Introduction to Teichmüller theory, old and new

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Teichmüller theory is one of those few wonderful subjects which bring together, at an equally important level, fundamental ideas coming from different fields. Among the fields related to Teichmüller theory, one can surely mention complex analysis, hyperbolic geometry, the theory of discrete groups, algebraic geometry, low-dimensional topology, differential geometry, Lie group theory, symplectic geometry, dynamical systems, number theory, topological quantum field theory, string theory, and there are many others.

Let us start by recalling a few definitions.

Let $S_{\mathfrak{g},\mathfrak{p}}$ be a connected orientable topological surface of genus $\mathfrak{g} \geq 0$ with $\mathfrak{p} \geq 0$ punctures. Any such surface admits a complex structure, that is, an atlas of charts with values in the complex plane $\mathbb{C}$ and whose coordinate changes are holomorphic. In the classical theory, one considers complex structures $S_{\mathfrak{g},\mathfrak{p}}$ for which each puncture of $S_{\mathfrak{g},\mathfrak{p}}$ has a neighborhood which is holomorphically equivalent to a punctured disk in $\mathbb{C}$. To simplify the exposition, we shall suppose that the orientation induced on $S_{\mathfrak{g},\mathfrak{p}}$ by the complex structure coincides with the orientation of this surface. Homeomorphisms of the surface act in a natural manner on atlases, and two complex structures on $S_{\mathfrak{g},\mathfrak{p}}$ are said to be equivalent if there exists a homeomorphism of the surface which is homotopic to the identity and which sends one structure to the other. The surface $S_{\mathfrak{g},\mathfrak{p}}$ admits infinitely many non-equivalent complex structures, except if this surface is a sphere with at most three punctures.

To say things precisely, we introduce some notation. Let $\mathcal{C}_{\mathfrak{g},\mathfrak{p}}$ be the space of all complex structures on $S_{\mathfrak{g},\mathfrak{p}}$ and let $\mathrm{Diff}^+(S_{\mathfrak{g},\mathfrak{p}})$ be the group of orientation-preserving diffeomorphisms of $S_{\mathfrak{g},\mathfrak{p}}$. We consider the action of $\mathrm{Diff}^+(S_{\mathfrak{g},\mathfrak{p}})$ by pullback on $\mathcal{C}_{\mathfrak{g},\mathfrak{p}}$. The quotient space $\mathcal{M}_{\mathfrak{g},\mathfrak{p}} = \mathcal{C}_{\mathfrak{g},\mathfrak{p}}/\mathrm{Diff}^+(S_{\mathfrak{g},\mathfrak{p}})$ is called Riemann's moduli space of deformations of complex structures on $S_{\mathfrak{g},\mathfrak{p}}$. This space was considered by G. F. B. Riemann in his famous paper on Abelian functions, *Theorie der Abel'schen Functionen*, Crelle's Journal, Band 54 (1857), in which he studied moduli for algebraic curves. In that paper, Riemann stated, without giving a formal proof, that the space of deformations of equivalence classes of conformal structures on a closed orientable surface of genus $\mathfrak{g} \geq 2$ is of complex dimension $3\mathfrak{g} - 3$.

The Teichmüller space $\mathcal{T}_{\mathfrak{g},\mathfrak{p}}$ of $S_{\mathfrak{g},\mathfrak{p}}$ was introduced in the 1930s by Oswald Teichmüller. It is defined as the quotient of the space $\mathcal{C}_{\mathfrak{g},\mathfrak{p}}$ of complex structures by the group $\mathrm{Diff}^+_0(S_{\mathfrak{g},\mathfrak{p}})$ of orientation-preserving diffeomorphisms of $S_{\mathfrak{g},\mathfrak{p}}$ that are isotopic to the identity. The group $\mathrm{Diff}^+_0(S_{\mathfrak{g},\mathfrak{p}})$ is a normal subgroup of $\mathrm{Diff}^+(S_{\mathfrak{g},\mathfrak{p}})$, and the quotient group $\Gamma_{\mathfrak{g},\mathfrak{p}} = \mathrm{Diff}^+(S_{\mathfrak{g},\mathfrak{p}})/\mathrm{Diff}^+_0(S_{\mathfrak{g},\mathfrak{p}})$ is called the mapping class group of $S_{\mathfrak{g},\mathfrak{p}}$ (sometimes also called the modular group, or the Teichmüller modular group) of $S_{\mathfrak{g},\mathfrak{p}}$. The mapping class group $\Gamma_{\mathfrak{g},\mathfrak{p}}$ acts on the Teichmüller space $\mathcal{T}_{\mathfrak{g},\mathfrak{p}}$, and the quotient of $\mathcal{T}_{\mathfrak{g},\mathfrak{p}}$ by this action is Riemann's moduli space $\mathcal{M}_{\mathfrak{g},\mathfrak{p}}$.

During a remarkably brief period of time (1935–1941), Teichmüller wrote about thirty papers which laid the foundations of the theory which now bears his name. After Teichmüller's death in 1943 (at the age of 30), L. V. Ahlfors, L. Bers, H. E. Rauch and several of their students and collaborators started a project that provided a solid grounding for Teichmüller's ideas. The realization of this project took more than two decades, during which the whole complex-analytic theory of Teichmüller space was built. In the 1970s, W. P. Thurston opened a new and wide area of research by introducing beautiful techniques of hyperbolic geometry in the study of Teichmüller space and of its asymptotic geometry. Thurston's work highlighted this space as a central object in the field of low-dimensional topology. Thurston's ideas are still being developed and extended today by his students and followers. In the 1980s, there also evolved an essentially combinatorial treatment of Teichmüller and moduli spaces, involving techniques and ideas from high-energy physics, namely from string theory. The current research interests in Teichmüller theory from the point of view of mathematical physics include the quantization of Teichmüller space using the Weil–Petersson symplectic and Poisson geometry of this space, as well as gauge-theoretic extensions of these structures. The quantization theories are also developed with the purpose of finding new invariants of hyperbolic 3-manifolds.

My aim in the rest of this introduction is twofold: first, to give an exposition of the structure of Teichmüller theory that is studied in this volume of the Handbook, and second, to give an overview of the material contained in this volume. The goal is to give a flavour of the subject. Of course, the exposition will be very sketchy at some places.[1]

1 An overview of the structure of Teichmüller space

Teichmüller space $\mathcal{T}_{\mathfrak{g},\mathfrak{p}}$ has a natural topology which makes it homeomorphic to an open ball of dimension $6\mathfrak{g} - 6 + 2\mathfrak{p}$. It has a complex structure which is induced by an embedding of this space as a bounded pseudo-convex domain of holomorphy in a Banach space. It carries several interesting metrics, in addition to the usual metrics that are associated to a bounded domain of holomorphy viz. the Carathéodory, the Kobayashi and the Bergman metrics. It also has a symplectic and a Poisson structure, various boundary structures and several other structures which I now briefly describe.

[1] In order not to overload this introduction, I decided not to give the complete references for the results that I cite. Most of the references are given in the chapters that follow this introduction and, furthermore, with the availability of electronic supports, it is nowadays easy to find the exact reference for a result, given its statement, its author's name and the year it was written.

1.1 Complex structure

Let $S_{\mathfrak{g},\mathfrak{p}}$ be as before a surface of genus $\mathfrak{g} \geq 0$ with $\mathfrak{p} \geq 0$ punctures. If the Euler characteristic of $S_{\mathfrak{g},\mathfrak{p}}$ is negative, then the Teichmüller space $\mathcal{T}_{\mathfrak{g},\mathfrak{p}}$ of $S_{\mathfrak{g},\mathfrak{p}}$ is a complex manifold of complex dimension $3\mathfrak{g} - 3 + \mathfrak{p}$. The bases of the complex analytic theory of Teichmüller space were developed by Ahlfors and Bers. There are several ways of defining the complex structure of Teichmüller space. All of them use deep results from analysis, and giving the details of this theory would be far beyond the scope of this introduction. We only mention that there is a famous embedding (called Bers' embedding) which realizes $\mathcal{T}_{\mathfrak{g},\mathfrak{p}}$ as an open bounded pseudo-convex (but not strictly pseudo-convex) domain in $\mathbb{C}^{3\mathfrak{g}-3+\mathfrak{p}}$. This embedding depends on the choice of a basepoint in Teichmüller space. Classically, the holomorphic cotangent space at a point of Teichmüller space is described by the Kodaira–Spencer deformation theory, which parametrizes the deformations of a conformal structure in terms of the sheaf of holomorphic vector fields on the given Riemann surface. In Bers' embedding of $\mathcal{T}_{\mathfrak{g},\mathfrak{p}}$, the holomorphic cotangent space at a point is identified with the vector space of holomorphic quadratic differentials with simple poles at the punctures on a Riemann surface representing the point. By the theory of quasiconformal mappings, any complex structure on a surface is specified by a Beltrami differential of norm less than one, and this leads to a description of the holomorphic tangent space at a point of Teichmüller space as a vector space of Beltrami differentials of norm less than one divided by a subspace of differentials which induce trivial deformations. We note that besides providing a description of the complex structure of Teichmüller space, these descriptions of the tangent and cotangent spaces are used to give an infinitesimal description of the Teichmüller metric which we recall below, and that as such, the tangent and cotangent spaces are Banach spaces and not Hilbert spaces, which reflects the fact that the Teichmüller metric is a Finsler metric and not a Riemannian metric. We also note that the descriptions of the tangent and cotangent spaces by holomorphic quadratic differentials and Beltrami differentials are key tools for the definition of Hermitian metrics on Teichmüller space, for instance the Weil–Petersson metric which we discuss below.

