 (12.52). *Let G be a connected pro-Lie group and $\mathfrak{h}$ a closed subalgebra of $\mathfrak{g} = \mathfrak{L}(G)$. Then $\mathfrak{h}$ is maximal compactly embedded in $\mathfrak{g}$, respectively, maximal compactly embedded abelian in $\mathfrak{g}$ if and only if there is a maximal compactly embedded connected subgroup H of G, respectively, a maximal compactly embedded connected abelian subgroup H of G such that $\mathfrak{L}(H) = \mathfrak{h}$.*

The Conjugacy of Maximal Compact Connected Subgroups

On our way to maximal compact connected subgroups of a connected pro-Lie group we are now reaching the crucial step (12.53):

Theorem 99 (Maximal Compactly Embedded Connected Subgroups: Existence and Conjugacy; 12.53). *Each connected pro-Lie group G contains maximal compactly embedded connected subgroups, respectively, maximal potentially compact connected abelian subgroups, and these are conjugate in G under inner automorphisms.*

The desired result on maximal connected compact, respectively, maximal compact connected abelian subgroups is now a relatively easy corollary.

Corollary 100 (Maximal Compact Connected Subgroups: Existence and Conjugacy; 12.54). *Each connected pro-Lie group G contains maximal compact connected subgroups, respectively, maximal compact connected abelian subgroups. Maximal compact connected subgroups, respectively, maximal compact connected abelian subgroups, are conjugate in G under inner automorphisms.*

Intersecting a conjugacy class of closed algebras yields an ideal, and intersecting a conjugacy class of closed subgroups yields a normal subgroup. These simple steps provide us now with these results:

Theorem 101 (The Largest Compactly Embedded Connected Normal Subgroup of a Pro-Lie Group; 12.56). *Let G be a connected pro-Lie group and $R(G)$ its radical. Let $Z(G)_0 = VC$ denote the direct product decomposition of the identity component of the center into a vector group factor V and the maximal compact subgroup $C = \mathrm{comp}(Z(G)_0)$ according to the Vector Group Splitting Lemma* 22 *for Connected Abelian Pro-Lie Groups. Then the following assertions hold.*

(i) *There is a unique maximal compactly embedded connected normal subgroup* $\mathrm{MaxCE}(G)$.
(ii) $\mathfrak{L}(\mathrm{MaxCE}(G)) = \mathfrak{m}(\mathfrak{g})$, *the algebraic commutator subgroup* $\mathrm{MaxCE}(G)'$ *is compact and agrees with* $\exp\mathfrak{m}(\mathfrak{g})'$, *and* $\mathrm{MaxCE}(G) = Z(G)_0 \exp\mathfrak{m}(\mathfrak{g})'$. *There is a quotient morphism* $V \times C \times \mathrm{MaxCE}(G)' \to \mathrm{MaxCE}(G)$, *given by* $(v, z, c) \mapsto zc$ *whose kernel is isomorphic to* $C \cap \mathrm{MaxCE}(G)'$.
(iii) $(\mathrm{MaxCE}(G) \cap R(G))_0 = Z(G)_0 = Z(\mathrm{MaxCE}(G))_0$ *and* $\mathrm{MaxCE}(G)' = (\mathrm{MaxCE}(\mathfrak{g}) \cap S)_0$ *for each Levi factor S of G.*
(iv) *There is a unique maximal compact connected normal subgroup* $\mathrm{MaxC}(G)$ *and* $\mathrm{MaxC}(G) = \mathrm{comp}(\mathrm{MaxCE}(G)) = C\,\mathrm{MaxCE}(G)'$.
(v) *The factor group* $G/\mathrm{MaxC}(G)$ *has no nontrivial compact connected normal subgroups.*

Corollary 102 (12.57). *Let G be a connected pro-Lie group.*

(i) *G contains a largest compactly embedded connected abelian normal subgroup which is central, namely,*

$$Z(\mathrm{MaxCE}(G))_0 = Z(G)_0,$$

the identity component of the center of $\mathrm{MaxCE}(G)$ *and of G itself.*
(ii) *Similarly, G contains a unique largest compact connected abelian normal subgroup which is also central, namely,*

$$Z(\mathrm{MaxC}(G)_0) = \mathrm{comp}(Z(\mathrm{MaxCE}(G))_0) = \mathrm{comp}(Z(G)_0).$$