It follows from the definitions that the mapping class group $\Gamma_{\mathfrak{g},\mathfrak{p}}$ preserves this complex structure, that is, it acts on Teichmüller space by biholomorphic maps. Conversely, by a result of H. L. Royden, any biholomorphic automorphism of Teichmüller space is induced by an element of the mapping class group. The quotient moduli space $\mathcal{M}_{\mathfrak{g},\mathfrak{p}}$ is a complex orbifold of dimension $3\mathfrak{g} - 3 + \mathfrak{p}$.

1.2 Metric structures

Teichmüller space carries various metrics, each arising naturally from a particular viewpoint on the space. There are three metrics on Teichmüller space which will be considered in some detail in this volume: Teichmüller's metric, the Weil–Petersson metric and Thurston's asymmetric metric (which, because of its asymmetry, is not a metric in the usual sense of the word). I will say a few words about each of these metrics.

Teichmüller's metric. This metric is obtained by first defining the distance between two conformal structures g and h on the surface $S_{\mathfrak{g},\mathfrak{p}}$ to be $\frac{1}{2}\inf_f \log K(f)$, where the infimum is taken over all quasiconformal homeomorphisms $f\colon (S_{\mathfrak{g},\mathfrak{p}}, g) \to (S_{\mathfrak{g},\mathfrak{p}}, h)$ that are isotopic to the identity and where $K(f)$ is the quasiconformal dilatation of f. Teichmüller showed that the infimum is realized by a quasiconformal homeomorphism, and he gave a description of this homeomorphism in terms of a quadratic differential on the domain conformal surface $(S_{\mathfrak{g},\mathfrak{p}}, g)$. This distance function on the set of conformal structures is invariant by the action of the group of diffeomorphisms isotopic to the identity on each factor, and it induces a distance function on Teichmüller space $\mathcal{T}_{\mathfrak{g},\mathfrak{p}}$, which is Teichmüller's metric. Teichmüller's metric is a complete Finsler metric which is not Riemannian unless the surface is a torus, in which case Teichmüller space, equipped with Teichmüller's metric, is isometric to the 2-dimensional hyperbolic plane. Teichmüller's metric is geodesically convex, that is, any two points are joined by a unique geodesic segment. The metric is also uniquely geodesic, that is, the geodesic segment joining two arbitrary points is unique. Furthermore, any geodesic segment can be extended in a unique way (up to parametrization) to a geodesic line. For some time, it was believed that Teichmüller's metric was nonpositively curved in the sense of Busemann, that is, that the distance function between two parametrized geodesics is convex. Indeed, in 1959, S. Kravetz published a paper containing this statement as one of the main results. In 1971, M. Linch found a mistake in the arguments of Kravetz and in 1975, H. Masur showed that the statement is false, by providing examples of two geodesic rays whose associated distance function is bounded, and therefore is not convex. By a result of Masur and Wolf (1995), Teichmüller's metric is not Gromov hyperbolic. Despite these facts, much of the work that has been done on Teichmüller's metric is motivated by analogies with metrics on simply connected manifolds of negative curvature. We can mention in this connection that there is a Teichmüller geodesic flow which preserves a natural measure, which induces a quotient geodesic flow on moduli space, and that this quotient flow, like the geodesic flow of a finite volume hyperbolic manifold, is ergodic. (This result was obtained independently in 1982 by H. Masur and W. Veech.)

The mapping class group $\Gamma_{\mathfrak{g},\mathfrak{p}}$ of $S_{\mathfrak{g},\mathfrak{p}}$ acts on $\mathcal{T}_{\mathfrak{g},\mathfrak{p}}$ by isometries of the Teichmüller metric and, conversely, by a result of Royden (1970), every isometry group of the Teichmüller metric is induced by an element of the mapping class group. At the same time, Royden showed that the Teichmüller metric can be recovered directly from the complex structure of Teichmüller space, by proving that this metric coincides with the Kobayashi metric of this space. (In fact, Royden proved the first result as a corollary of the second.)

The Weil–Petersson metric. This metric is a Riemannian metric which is also closely related to the complex structure of Teichmüller space. Its definition starts with an L^2-norm on the space of quadratic differentials at each point X of Teichmüller space, considered as the holomorphic cotangent space at that point. The definition of this norm makes use of the hyperbolic metric of a surface representing this point. More precisely, this norm is given by the formula $\|\phi\|^2 = \int_X \rho^{-2}(z)|\phi(z)|^2|dz|^2$, with $\rho(z)|dz|$ being the length element on the point X represented by a hyperbolic surface. The Weil–Petersson inner product on the tangent space is then defined by taking a dual of this L^2 inner product, the duality between the tangent and cotangent spaces being defined through a natural pairing between quadratic differentials and Beltrami differentials. Several chapters of this Handbook deal with this metric, and we shall list here some of its important geometric features. L. Ahlfors showed that the Weil–Petersson metric is Kähler. It has variable negative sectional curvature (a result proved independently by A. Tromba and S. Wolpert). It is geodesically convex but it is not complete (a result obtained by Masur and by Wolpert). Masur studied the completion of this metric, and he described the points on the frontier of this completion as Riemann surfaces with nodes [2]. More precisely, Masur identified the Weil–Petersson completion with the augmented Teichmüller space, whose frontier is a stratified union of lower-dimensional spaces, each of which is the Teichmüller space of a nodal surface. He showed that in some precise sense the tangential component of the Weil–Petersson metric on Teichmüller space extends to the Weil–Petersson metric of these boundary Teichmüller spaces. The sectional curvature of the Weil–Petersson metric is unbounded (its supremum is zero and its infimum is $-\infty$), except in the cases where the complex dimension of Teichmüller space is 1. Ahlfors showed that the holomorphic sectional curvature of the Weil–Petersson metric is negative. Wolpert showed that the holomorphic sectional curvature and the Ricci curvature of this metric have negative upper bounds (proving a conjecture made by Royden), with these upper bounds expressible in terms of the topological type of the surface. The mapping class group acts by biholomorphic isometries on Teichmüller space equipped with the Weil–Petersson metric, and this metric descends to a metric on moduli space. Masur and Wolf proved (2002) that conversely, every Weil–Petersson isometry is induced by an element of the mapping class group, except for a few surfaces of low genus. The untreated cases were completely analyzed later on by J. Brock and D. Margalit (2004). J. Brock and B. Farb (2004) showed that the Weil–Petersson metric is not Gromov hyperbolic, except if the surface $S_{\mathfrak{g},\mathfrak{p}}$ is a torus with at most two punctures or a sphere with at most five punctures. In all the other cases, the space has higher rank in the sense of Gromov. S. Yamada (2001) showed that the Weil–Petersson completion of the Weil–Petersson metric is a CAT(0)-space (that is, a non-positively curved space in the sense of Cartan–Alexandrov–Toponogov). Brock initiated a new point of view on the

[2]We note that surfaces with nodes appear naturally in complex analysis. A node is the simplest type of a singularity of a complex curve. More precisely, while a smooth point of a complex curve is a point which admits a neighborhood which is biholomorphically equivalent to an open disk in $\mathbb{C}$, a node is a point which admits a neighborhood which is biholomorphically equivalent to a space obtained by taking a disjoint union of two disks in $\mathbb{C}$ and gluing them along a point. Riemann surfaces with nodes are complex curves whose points are either smooth points or nodes, and they appear naturally as degenerations of smooth complex curves (that is, of Riemann surfaces). Thus, it is not surprising that surfaces with nodes appear as boundary structures of spaces of Riemann spaces.