(iii) *The factor group* $G/\mathrm{comp}(Z(G)_0)$ *has no nontrivial compact connected central subgroups.*

Recall the three-dimensional Heisenberg group $N = (\mathbb{R} \cdot X + \mathbb{R} \cdot Y + \mathbb{R} \cdot Z, *)$, $[X, Y] = Z$, $x * y = x + y + \frac{1}{2} \cdot [x, y]$. Let $G = N/\mathbb{Z} \cdot Z$. Then $Z(G) = \mathrm{MaxC}(G) = C(G) = (\mathbb{R} \cdot Z)/(\mathbb{Z} \cdot Z) \cong \mathbb{R}/\mathbb{Z} = \mathbb{T}$. But $G/\mathrm{MaxC}(G) \cong \mathbb{R}^2$ is abelian and therefore procompact; that is $C(G/C(G)) = G/C(G) = \mathbb{R}^2 \neq \{0\}$. Thus, in general, $G/C(G)$ and $G/\mathrm{MaxC}(G)$ may have compactly embedded normal subgroups.

Sometimes it facilitates notation if one introduces a new name.

Definition 103 (12.58). A group G will be called *compactly simple*, if it is a topological group in which every compact normal subgroup is singleton.

With this terminology we finally get:

Theorem 104 (Largest Compact Normal Subgroup of a Pro-Lie Group; 12.59). *Every connected pro-Lie group G has a unique largest compact normal subgroup* $\mathrm{MaxK}(G)$ *and $G/\mathrm{MaxK}(G)$ is a compactly simple connected pro-Lie group.*

Yamabe's Theorem ([206], [207]) states that *every locally compact almost connected group is a pro-Lie group*. In a locally compact pro-Lie group G, all sufficiently small members of the standard filter basis $\mathcal{N}(G)$ are compact. In view of these two facts we see that

a locally compact almost connected Lie group is compactly simple only if it is a compactly simple Lie group.

Naturally, we now wish to know as much as possible about compactly simple groups. The following is a step in this direction.

Proposition 105 (12.60). *Let G be a connected pro-Lie group without nontrivial compact central subgroups. Then the following conclusions hold.*

(i) *Every compact connected normal subgroup is semisimple and center-free.*
(ii) *The center $Z(G)$ of G is an abelian pro-Lie group isomorphic to $Z(G)_0 \times H$, where the identity component $Z(G)_0$ of the center $Z(G)$ is a vector group isomorphic to $\mathbb{R}^I$ for some set I and where H is a totally disconnected subgroup of $Z(G)$ which contains no compact subgroups.*
(iii) *The nilradical $N(G)$, that is, the largest pronilpotent connected closed normal subgroup, is simply connected and thus is isomorphic to $(\mathfrak{n}(\mathfrak{g}), *)$ where $\mathfrak{n}(\mathfrak{g})$ is the nilradical of $\mathfrak{g} = \mathfrak{L}(G)$.*
(iv) *The radical $R(G)$ is a semidirect product $V \rtimes K$ of a simply connected normal subgroup V containing $N(G)$ and a compact connected group.*

The Analytic Subgroups Whose Lie Algebras Equal That of the Full Group

We have the ingredients for significant results on the unique minimal analytic subgroup $A(\mathfrak{g}, G) = \langle \exp_G \mathfrak{g} \rangle$ with full Lie algebra $\mathfrak{g}$ in any pro-Lie group G. We discover, that a certain characteristic, but not necessarily connected, supergroup of the radical $R(G)$ plays an important role.

Definition 106 (12.63). Let G be a connected pro-Lie group, let $q\colon G \to G/R(G)$ denote the quotient map and define $Q(G) \stackrel{\text{def}}{=} q^{-1}(\text{KZ}(G/R(G)))$. Call $Q(G)$ the *extended radical* of G.

So the extended radical is an extension of the radical by a compact totally disconnected abelian group. Of course we would like to know some typical properties of this extended radical.

Proposition 107 (12.64). *For a connected pro-Lie group G, the extended radical $Q(G)$ satisfies the following conditions:*

(i) *$Q(G)$ is a prosolvable characteristic subgroup of G whose identity component is the radical $R(G)$.*
(ii) *$Q(G)$ contains a closed subgroup V that is normal in $Q(G)$, contains the nilradical $N(G)$ of G and is simply connected modulo* $\text{KZ}(G)_0$.
(iii) *$Q(G)$ contains a maximal compact abelian subgroup K such that $Q(G) = VK$ and $V \cap K =$* $\text{KZ}(G)_0$.
(iv) *K contains* $\text{KZ}(G)$.
(v) *There is a totally disconnected compact subgroup D of K such that $K = A(\mathfrak{L}(K), K)D$ and $Q(R) = R(G)D$.*

Now we are ready for a completely general result on the dense characteristic subgroup $A(\mathfrak{g}, G) = \langle \exp_G \mathfrak{g} \rangle$ of a connected pro-Lie group G.

Theorem 108 (Supplementing the Minimal Analytic Subgroup Generated by the Full Lie Algebra; 12.65). *Let G be a connected pro-Lie group and K a maximal compact subgroup of the extended radical $Q(R)$. Then*

$$G = K \cdot A(\mathfrak{g}, G).$$

There is a totally disconnected compact subgroup D of K such that

$$G = D \cdot A(\mathfrak{g}, G) \quad \textit{and} \quad Q(G) = R(G)D.$$

There is a connected compact abelian subgroup C of G containing K. In particular, $G = C \cdot A(\mathfrak{g}, G)$ is generated by divisible groups.

This result has many consequences. One immediate outcome is the following:

Corollary 109 (12.66). *The abstract group $G/A(\mathfrak{g}, G)$ is abelian. The algebraic commutator subgroup of G is contained in $A(\mathfrak{g}, G)$.*

The next corollary, however, requires the full power of Open Mapping Theorem for Almost Connected Pro-Lie Groups (Theorem 8). In addition one has to ascertain that the group G acts automorphically on the simply connected universal group $\widetilde{G}$ via a homomorphism $\alpha\colon G \to \text{Aut}(\widetilde{G})$.

Corollary 110 (The Resolution Theorem for Connected Pro-Lie Groups; 12.68). *Let G be a connected pro-Lie group. Then there is a totally disconnected compact subgroup D of the extended radical and a quotient morphism of pro-Lie groups*

$$\delta\colon \widetilde{G} \rtimes_\alpha D \to G, \quad \delta(x,k) = \pi_G(x)k.$$

The morphism

$$x \mapsto (x^{-1}, \pi_G(x))\colon \pi_G^{-1}(D) \to \ker \delta$$

is an isomorphism of prodiscrete groups. The morphism δ induces an isomorphism $\mathfrak{L}(\delta)$ of pro-Lie algebras. There is an exact sequence

$$1 \to \pi^{-1}(D) \to \widetilde{G} \rtimes D \xrightarrow{\delta} G \to 1.$$

The following consequence is a kind of Levi–Mal'cev decomposition theorem on the group level. We recall that it is rather frustrating in general to imitate the clean Levi–Mal'cev splitting of the Lie algebra on the group level.

Corollary 111 (12.69). *Let G be a connected pro-Lie group and let $\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) + \mathfrak{s}$ be a Levi–Mal'cev decomposition. Then $G = Q(G) \cdot A(\mathfrak{s}, G)$, $A(\mathfrak{s}, G) = \langle \exp_G \mathfrak{s} \rangle$.*

Under special conditions more elegant conclusions may be drawn, like for instance the following:

Theorem 112 (12.70). *Let G be a connected pro-Lie group and let $\mathfrak{g} = \mathfrak{r}(\mathfrak{g}) + \mathfrak{s}$ be a Levi decomposition of its Lie algebra. Assume the following hypotheses:*

(i) $[\mathfrak{r}(\mathfrak{g}), \mathfrak{s}] = \{0\}$, *and*
(ii) $R(G)/\operatorname{KZ}(G)_0$ *is simply connected.*

Then $G = \operatorname{KZ}(G) \cdot A(\mathfrak{g}, G)$.