Weil–Petersson metric, by establishing a quasi-isometry between this metric and the metric on the vertices of the pants graph of the surface induced by the simplicial metric on the 1-skeleton. This work has connections with volumes of quasi-Fuchsian hyperbolic three-manifolds, and it also leads to beautiful results on the global behaviour of Weil–Petersson geodesics, quasi-geodesics of the Weil–Petersson metric being in correspondence with quasi-geodesics in the pants graph. There is a result which has a similar flavour, on the comparison between quasi-geodesics of the Teichmüller metric, and the quasi-geodesics in the complex of curves, which is described in Chapter 10 by Ursula Hamenstädt.

There have been other recent breakthroughs on the Weil–Petersson metric, by S. Wolpert, R. Wentworth, S. Yamada, M. Mirzakhani, B-Y. Chen, Z. Huang, joint work by S. Y. Cheng and S. T. Yau and by K. Liu, X. Sun and S. T. Yau, and by others.

We also mention that the Weil–Petersson metric can be recovered through Bonahon's embedding of Teichmüller space into the space of geodesic currents.

Recent work on the Weil–Petersson metric will be reported on in Volume II of this Handbook.

Thurston's asymmetric metric. This metric was introduced by Thurston in 1986, in his paper *Minimal Stretch maps between hyperbolic surfaces* (which is still in an unpublished form). Here, one considers Teichmüller space as the set of isotopy classes of complete finite-area hyperbolic structures on $S_{\mathfrak{g},\mathfrak{p}}$. If g and h are such structures, one defines their "distance" to be the logarithm of the infimum of Lipschitz constants of all diffeomorphisms $\varphi\colon (S_{\mathfrak{g},\mathfrak{p}}, g) \to (S_{\mathfrak{g},\mathfrak{p}}, h)$ which are isotopic to the identity, the Lipschitz constant of φ being, by definition, $\sup \frac{d_h(\varphi(x),\varphi(y))}{d_g(x,y)}$, the supremum being taken over all distinct pairs of points x and y in $S_{\mathfrak{g},\mathfrak{p}}$. This distance function on the set of hyperbolic structures is invariant by the action on each factor of the group $\mathrm{Diff}_0^+(S_{\mathfrak{g},\mathfrak{p}})$ of diffeomorphisms of $S_{\mathfrak{g},\mathfrak{p}}$ that are isotopic to the identity, and it induces a function on the Teichmüller space $\mathcal{T}_{\mathfrak{g},\mathfrak{p}}$ of $S_{\mathfrak{g},\mathfrak{p}}$, which we call Thurston's asymmetric metric. This is not a metric in the usual sense of the word since, as its name indicates, it does not satisfy the symmetry axiom. Despite this asymmetry (and in some cases because of this asymmetry), Thurston's metric has beautiful geometric properties. Several of these properties are expounded in Chapter 2 of this volume.

Let us note here that the study of asymmetric metrics has a long history, and we can mention in this respect the name of Herbert Busemann, who did a systematic study of such metrics, starting in the 1940s. The methods of synthetic geometry in the sense of Busemann can be used to investigate Thurston's asymmetric metric, for instance in the study of the behaviour of its geodesics, its isometries, its visual boundaries and so on. The techniques that are used by Thurston in the study of his asymmetric metric are those of basic hyperbolic geometry, and it is most satisfying to see that with these basic elementary techniques, a beautiful picture of a whole theory arises. There are several good questions which one can ask about this metric, and some of them are listed in Chapter 2.

The Carathéodory, the Kobayashi and the Bergman metrics. The Carathéodory and the Kobayashi "metrics" are semi-metrics[3] that are classically defined on any complex space, and that are invariant under biholomorphic self-mappings of this space. In the case of Teichmüller space, these semi-metrics are genuine metrics. Let us first recall the definitions, since they are easy to state and are most appealing from the point of view of synthetic geometry.

Let X be a complex space and let D be the unit disk in $\mathbb{C}$ equipped with its Poincaré metric. The Carathéodory semi-metric on X is defined by the formula $d_{\mathcal{C}}(x,y) = \sup_f d_D\big(f(x), f(y)\big)$ for all x and y in X, where the supremum is taken over all holomorphic maps $f\colon X \to D$.

The Kobayashi semi-metric on X is defined by a dual construction. One first defines a map $d'_{\mathcal{K}}\colon X \times X \to \mathbb{R}$ by $d'_{\mathcal{K}}(x,y) = \inf_{f,a,b} d_D(a,b)$ for all x and y in X, the infimum being taken over all holomorphic maps $f\colon D \to X$ and over all a and b in D satisfying $f(a) = x$ and $f(b) = y$. The map $d'_{\mathcal{K}}$ does not necessarily satisfy the triangle inequality, and the Kobayashi semi-metric $d_{\mathcal{K}}$ is defined as the largest semi-metric on X satisfying $d_{\mathcal{K}} \leq d'_{\mathcal{K}}$.

We already mentioned that H. L. Royden proved in 1970 that the Kobayashi metric of Teichmüller space coincides with the Teichmüller metric. In 1974, C. J. Earle proved that the Carathéodory metric on Teichmüller space is complete, and he raised the question of whether this metric coincides with the Teichmüller metric. He also asked the weaker question of whether this metric on Teichmüller space is proper (that is, whether every closed bounded set is compact). Both questions were answered by S. L. Krushkal'. In 1976, Krushkal' proved that the Carathéodory metric on Teichmüller space is proper, and in 1981 he proved that the Carathéodory metric does not coincide with the Teichmüller metric, unless the complex dimension of this space is one.

We also note that the Carathéodory and the Kobayashi metrics on a complex space have infinitesimal descriptions, and that in general they are Finsler and not Riemannian metrics.

The Bergman metric is a Kähler semi-metric that is associated to any bounded domain of holomorphy. Its definition uses the Bergman Kernel. The Bergman metric of Teichmüller space (seen as a bounded domain through Bers' embedding) is a genuine metric.

By a result of K. T. Hahn (1976), the Carathéodory metric on any bounded domain in $\mathbb{C}^N$ is bounded from above by the Bergman metric. Therefore, by using Earle's result on the completeness of the Carathéodory metric, the Bergman metric is complete. B.-Y. Chen showed (2004) that the Bergman metric of Teichmüller space is equivalent to the Teichmüller metric in the sense that there exists a positive constant C such that $\frac{1}{C} d_T \leq d_B \leq C d_T$, where d_T and d_B denote respectively the Teichmüller and the Bergman metrics. This result was also obtained, at about the same time, by K. Liu, X. Sun and S.-T. Yau.

[3] Recall that a semi-metric $d\colon X \times X \to [0, \infty[$ satisfies the axioms of a metric except possibly for the separation axiom $d(x,y) = 0 \Rightarrow x = y$.

A survey of the Carathéodory, the Kobayashi and the Bergman metrics on Teichmüller space is contained in a chapter by A. Fletcher and V. Markovich in Volume II of this Handbook.

Other metrics. There are many other interesting metrics on Teichmüller space. We mention a few of them, although they are not considered in this volume. S. Y. Cheng, and S. T. Yau showed that there exists a complete Kähler–Einstein metric of negative Ricci curvature on any bounded pseudo-convex domain in $\mathbb{C}^N$ (1975). Using Bers' embedding, this result shows that there is a unique complete Kähler–Einstein metric on Teichmüller space. This metric is invariant under the action of the mapping class group. It has constant negative scalar curvature, it descends to a complete Kähler–Einstein metric on moduli space and it has been thoroughly studied by S. Y. Cheng, N. Mok and S. T. Yau, and by algebraic geometers. In 2004, K. Liu, X. Sun and S.-T. Yau showed that this Kähler–Einstein metric is equivalent to the Teichmüller metric (a result which had been conjectured by Yau in 1986), which implies that the Bergman and the Kähler–Einstein metrics are equivalent.

We also mention a metric defined in the late 1990s by C. McMullen's (a metric which is now called McMullen's metric), which is a complete Kähler metric of bounded sectional curvature. It is obtained as a kind of interpolation between the Teichmüller and the Weil–Petersson metric. McMullen showed that this metric is equivalent to Teichmüller's metric and that it is Kähler-hyperbolic in the sense of Gromov.

Other complete Kähler metrics on Teichmüller space have been considered quite recently by K. Liu, X. Sun and S.-T. Yau. These authors asked a number of interesting questions about these metrics: computation of the L^2-cohomology groups, Ricci flow between these metrics, index theory, representation of the mapping class group in the middle-dimensional L^2-cohomology, and so on.

Let us note that a recent paper by S.-K. Yeung (Quasi-isometry of metrics on Teichmüller space, 2005) reports on the state of the art concerning quasi-isometries and equivalences between several of the metrics on Teichmüller space that we mentioned above. The same paper adds new results on that subject.