The preceding results motivate a further definition:

Definition 113 (12.72). A connected pro-Lie group G will be called *centrally supplemented* if

$$G = Z(G)A(\mathfrak{g}, G) = Z(G)\pi_G(\widetilde{G}).$$

By Theorem 112, for example, all reductive connected pro-Lie groups and all prosolvable pro-Lie groups which are simply connected modulo the maximal connected central compact subgroup are centrally supplemented. The Center-Free Embedding Lemma 35 gives us plenty center-free metabelian connected pro-Lie groups G in which $A(\mathfrak{g}, G)$ is a proper subgroup and which, therefore, are not centrally supplemented.

For centrally supplemented connected pro-Lie groups, there exist more elegant versions of Theorem 108 and Corollary 110, namely:

Theorem 114 (The Resolution Theorem for Centrally Supplemented Pro-Lie Groups; 12.74). *Let G be a connected centrally supplemented pro-Lie group. Then there is a prodiscrete central subgroup D of G such that there is a quotient morphism*

$$\delta\colon \widetilde{G} \times D \to G, \quad \delta(x,d) = \pi_G(x)d$$

inducing an isomorphism $\mathfrak{L}(\delta)$ *of Lie algebras, and the kernel* $\ker\delta$ *is prodiscrete and isomorphic to* $\pi^{-1}(D) \cap A(\mathfrak{g}, G)$. *There is an exact sequence*

$$1 \to \pi^{-1}(D) \to \widetilde{G} \times D \xrightarrow{\delta} G \to 1.$$

We noted in discussing the universal morphism $\pi_G\colon \widetilde{G} \to G$ that already the example of compact connected abelian groups shows that for non-Lie groups, π_G fails rather significantly to be a universal covering morphism (which it is if G is a Lie group!). In general it is neither surjective, nor open, let alone a local isomorphism. The morphism δ in the Resolution Theorems is as close as one can get to adjust π_G in such a fashion that something resembling a universal covering results: At least it is surjective, open, and has a prodiscrete kernel. Resolution Theorems for compact abelian groups and compact groups were described for the first time in our book [102] in Chapters 8 (Theorem 8.20) and 9 (Theorem 9.51).

One of the applications we are making of (a special case of) the Resolution Theorems is that we can prove a fact on maximal compact subgroups of connected pro-Lie groups that is overdue. We know that maximal compact *connected* subgroups exist and are conjugate, and we know that a unique maximal compact *normal* subgroup exists. Yet up to this point we do not know whether maximal compact subgroups exist. Now in this direction we show that maximal compact subgroups of connected pro-Lie groups exist and are connected, and therefore are all conjugate. However, the situation is really better. We shall come up with a significant result, that illustrates quite well our motivation to invest so much energy into the investigation of compact and notably maximal compact subgroups. The core theorem is the following.

Theorem 115 (Theorem on the Maximal Compact Subgroups; 12.81). *Let G be an arbitrary connected pro-Lie group. Then:*

(i) *G has at least one maximal compact subgroup C.*
(ii) *Every maximal compact subgroup is connected.*
(iii) *All maximal compact subgroups are conjugate under inner automorphisms.*
(iv) *There exists a set J and a homeomorphism $\varepsilon\colon C \times \mathbb{R}^J \to G$ such that $\varepsilon(C \times \{0\}) = C$. Also $G_a = \varepsilon(C_a \times \mathbb{R}^J) = \langle \exp_G \mathfrak{g} \rangle$.*

None of these statements is obvious. Our proof gives many more details about Statement (iv). This last one, taken together with the Borel–Scheerer–Hofmann Splitting Theorem (see [102, Theorem 9.39], and see also Theorem 11.8 in this book) gives the following statement:

Theorem 116 (Theorem on the Topological Splitting of Pro-Lie Groups; 12.87). *Every connected pro-Lie group is homeomorphic to a direct product of a compact connected semisimple group, a compact connected abelian group, and a space $\mathbb{R}^J$ for a set J.*

Clearly, this gives us a characterisation of the local compactness of a connected pro-Lie group. The cardinal of J is invariantly attached to G, and we are justified (as we shall see in detail) to call this cardinal the dimension of the quotient space G/C.

We shall call this cardinal the *manifold rank* of G. The group G is locally compact if and only if its manifold rank is finite.

These theorems will be anticipated a few times in the course of the book, so for instance in the structure theorem for connected abelian pro-Lie groups (Theorem 23) and Lemma 22, or the Structure Theorem 36 for Simply Connected Pro-Lie Groups.

An obvious but striking consequence is that all the information that algebraic topology (homotopy, homology, cohomology) gives us on compact connected groups yields this precise information on connected pro-Lie groups, since according to the preceding theorems each maximal compact subgroup C of a connected pro-Lie group G is a homotopy deformation retract and thus is homotopically equivalent (that is, isomorphic in the homotopy category) to G.

Another consequence will be that a connected pro-Lie group G is locally compact if and only if the linear codimension of the Lie algebra $\mathfrak{L}(C)$ of a maximal compact subgroup C of G is finite. This is equivalent to saying that the factor group $G/\operatorname{MaxK}(G)$ of G modulo the unique largest compact normal subgroup is a Lie group. (The proof of this fact uses Yamabe's Theorem.)

It does perhaps not come as a surprise that with the powerful tools that these results provide one can successfully revisit the scenes of earlier investigations. One example is the Open Mapping Theorem which we presented early in this overview, namely, in Theorem 8, being aware that it requires tools that become available at a later stage. The setting of Open Mapping Theorems is always the same: We have a surjective morphism $f: G \to H$ of topological groups and we are hoping for additional sufficient conditions which will allow us to conclude that f is an open mapping and thus is equivalent to a quotient morphism. The perennial illustration is the identity map $\mathbb{R}_d \to \mathbb{R}$ from the discrete additive group of real numbers to the additive group of real numbers in its natural topology. This is a bijective morphism between abelian Lie groups that fails to be open. If G and H are pro-Lie groups we have seen in Theorem 8 that if G/G_0 is compact then f is open. We will show the following result.

Theorem 117 (Alternative Open Mapping Theorem; 12.85). *Assume that $f: G \to H$ is a surjective morphism from a pro-Lie group G onto a connected pro-Lie group H and that the quotient group $G/\ker f$ is complete. If the morphism $\mathfrak{L}(f): \mathfrak{L}(G) \to \mathfrak{L}(H)$ of pro-Lie algebras is surjective, then f is open.*

In Theorem 4.20 and its corollaries which we previewed in Theorem 15 above we shall show that if f is open, then $\mathfrak{L}(f)$ is a quotient morphism. We can express the gist of the situation by saying

> *A bijective morphism $f: G \to H$ of pro-Lie groups, of which at least one is connected, is an isomorphism if and only if it induces an isomorphism $\mathfrak{L}(f): \mathfrak{L}(G) \to \mathfrak{L}(H)$ of Lie algebras.*

When formulated in this fashion it becomes evident that this circle of ideas belongs genuinely to a Lie theory of pro-Lie groups.

This last theorem permits us to prove a structure theorem, well known but not trivial for Lie groups and a hard fact for pro-Lie groups:

Theorem 118 (12.87). *The center of a connected pro-Lie group is contained in some closed connected abelian subgroup.*

Part 7. Local Splitting According to Iwasawa

We already referred to Iwasawa's paper [120] of 1949 as one of the most influential contributions to the structure theory of topological groups. One of the results in that paper turned out to be immensely practical and like no other result it illustrates how the structure theory of locally compact groups reduces largely (but not completely!) to the structure theory of compact groups and classical Lie theory.

Theorem 119 (Iwasawa's Local Splitting Theorem for Connected Locally Compact Groups). *Any identity neighborhood of a connected locally compact group G contains a compact normal subgroup N such that G/N is a Lie group and that the groups G and $N \times G/N$ are locally isomorphic.*

This enables us to produce a quotient morphism from $N \times \widetilde{G/N}$ to G, where $\widetilde{L}$ is the universal covering group of a connected Lie group L and where the kernel of this morphism is discrete. In general, N is not connected, but this quotient map is the next best thing to a covering morphism.

Any general theory of pro-Lie groups will have to be measured by the elucidation that it offers of this result. Does it generalize to connected pro-Lie groups? The answer, alas, is no. In Chapter 13, which deals with this topic we illustrate this negative situation by producing a center-free pronilpotent pro-Lie group, a metabelian center-free pro-Lie group and a class two nilpotent pro-Lie group, none of which permits locally splitting in the sense of the Iwasawa Local Splitting Theorem. The latter example is *the* Heisenberg group of the realm of pro-Lie groups. So what, if anything, is the obstruction that prevents us from proving a straightforward generalisation?