1.3 Symplectic and Poisson structures

Teichmüller space has a natural symplectic structure ω which is induced by its Weil–Petersson Kähler metric. This structure is invariant by the action of the mapping class group and it descends to a symplectic structure on moduli space. The most geometrically appealing description of this structure is certainly the one given by Scott Wolpert in the early 1980s, in terms of the Fenchel–Nielsen coordinates. Specifically, if $\alpha_1, \dots, \alpha_n$ is a set of homotopy classes of simple closed curves defining a pants decomposition of the surface, and if $\tau_1, \dots, \tau_n$ and $l_1, \dots, l_n$ are respectively the twist and length parameters associated to these curves, then it is known that the parameters (τ_i, l_i), $i = 1, \dots, n$ are global coordinates for Teichmüller space; they are called the Fenchel–Nielsen (twist-length) coordinates. Wolpert proved that the Weil–Petersson symplectic form is given by $w = \sum_{i=1}^{n} d\tau_i \wedge dl_i$ (which implies in particular that this twist-length description of the symplectic structure is independent of the choice of the pants decomposition). We note that in the definition of the twist parameter, one should be careful in counting right twists positively, and in twisting at time t a total length of t.

Wolpert obtained several other beautiful formulae concerning this symplectic structure, and we state a few of them. Let $t(\alpha)$ denote the Fenchel–Nielsen vector field on Teichmüller space associated to a homotopy class of simple closed curves α on the surface and $l(\alpha)$ denote the hyperbolic length function on $\mathcal{T}$ associated to α. Then, Wolpert proved the following:

– A "duality formula": $\omega(t(\alpha), \cdot) = -dt(\alpha)$.

– A "cosine formula": $w(t(\alpha), t(\beta)) = t(\alpha)l(\beta) = \sum_p \cos\theta_p$. Here, $t(\alpha)l(\beta)$ is the natural action of a vector field on a function, p varies over the set of intersection points of two fixed representatives of α and β which are in minimal position and θ_p is the angle of the intersection at the point p of these representatives, with appropriate orientations.

– A formula for the Poisson bracket. First, Wolpert defines a normalized vector field $T_\alpha = 4(\sinh l(\alpha)/2)t(\alpha)$. The T_α's form a Lie algebra over the integers, and the formula for the Lie bracket is $[T_\alpha, T_\beta] = \sum_p T_{(\alpha_p\beta)^+} - T_{(\alpha_p\beta)^-}$, where p varies as above over the set of intersection points of two fixed closed curves in minimal position representing α and β and where $(\alpha_p\beta)^+$ and $(\alpha_p\beta)^-$ are homotopy classes of closed curves obtained by modifying by local surgeries the union of the curves representing α and β at the intersection point p.

We finally note that there are nice formulae for the pull-back of the symplectic form ω on decorated Teichmüller space of a punctured surface, obtained by Penner, which also have a geometric flavour. As we shall see below, Penner's formulae have been used in the quantization theory of Teichmüller spaces of punctured surfaces.

1.4 Boundary structures

Although the topology of Teichmüller space is very simple (the space is homeomorphic to a ball), the boundary structure of this space is very rich and highly nontrivial. In fact, there are several interesting boundaries. Every time Teichmüller space is embedded in some function space (for instance, by Bers' embedding as a bounded domain in a Banach space, or by Thurston's embedding in the space of geodesic length functions, or by Bonahon's embedding in the space of geodesic currents, or by its embeddings in various spaces of representations), one can define a boundary structure for this space, by taking the closure of the image of the embedding. In all the cases mentioned, the boundary points have a geometric significance, as degenerate Riemann surfaces (for instance surfaces with nodes), or as projective classes of measured foliations, or as degenerate representations defined by group actions on $\mathbb{R}$-trees, and so on. In some cases, the closure of the image is compact, and the boundary structure defines a compactification of Teichmüller space. In some cases, the boundary is homeomorphic to a sphere, and in other cases it is not. It can happen that the boundary structure of Teichmüller space defines a boundary structure to Riemann's moduli space and vice versa. We shall review here some of the boundary structures of Teichmüller space and moduli space.

Thurston's boundary. Thurston's boundary of Teichmüller space is induced from the embedding of this space in the space $P\mathbb{R}_+^{\mathcal{S}}$ of projective classes of nonnegative functions on the set $\mathcal{S}$ of homotopy classes of essential simple closed curves on the surface, by means of the geodesic length functions. The boundary points are projective classes of compactly supported measured geodesic laminations, embedded in $P\mathbb{R}_+^{\mathcal{S}}$ by means of the geometric intersection functions. The mapping class group action on Teichmüller space extends continuously to the union of Teichmüller space with is Thurston boundary.

We note that Thurston's boundary can also be recovered using Bonahon's embedding of Teichmüller space into the space of geodesic currents.

Bers' boundary. Bers' boundary of Teichmüller space is induced from the Bers embedding of this space in the Banach space of quadratic differentials on a given Riemann surface. Bers showed that the closure of this embedding is compact, and this defines Bers' compactification of Teichmüller space. Bers' embedding depends upon the choice of a basepoint in Teichmüller space, but any two such embeddings are biholomorphically equivalent. However, Kerckhoff and Thurston showed that the boundary structure that one obtains in this way depends upon the choice of the basepoint. They proved that the mapping class group action on Teichmüller space does not extend continuously to an action on the union of this space with its Bers boundary. In fact, Kerckhoff and Thurston's work (1990) gives a new description of Bers' boundary in the setting of geometric convergence in the space of representations of the fundamental group of the surface in $\mathrm{PSL}(2, \mathbb{C})$. Their results on Bers' boundary are obtained after translating the questions into the context of quasi-Fuchsian groups. By Kerckhoff and Thurston's results, Bers' boundary and Thurston's boundary are distinct.

The visual boundaries. Different visual boundaries of Teichmüller space can be associated to the various metrics on this space. We recall that the visual boundary of a complete metric space (X, d) is the space of equivalence classes of geodesic rays, where two geodesic rays $\gamma_1 : [0, \infty[\to X$ and $\gamma_2 : [0, \infty[\to X$ are considered equivalent if the set $\{d(\gamma_1(t), \gamma_2(t)), t \in [0, \infty[\}$ is bounded. As a matter of fact, one usually considers the visual boundary at a given point x in X, by taking the set of geodesic rays emanating from x. There is a natural topology (the "cone topology") on the visual boundary at x. In the case where the space X is uniquely geodesic, there is a topology on the union of X with its visual boundary at x (which makes this visual "boundary" indeed a boundary). This topology is obtained by embedding X into the space of all (that is, finite or infinite) geodesics emanating from x, each point in X being identified with the geodesic that joins it to x. Of course, the first natural question arising from such a definition is: To what extent does the visual boundary depend on the choice of x?

The visual boundary of Teichmüller's space equipped with Teichmüller's metric was studied in the late 1970s by Kerckhoff, who showed that the action of the mapping class group on Teichmüller space does not extend continuously to the Teichmüller visual boundary.

The Weil–Petersson visual boundary has recently been studied by J. Brock, who worked on the analogies of this visual boundary with the boundary defined by taking the completion of the Weil–Petersson metric, and with the Bers boundary of Teichmüller space. Masur had already worked on the comparison between these last two boundaries in 1982. Brock's work starts with the known observation that some Weil–Petersson geodesic rays have finite length, a fact related to the non-completeness of the Weil–Petersson metric. In this work, the definition of the visual boundary is adapted to the setting of a non-complete space. The finite-length geodesics can be made to converge to surfaces with cusps, and this convergence is comparable to the convergence to surfaces with nodes that appears in the works of Bers and Abikoff. Brock also showed that the action of the mapping class group on Teichmüller space does not extend continuously to the Weil–Petersson visual boundary. Again, this is analogous to the result by Kerckhoff and Thurston that we mentioned above, showing that the action of the mapping class group does not extend continuously to Bers' boundary. Brock showed that despite these analogies, the Weil–Petersson boundary is distinct from Bers' boundary.

Finally, we note that the visual boundary of Thurston's asymmetric metric is another interesting object which so far has not been given the attention it deserves.

Other boundary structures. There are boundary structures for Riemann's moduli space, some of them induced from boundary structures for Teichmüller space, and some of them belonging only to the realm of moduli space. The compactification of moduli space by stable curves is a projective variety (studied as such by Wolpert and others). It is a quotient space of Teichmüller space equipped with its bordification by surfaces with nodes (sometimes called augmented Teichmüller space, with boundary called Abikoff's augmented boundary) by the action of the mapping class group. Note that this boundary structure of Teichmüller space is not a compactification of this space, but the quotient of this structure by the mapping class group is a compactification of moduli space.

Sometimes, the same boundary structure for Teichmüller space is studied from various points of view. When Riemann surfaces are viewed as algebraic surfaces, the boundary structure can be studied in terms of degeneration of algebraic surfaces, and so on. For instance, the augmented moduli space (and finite coverings of this space), whose boundary points are surfaces with nodes, has been studied by Deligne, Mumford, Knudsen and Mayer from the point of view of algebraic geometry, by Abikoff from the point of view of (geometric) complex analysis, and by Marden in relation with hyperbolic 3-manifold theory. Due to the fact that people working in different domains use different languages, it is sometime difficult to see the relations between the various results.

Let us finally mention that there are boundary structures for Schottky space, for general representation spaces, and for several other spaces related to Teichmüller space.

1.5 The harmonic maps approach

The study of harmonic maps is a classical subject in analysis and geometry, and these maps have been used in the last three decades as an important tool in the study of Teichmüller space. The definition of a harmonic map is based on the notion of the energy of a map. The energy of a map between two (say compact smooth) manifolds is the integral over the domain of the square of the derivative of the map. This quantity is regarded as a measure of the average squared stretching of the map. A map between two Riemannian manifolds is harmonic if it is a critical point of the energy functional. The collection of maps between the two manifolds over which the critical point is searched for is usually taken to be the set of maps in a given homotopy class of maps. Harmonic maps are solutions of an Euler–Lagrange equation, defined by a second-order elliptic partial differential operator. In some sense, harmonic maps generalize closed geodesics (which are harmonic maps with the domain manifold being the circle). They also generalize totally geodesic maps and harmonic functions (which are harmonic maps whose range is Euclidean space). The study of Teichmüller space using harmonic maps started with the work of M. Gerstenhaber and H. E. Rauch in 1954. In particular, there are harmonic map approaches to the Teichmüller metric and to the Weil–Petersson metric. The list of people that have worked in this domain is very long; just to give a few names, we mention E. Reich, C. Earle, J. Eells, J. Jost, A. Tromba, M. Wolf, Y. Minsky, C. Mese, G. Daskalopoulos, R. Wentworth and L. Katzarkov.

The classes of harmonic maps that have been used in Teichmüller theory include harmonic maps between surfaces, harmonic maps from surfaces to $\mathbb{R}$-trees (as limits of harmonic maps between surfaces; these maps can be used to describe degenerations of elements of Teichmüller space), and, more recently, harmonic maps from surfaces to Teichmüller space itself (equipped with various metrics).

A basic feature of the study of harmonic maps between surfaces is that the energy of such a map only depends on the conformal class of the domain surface, and not on its metric. In the harmonic maps approach to Teichmüller theory, the domain surface is a Riemann surface and the target surface is generally equipped with a hyperbolic metric. Sometimes, the target surface is equipped with a singular flat metric. For instance, E. Kuwert (1996) proved that a Teichmüller map is the unique harmonic map in its isotopy class when the target surface is equipped with the singular flat metric defined by the terminal quadratic differential of the map.

Thus, one can distinguish the following three approaches to Teichmüller theory:

– the approach where the elements of Teichmüller space are homotopy classes of conformal structures on a surface, and the optimal maps between two such structures are those that have minimum quasiconformal constant;

– the approach where the elements of Teichmüller space are homotopy classes of hyperbolic structures, the optimal maps between two such structures being those that have minimum Lipschitz constant;

– the harmonic maps approach where the domain surface is equipped with a conformal structure and the target surface is equipped with a hyperbolic structure, and where the harmonic maps are the optimal maps in the sense that they minimize energy.

Of course, there are very good questions concerning relations between the three approaches. The harmonic maps approach has the advantage of using, at the same time, the conformal and the hyperbolic point of view on Teichmüller space, and making links between them.

M. Wolf and A. E. Fischer together with A. Tromba developed theories that describe Teichmüller space in terms of harmonic maps. The approach of Fischer and Tromba uses techniques of analysis, whereas Wolf's approach is more geometric. Wolf gave a harmonic map description of the Weil–Petersson metric. He also studied harmonic maps between a fixed Riemann surface and families of degenerating hyperbolic surfaces. His approach includes a harmonic map description of Thurston's boundary. This approach involves maps from surfaces to $\mathbb{R}$-trees, and it uses the work of Gromov and Schoen on harmonic maps from surfaces to singular spaces. We can also mention here the work of Y. Minsky, who studied the behaviour of families of harmonic maps between surfaces when the domain surface varies along a Teichmüller geodesic, and other related problems.

We finally mention joint work by Daskalopoulos, Katzarkov and Wentworth on harmonic maps from surfaces to Teichmüller space equipped with the Weil–Petersson metric and with McMullen's metric, and joint work by Daskalopoulos, Dostoglou and Wentworth in which they give a harmonic map interpretation of the Morgan–Shalen compactification of the character variety of representations of the fundamental group of a surface into $\mathrm{SL}(2, \mathbb{C})$.

By looking at the work done by the various people working in the area, one gets the impression that all the important features of Teichmüller theory can be recovered using the theory of harmonic maps.

1.6 The group theory

There are several classes of groups that are intimately connected to Teichmüller theory. Of course, one thinks primarily of mapping class groups of surfaces and their subgroups, but one can consider more generally the mapping class groups of general C^∞ manifolds, and their subgroups. Another important class of groups related to Teichmüller theory is the class of Kleinian groups, with the related Fuchsian, quasi-Fuchsian, Schottky groups, and other groups defined by representations of the fundamental group of the surface. Of course, all these groups are also studied for their own sake, that is, independently of their relation to Teichmüller space.

Mapping class groups. The groups that play the most prominent role in Teichmüller theory are certainly the mapping class groups.[4] Important special cases of mapping class groups include the mapping class groups of punctured disks, which can be identified with braid groups.

The study of the mapping class group has a long history that starts with that of low-dimensional topology. There is a large number of interesting open problems related to the algebraic structure of mapping class groups, to their action on various spaces, and to the structure of their subgroups. An impressive list of such problems appears in a book edited by B. Farb (American Mathematical Society, 2006). An important subgroup of the mapping class group is the Torelli subgroup, consisting of the mapping classes that induce the identity on the homology of the surface, but there are several others. In Chapter 8 of this volume, L. Mosher gives an overview of the geometry and dynamics of actions of several classes of subgroups of the mapping class group on Teichmüller space equipped with its Thurston boundary.

As already mentioned, the mapping class group of a surface is also the isometry group of the Teichmüller metric and of the Weil–Petersson metric on the Teichmüller space of that surface, and it is conceivable that similar results hold for other metrics on that space. The most enlightening study of the mapping class group is certainly the one that Thurston made through the analysis of the action of that group on the compactification of Teichmüller space by the space of projective classes of measured foliations on the surface. Thurston's topological classification of the elements of the mapping class group, which is surveyed in Chapter 8 by Mosher, has counterparts in terms of the metric structures of Teichmüller space. In this connection, we recall that after Thurston's work was completed, Bers worked out a similar classification of mapping classes which is based on the action of the mapping class group equipped with the Teichmüller metric. Bers obtained this classification by analyzing the minimal set and the displacement function associated to a mapping class acting by isometries. A similar metric classification was recently obtained by Daskalopoulos and Wentworth, this time with respect to the Weil–Petersson metric. This classification is surveyed in Chapter 1 of this volume.

There are several other interesting actions of the mapping class group that were studied in the last few decades. For instance, Hatcher and Thurston proved in 1980 that the mapping class group of a closed surface is finitely presented, by studying the action of that group on a two-dimensional simplicial complex whose vertices are cut systems on the surface, that is, systems of isotopy classes of disjoint closed curves whose complement is a sphere with holes. The edges of this complex correspond to certain moves between cut systems, called elementary moves, and the two-dimensional cells correspond to certain cycles of moves, which are of three types: triangles, rectangles and pentagons. E. Irmak and M. Korkmaz proved recently that the automorphism

[4] There are some variations in the terminology of mapping class groups. Usually, the term *extended mapping class group* of a surface designates the group of isotopy classes of diffeomorphisms of that surface. The mapping class group is then the group of isotopy classes of orientation-preserving diffeomorphisms of an oriented surface. There are also various subclasses, depending on whether the surface is closed or not, with or without boundary components, whether the group fixes a distinguished point or not, and so on.

group of the Hatcher–Thurston complex of a compact oriented surface of positive genus is isomorphic to the extended mapping class group of the surface modulo its center. Other actions of mapping class groups which have been thoroughly studied include the action on the complex of curves, a flag complex whose n-simplices are collections of $(n+1)$ distinct isotopy classes of essential disjoint simple closed curves on the surface. This complex was introduced in 1979 by W. Harvey, with the idea of including it as a boundary structure of Teichmüller space. We also mention the actions of the mapping class group on the complex of pants, on the complex of nonseparating curves, on the complex of separating curves and the complex of domains, and there are several others. We already mentioned that Brock found a precise relation between the Weil–Petersson metric on Teichmüller space and the metric on the vertices of the complex of pants, induced by the simplicial metric on the 1-skeleton. The coarse geometries of these complexes equipped with their natural simplicial metric have also been studied for their own sake, that is, independently of their relation to the mapping class group. In Chapter 10 of this volume, Ursula Hamenstädt studies the action of the mapping class group on the complex of curves. She gives a new proof of a result of Masur and Minsky stating that this complex is hyperbolic in the sense of Gromov, and she discusses the relation between the geometry of this complex and the geometry of Teichmüller space.