Let G be an arbitrary pro-Lie group. Note right away that we do not insist that it be connected. Its identity component G_0 has a largest connected normal pronilpotent subgroup $N(G_0)$ which we call the *nilradical* which we already mentioned several times. Naturally, it contains the identity component $Z(G_0)_0$ of the center of G_0. We shall call the quotient $N(G_0)/Z(G_0)_0$ the *nilcore* of G, and we shall also abbreviate it by $\mathcal{N}(G)$. We shall show (13.16) that the nilcore *always* is a simply connected pronilpotent pro-Lie group. We know from Theorem 34 and its context that its Lie algebra $\mathfrak{n}$ has an everywhere defined Campbell–Hausdorff multiplication

$$* : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}, \quad X * Y = X + Y + \tfrac{1}{2}[X, Y] + \cdots$$

such that the exponential function $\exp\colon (\mathfrak{n}, *) \to \mathcal{N}(G)$ defines an isomorphism of pro-Lie groups. The Lie algebra of the nilcore determines its structure completely. It is homeomorphic to the weakly complete topological vector space underlying $\mathfrak{n}$ which is isomorphic to $\mathbb{R}^J$ for a set J with a uniquely defined cardinality which we also call the *nildimension* of G, written $\nu(G)$. That is a cardinal valued invariant of G.

The counterexamples mentioned above all have infinite nildimension. We show in Chapter 13 the following:

Theorem 120 (The Local Splitting Theorem for Pro-Lie Groups; 13.19). *Let G be a pro-Lie group and assume its nildimension $\nu(G)$ to be finite. Then every identity neighborhood contains an almost connected normal pro-Lie subgroup N such that G/N is a Lie group and the groups G and $N \times G/N$ are locally isomorphic.*

No connectivity is required, but apart from the hypothesis that nildimension be finite nothing is required – except, of course, that we are talking about pro-Lie groups. We recognize, that the obstruction is the nilcore. On the other hand, we should also recognize that it is easy to visualize even nilpotent pro-Lie groups of arbitrary nildimension that do have local splitting, for instance, if H_3 is the three-dimensional Heisenberg group, then $G \stackrel{\text{def}}{=} H_3^J$ for an infinite set J is a class two nilpotent pro-Lie algebra of nildimension $\nu(G) = \text{card}\, J$, and by its very construction, G has local splitting.

The hypothesis $\nu(G) < \infty$ implies global structural results:

Theorem 121 (The Finite Nildimension Theorem;13.19, 13.22). *Let G be a pro-Lie group. Then the following statements are equivalent:*

(i) *The nildimension $\nu(G)$ is finite.*
(ii) *G is locally isomorphic to the product of a closed normal almost connected subgroup with a reductive identity component, and a connected Lie group.*

If these conditions are satisfied, then there is a compact totally disconnected central subgroup D of G and an open surjective morphism $\mu\colon D \times \widetilde{G} \to G$, $\mu(d, x) = d\pi_G(x)$ with a prodiscrete kernel.

However, in Theorems 120 and 121, the hypothesis $\nu(G) < \infty$ is acceptable insofar as it is implied by the assumption that G be locally compact. Indeed: if G is locally compact so is its nilcore $\mathcal{N}(G)$, and since we saw that $\mathcal{N}(G)$ is homeomorphic to $\mathbb{R}^J$ with card $J = \nu(G)$ we see that $\mathcal{N}(G)$ is locally compact iff $\nu(G) < \infty$. Therefore we deduce the Iwasawa Local Splitting Theorem for Locally Compact Groups in the following form which is more general than the classical version:

Corollary 122 (Iwasawa's Local Splitting Theorem Revisited). *Let G be a* locally compact *pro-Lie group. Then there is an open subgroup $\mathbb{G}$ of G and there are arbitrarily small compact normal subgroups N of $\mathbb{G}$ such that $\mathbb{G}/N$ is a Lie group and G and $N \times \mathbb{G}/N$ are locally isomorphic.*

Classically, the Iwasawa Local Splitting Theorem 119 required the hypothesis of connectivity. The presentation of the local splitting theory in the frame work of pro-Lie group theory, culminating in Theorem 120 and Corollary 118 does not require this hypothesis, thanks to an effective pro-Lie theory.

Pro-Lie-tarians of the world: unite!
Karl Morrix, 2007