Outer automorphism groups. An inner automorphism of a group G is an automorphism of the form $g \mapsto h^{-1}gh$, where h is a fixed element of G. The outer automorphism group of G is the quotient group of the automorphism group of G by the action of the inner automorphism group of G.

Automorphism and outer automorphism groups of free groups are basic objects of study in combinatorial group theory. They were already extensively investigated by J. Nielsen and W. Magnus, in the first quarter of the twentieth century. There is a close relation between mapping class groups of surfaces and outer automorphism groups. For instance, in the particular case of a closed surface, the extended mapping class group of the surface is isomorphic to the outer automorphism group of its fundamental group. In the case of a surface with one puncture, the fundamental group is a free group, and there is a natural map from the extended mapping class group of the surface to the outer automorphism group of its fundamental group. This map is explicited in Chapter 7, written by S. Morita. The study of the outer automorphism of a free group acquired a very geometric character after the introduction in the 1980s, by M. Culler and K. Vogtmann, of a space called *Outer space*, on which this group acts. There are parallels between the action of the outer automorphism of a free group on Outer space, and the action of the mapping class group on Teichmüller space.

Kleinian groups A Kleinian group G is a discrete group of orientation-preserving Möbius transformation of hyperbolic 3-space $\mathbb{H}^3$.[5] A Fuchsian group is a Kleinian

[5] The terminology here is not universally established. For instance, some authors ask that a Kleinian group acts properly discontinuously on some non-empty open subset on the Riemann sphere seen as the boundary at infinity of $\mathbb{H}^3$, and some other authors require that the group is finitely generated.

group whose action on $S^2 = \partial\mathbb{H}^3$ preserves a round disk. A quasi-Fuchsian group is obtained from a Fuchsian group by conjugation of the action by a quasiconformal homeomorphism of $\mathbb{H}^2$. The quotient space of the action of a Kleinian group on $\mathbb{H}^3$ is a hyperbolic 3-orbifold (and it is a manifold if Γ is torsion-free). Under certain conditions, the picture of the action can be made more complete: the Kleinian group on $\mathbb{H}^3 \cup S^2$ gives rise to a hyperbolic 3-manifold with nonempty boundary consisting of a finite number of surfaces of finite type carrying a canonical conformal structure, and the deformation theory of the hyperbolic 3-manifold can be studied through the deformation theory of its boundary surfaces. More precisely, given a Kleinian group G, it has an induced action on the boundary at infinity $\partial\mathbb{H}^3 = S^2$. This action decomposes S^2 into the union of two subsets, the ordinary set and the limit set. The *ordinary set* (or domain of discontinuity) $\Omega \subset S^2$ is the largest open subset of the sphere on which the group G acts properly discontinuously. The *limit set* Λ of the action is the set of accumulation points in S^2 of the orbit of any point in $\mathbb{H}^3$. From the point of view of the theory of hyperbolic 3-manifolds with boundary, one considers the action of G on the union $\mathbb{H}^3 \cup \Omega$. Ahlfors' finiteness theorem ensures that under some mild hypotheses, the quotient of $\mathbb{H}^3 \cup \Omega$ by this action is a 3-manifold with boundary, with the boundary consisting of a finite union of connected Riemann surfaces of finite type. The 3-manifold is equipped with a quotient hyperbolic structure induced from that of $\mathbb{H}^3$, and each boundary component is equipped with a complex structure (in fact, with a complex projective structure) induced from that of S^2. There is an interplay between the deformation theory of the hyperbolic 3-manifold and the union of the Teichmüller spaces of the boundary components. The classical approach to this study consists in applying Teichmüller's quasiconformal deformation theory in succession to the boundary components. For instance, in the case where the Kleinian group is a quasi-Fuchsian group, a theorem of Bers says that the hyperbolic structure of the 3-manifold is completely determined by the conformal structure of the boundary, which as a matter of fact consists of two conformal structures on the same topological surface. Thus, quasi-Fuchsian space is parametrized by the product of two copies of Teichmüller space.

We also note that the relation of Teichmüller theory with Kleinian groups involves the deformation theory of the projective structure of the boundary, and that the general deformation theory of Kleinian groups can be studied from the point of view of representation theory.

1.7 Representation theory

By the uniformization theorem, every Riemann surface of negative Euler characteristic can be defined as a complex structure induced by some hyperbolic metric. In the case of a surface without boundary, the hyperbolic metric can be regarded, by lifting the hyperbolic structure to the universal cover, as a discrete group of orientation-preserving isometries of hyperbolic 2-space $\mathbb{H}^2$, and such a group is well-defined up to conjugacy. From this observation, Teichmüller space can be studied as a set of conjugacy classes of representations of the fundamental group of the surface into the group $\mathrm{PSL}(2,\mathbb{R})$ of orientation-preserving isometries of $\mathbb{H}^2$.

More generally, one can study the moduli spaces of representations of the fundamental group of a surface S in a Lie group G, that is, the space of homomorphisms of $\pi_1(S)$ into G up to conjugation. The classical case is $G = \mathrm{PSL}(2,\mathbb{R})$, as described above. The Teichmüller space of a closed surface S of genus ≥ 2 is a connected component of this moduli space. This component consists of the conjugacy classes of discrete and faithful representations. W. Goldman showed that in this case, the number of connected components of the moduli space of representations is equal to $4g - 3$.

There are other important cases for the Lie group G. These include the case where $G = \mathrm{PSL}(2,\mathbb{C})$ (which is the setting of Kleinian representations, which includes several subsettings, like quasi-Fuchsian representations, Schottky representations, and so on). There is also the case where G is a split semisimple real group, whose study is the subject of "higher Teichmüller theory". It is worth saying a few words about this theory, since it has recently attracted much interest.

It is generally considered that higher Teichmüller theory was initiated by the work of Nigel Hitchin, who proved in the early 1990s that the space of conjugacy classes of discrete faithful representations in the space of conjugacy classes of all representations of the fundamental group of a compact oriented surface of genus ≥ 2 in a semisimple split real Lie group (namely $\mathrm{SL}(n,\mathbb{R})$ or $\mathrm{Sp}(2n,\mathbb{R})$) is a contractible component, a result which is regarded as a generalization of the fact that Teichmüller space is contractible. As mentioned above, in the case $n = 2$, this component is the Teichmüller space of the surface. In the case $n = 3$, the corresponding component is, by a result of Goldman and Choi, the moduli space of convex real projective structures on the surface. There are several good questions in higher Teichmüller theory, some of them independent of classical Teichmüller theory; some concern the action of the mapping class group on components other than the one discovered by Hitchin. V. Fock and A. Gontcharov recently studied "positive" representations of the fundamental group of a closed surface into a split semisimple algebraic group G with trivial center. They proved that these representations are faithful and discrete and that the moduli space of such representations is an open cell in the space of all representations. For $G = \mathrm{SL}(2,\mathbb{R})$, this space is the classical Teichmüller space. Higher Teichmüller theory has also been developed by F. Labourie. M. Burger, A. Iozzi and A. Weinhard recently undertook a study of representations of the fundamental group of the surface into semisimple Lie groups of Hermitian type. Let us note that one can also study representations of fundamental groups of surfaces in compact Lie groups like $\mathrm{SU}(2)$; in this case, the space of representations is compact.

1.8 Dessins d'enfants

A natural question in the algebro-geometric point of view on Teichmüller space is to understand the Riemann surfaces that can be defined by polynomials with coefficients in number fields. For instance, one would like to understand how the collection of such surfaces is embedded in Riemann's moduli space, or how to recognize the algebraic and, more specially, the arithmetic surfaces. There are also interesting questions about the automorphisms of such surfaces, e.g. what are the stabilizers of these surfaces in the mapping class group, and so on.

Grothendieck's introduction of his theory of dessins d'enfants was a major step in this direction. This theory makes connections between Teichmüller theory, complex algebraic geometry, arithmetic geometry and the study of a class of combinatorial objects on the surface.

We now briefly discuss this theory, and the related space of Belyi affine curves. This space is an analytic variety equipped with a natural action of the *absolute Galois group* (also called the *universal Galois group* $\mathrm{Gal}(\overline{\mathbb{Q}})$) which, by definition, is the automorphism group of the field $\overline{\mathbb{Q}}$ of algebraic numbers. We note that the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}})$ is considered as one of the most mysterious groups in mathematics, and for this reason, producing new actions of this group is always interesting.

Grothendieck's approach to Teichmüller theory is based on the idea that the combinatorial techniques of hyperbolic geometry (starting with surface decompositions into hyperbolic pairs of pants) have a parallel in algebraic geometry over $\overline{\mathbb{Q}}$. In this approach, dessins d'enfants play a prominent role. A *dessin d'enfant* ("child's drawing") is a connected graph embedded in a compact topological surface, whose complementary components are simply connected and whose edges are colored black and white such that no edge has its two vertices of the same color.

This theory is outlined in Grothendieck's *Esquisse d'un programme*, a manuscript which has been widely circulated in the form of mimeographed notes since it was written in 1984 and which finally appeared in print (together with an English translation) in 1997.

The correspondence between dessins d'enfants and algebraic curves over $\overline{\mathbb{Q}}$ is a consequence of a theorem of G. Belyi. This theorem states that any algebraic curve over $\overline{\mathbb{Q}}$ can be realized as a ramified finite covering of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, with ramification locus contained in the set $\{0, 1, \infty\}$. In this way, an algebraic curve over $\overline{\mathbb{Q}}$ defines a dessin d'enfant, obtained as the preimage of $[0, 1] \subset \mathbb{P}^1(\mathbb{C})$ by the covering map. The color of a vertex is specified according to whether the image of that vertex is 0 or 1. The relation with algebraic curves over $\mathbb{C}$ stems then from the fact that a complex algebraic curve X can be represented by an algebraic curve over $\overline{\mathbb{Q}}$ if and only if there exists a non-constant holomorphic map $f : X \to \mathbb{P}^1(\mathbb{C})$ whose critical values are in $\overline{\mathbb{Q}}$. The surprising contribution of Belyi was to reduce the number of critical values to three. This result was first announced at the Helsinki ICM in 1978. It had been conjectured by Grothendieck, who later on declared that he was amazed by the simplicity of Belyi's proof of this result. The converse result, that is, the fact that to a dessin d'enfant one can associate a ramified covering over $\mathbb{P}^1(\mathbb{C})$ is due to Grothendieck himself (who proved it using Belyi's work).

Grothendieck's ideas on Teichmüller space have been essentially exploited by algebraic geometers, because the original language of Grothendieck is that of algebraic geometry, although Grothendieck claimed on several occasions (in particular in his *Esquisse d'un programme*) that he was amazed by Thurston's approach to Teichmüller space, and that he would have liked to see his theory expressed in a similar geometric language.

Volume II of this Handbook contains a chapter on the theory of dessins d'enfants, written by F. Herrlich and G. Schmithüsen.

Links between Grothendieck's theory and the geometric Teichmüller theory will certainly become more apparent in the near future.

1.9 Physics and quantum theory

E. Verlinde and H. Verlinde conjectured in 1990 that the space of conformal blocks of Liouville conformal field theory (which is a version of a two-dimensional Einstein quantum gravity) can be obtained by the quantization of the Teichmüller space of the corresponding Riemann surface. This was motivated by the developments of the quantization of Chern–Simons theory. The quantization theory of Teichmüller space makes the Verlindes idea precise and it provides an explicit Hilbert space and observable algebra for 2-dimensional quantum gravity.

More explicitly, the quantization of Teichmüller space $\mathcal{T}_{\mathfrak{g},\mathfrak{p}}$ produces a one-parameter family of noncommutative $*$-algebras, with an action of the mapping class group of the surface by $*$-algebra automorphisms. These algebras are parametrized by a quantization parameter $\hbar$, having the property that when $\hbar = 0$ the algebra coincides with the original commutative algebra of functions on $\mathcal{T}_{\mathfrak{g},\mathfrak{p}}$ while the first derivative of the commutator with respect to $\hbar$ at zero gives the Weil–Petersson Poisson structure on $\mathcal{T}_{\mathfrak{g},\mathfrak{p}}$.

Quantization theories of Teichmüller space were developed independently during the 1990s in joint work by L. Chekhov and V. Fock, and by R. Kashaev. The two theories are essentially equivalent up to technical details.

The quantization theory developed by Chekhov and Fock considers the Teichmüller space of a surface with holes as a (degenerate) Poisson manifold and uses Thurston's shear coordinates on that surface, whereas the quantization theory developed by Kashaev considers the decorated Teichmüller space and uses Penner's λ-length coordinates.

The quantization procedures produce (for $\hbar \neq 0$) semi-simple infinite-dimensional $*$-algebras, and unitary projective representations of a subgroup of the mapping class group in the $*$-representation spaces of these algebras. Technically, this representation acts on a Hilbert space of functions on $\mathbb{R}^n$ by certain compositions of Fourier transformations, multiplications by quadratic forms and multiplication by quantum dilogarithms.

Let us also mention that Kashaev recently developed a quantization theory of the moduli space of flat $\mathrm{PSL}(2,\mathbb{R})$-connections on a punctured surface with parabolic holonomy around the punctures. This moduli space is closely related to the classical

Teichmüller space of the surface, and in fact, two of its connected components are isomorphic to Teichmüller space.

J. Teschner's work proves the Verlindes' conjecture relating the quantization theories developed by Chekhov–Fock and by Kashaev to the Liouville conformal field theory.

Furthermore, the quantization of Teichmüller space may lead to new invariants of hyperbolic three-manifolds. A famous example in this respect is the volume conjecture by R. Kashaev, H. Murakami and J. Murakami which expresses the volume of any hyperbolic knot as a certain limit of a specific value of the colored Jones invariant associated with the quantum group $\mathrm{SU}(2)_q$, which in turn can be computed using Kashaev's dilogarithmic quantization theory.

Thus, the quantization theories establish new links between Teichmüller theory, algebra, algebraic geometry, representation theory and mathematical physics.

The present volume contains chapters written by the various people involved in the quantization theory, namely, Chekhov, Fock, Goncharov, Kashaev, Penner and Teschner.

We mention that S. Baseilhac and R. Benedetti worked out a finite-dimensional combinatorial version of the quantization of Teichmüller space, with the aim of constructing new invariants for 3-manifolds, and with a view towards relating these results to the volume conjecture (with the hope of proving this conjecture). Their techniques are partly inspired by those used by V. Turaev in his work on Chern–Simons theory. Volume II of this Handbook will contain a chapter written by Baseilhac and Benedetti on that subject.

Let us also note that F. Bonahon and X. Liu worked out a finite-dimensional version of the quantization theory of Chekhov and Fock.

2 An overview of the content of this volume

The overview of the various sections of this Handbook that I will give now is also intended to give an idea of the large variety of ideas that Teichmüller theory involves.

2.1 The metric and the analytic theory, 1

Chapter 1, written by Georgios Daskalopoulos and Richard Wentworth, gives an overview of several important aspects of Teichmüller theory from the point of view of the theory of harmonic maps. The subjects that are treated in this chapter include the geometry of the Weil–Petersson metric (geodesics, curvature, isometries, completion, length functionals, convexity and superrigidity) and the study of group actions on $\mathbb{R}$-trees. In particular, the authors describe a harmonic map construction of the Teichmüller map, a harmonic map interpretation of the theorem of Hubbard and Masur on the existence and uniqueness of a quadratic differential with a given equivalence class of horizontal foliation, and a harmonic map interpretation of the Thurston–Morgan–Shalen compactification of Teichmüller space. The chapter also contains an introduction to several basic topics in Teichmüller theory, including the uniformization theorem, quasiconformal maps, Beltrami differentials, the solution to the Beltrami equation, several proofs (including one that is based on harmonic maps) of Teichmüller's theorem stating that Teichmüller space is a cell, Nielsen realization, a Higgs bundle interpretation of Teichmüller space in relation to flat $SL(2, \mathbb{C})$ connections, and holomorphic convexity. The authors also discuss similarities and differences between the mapping class group and arithmetic lattices.

To make the exposition more accessible, the authors also give an introduction to the theory of harmonic maps, with an emphasis on harmonic maps between surfaces, between surfaces and trees and between surfaces and Teichmüller space. The exposition includes some classical results by Gerstenhaber and Rauch, the results of Eells and Sampson on the existence of harmonic maps in a given homotopy class of maps in the case where the range is compact and has nonpositive sectional curvature, Hartmann's uniqueness result in the case where the curvature is negative, and an overview of the relatively recent theory of Gromov and Schoen on harmonic maps between singular spaces and of the recent work by C. Mese on the proof of a conjecture made in 1954 by Gerstenhaber and Rauch on realizing extremal energy maps by Teichmüller maps.

Chapter 2 by Guillaume Théret and myself has two parts. The first part concerns Teichmüller's metric on Teichmüller space, and the second part concerns Thurston's asymmetric metric on this space. Teichmüller's metric regards Teichmüller space from the point of view of conformal geometry, whereas Thurston's asymmetric metric is based on considerations on Teichmüller space from the point of view of hyperbolic geometry. The two metrics are Finsler metrics, and they are studied in the same chapter with the aim of drawing parallels and differences between them. This chapter contains basic facts about the two metrics and about the asymptotic geometry they induce on Teichmüller space. The main results that are presented concern the behaviour of geodesic rays for the two metrics, in particular the convergence or non-convergence of certain classes of geodesic rays to points in Thurston's boundary. This includes results on the limiting behaviour of stretch lines, which are geodesics for Thurston's asymmetric metric, and on the limiting behaviour of anti-stretch lines, which are stretch lines traversed in the negative direction, and which in general are not geodesic lines for Thurston's asymmetric metric. The chapter also contains a review of a parametrization of Teichmüller space by a space of measured geodesic laminations that was introduced by Thurston, and which Thurston calls "cataclysm coordinates".

The results concerning Teichmüller's metric which are surveyed in Chapter 2 are classical, whereas some of the results concerning Thurston's asymmetric metric are new. We also discuss some open problems related to Thurston's asymmetric metric.

Chapter 3, written by Robert Penner, contains an exposition of a generalized notion of a decorated hyperbolic structure.

A *decoration* of a hyperbolic surface with cusps is the choice of a closed horocycle around each cusp. The space of homotopy classes of decorated hyperbolic structures is a fibre bundle over Teichmüller space, called decorated Teichmüller space. This space was introduced by R. Penner in 1987, and the idea turned out to be extremely fruitful. A particularly useful set of parameters that Penner defined are the λ-length coordinates of this space. These are defined as signed distances between distinguished horocycles, measured on the edges of an ideal triangulation of the surface. There is a very simple and useful description of the pull-back of the Weil–Petersson symplectic form on decorated Teichmüller space in terms of λ-length coordinates. These coordinates, together with their transformation laws (the so-called "Ptolemy equations"), were used successfully in the study of the combinatorial structure of Teichmüller space. For instance, Penner used them to construct a mapping class group invariant cell-decomposition of decorated Teichmüller space and to compute Weil–Petersson volumes of moduli spaces. We already mentioned that the decorated theory, together with the λ-length coordinates, have been used as essential tools in the quantization theory of Teichmüller space, in particular in the work of R. Kashaev.

In Chapter 3, λ-length coordinates are reviewed and are generalized in two directions, namely, in the setting of homeomorphisms of the circle, and in what the author calls the decorated Teichmüller theory of the punctured solenoid. Let us say a few words about these two theories.

Universal Teichmüller theory first appeared in the work of Ahlfors and Bers. This theory can be formulated in terms of quasisymmetric homeomorphisms of the unit circle. These mappings arise as homeomorphisms which are boundary values of quasiconformal maps of the hyperbolic disk. In the context of universal Teichmüller theory, quasisymmetric homeomorphisms of the unit circle are used to parametrize hyperbolic structures on the unit disk relative to the boundary, which in particular parametrize Teichmüller spaces of all finite type surfaces. λ-length coordinates are used to parametrize the space of cosets of the subgroup of Möbius transformations in the group of orientation-preserving homeomorphisms of the circle, which is identified with a suitable space of tessellations of the hyperbolic disk. Some elements of universal Teichmüller theory are surveyed in Chapter 3 of this volume.

The (hyperbolic) solenoid $\mathcal{H}$ was introduced in the 1990s by D. Sullivan as an inverse limit of the space of all branched covering of a closed Riemann surface. More precisely, let (S, x) be a pointed surface of genus > 1. If $\pi_i\colon (S_i, x_i) \to (S, x)$ and $\pi_j\colon (S_j, x_j) \to (S, x)$ are two pointed coverings of (S, x), then we write $\pi_i \leq \pi_j$ if the covering π_j factors through the covering $\pi_{j,i}\colon (S_j, x_j) \to (S_i, x_i)$ with $\pi_j = \pi_i \circ \pi_{j,i}$. Equipped with the relation $\leq$, the set of all coverings of (S, x) is inverse directed, and the solenoid $\mathcal{H}$ is the inverse limit of this directed set. This space $\mathcal{H}$ is equipped with a topology, viz. the subspace topology induced from the product topology on the infinite product of all closed surfaces in the finite coverings of (S, x). With this topology, the local structure of $\mathcal{H}$ is that of a plane times a Cantor set. The solenoid $\mathcal{H}$ can also be equipped with a complex structures, and in fact, like surfaces of negative Euler characteristic, it can be equipped with a family of inequivalent complex structures.

The space of complex structures on the solenoid was studied by Sullivan and others. There is a Teichmüller space $\mathcal{T}(\mathcal{H})$ of $\mathcal{H}$, which can be defined as a certain closure of the stack of the Teichmüller spaces of all closed surfaces of genus > 1. This space admits a complete metric, in analogy to Teichmüller's metric. The mapping class group of the solenoid $\mathcal{H}$ is the set of isotopy classes of quasiconformal maps $h\colon \mathcal{H} \to \mathcal{H}$, and there is a natural action of this group on $\mathcal{T}(\mathcal{H})$. Volume II of this Handbook contains a chapter on the solenoid, written by D. Šarić.

In Chapter 3, Penner discusses joint work with Šarić on the punctured solenoid. In analogy to Sullivan's definition, the punctured hyperbolic solenoid is defined in this setting as the inverse limit of all finite unbranched pointed covers, the branching being permitted only over the punctures. The Teichmüller space of the punctured solenoid is a separable Banach space admitting a complete Teichmüller metric. This is a universal object in the sense that one can canonically embed in this space every classical Teichmüller space over a hyperbolic punctured surface. There are again λ-length coordinates on an appropriate decorated Teichmüller space of the punctured solenoid, and the structure of decorated Teichmüller space for finite type surfaces survives mutatis mutandis for this punctured solenoid.

Chapter 4 by Jean-Pierre Otal contains an exposition of the quasiconformal deformation theory of Riemann surfaces, of Bers' construction of the complex structure of Teichmüller space and of the theory of geodesic currents and of Hölder distributions. The space of geodesic currents is a subspace of the space of Hölder distributions. Both spaces were introduced by F. Bonahon in the setting of Fuchsian groups of finite coarea. According to Bonahon's definition, a geodesic current is a positive locally finite measure on the space of geodesic lines in the universal covering of the surface, which is invariant by the action of the group of covering transformations on that space. We recall that if S^1 is the boundary at infinity of the unit disk, considered as the universal covering of the surface, then the space of geodesic lines in the universal covering of the surface is naturally identified with the set $(S^1 \times S^1 \setminus \Delta)/(\mathbb{Z}/2)$, where Δ is the diagonal set of $S^1 \times S^1$ and where $\mathbb{Z}/2$ acts by exchanging coordinates. The space of geodesic currents is a complete uniform space. The terminology of currents was used by Bonahon after Sullivan who introduced it in the study of dynamical systems, in analogy to de Rham's currents. Bonahon defined a topological embedding of Teichmüller space into the space of geodesic currents, which he called the Liouville map. This embedding gives a new and unifying point of view on Teichmüller space, from which one can recover, for instance, Thurston's compactification by adjoining to this space the space of asymptotic rays, which are identified with measured geodesic laminations on the surface. The space of Hölder distributions is the dual space of the space of Hölder continuous functions with compact support. The definition of the space of Hölder distributions was extended to the case of an arbitrary Fuchsian group by Šarić, who showed that the Liouville map, generalized to that setting, is also a topological embedding. As in Bonahon's setting, Šarić defined a Thurston-type boundary by adding to the image of the Liouville map the set of asymptotic rays, which can be identified with bounded measured laminations on the surface, but here, the closure of the image is in general not compact. Chapter 4 contains an exposition of this work of Šarić. The last part of this chapter contains new results by Otal on the analyticity of the Liouville map.

In Chapter 5, William Harvey considers the complex theory of moduli space of complex algebraic curves, and its relation with algebraic geometry and arithmetic geometry. This study mainly involves the following two interrelated topics:

— A discussion of Grothendieck's theory of dessins d'enfants and of Belyi's complex affine algebraic curves, with the action of the absolute Galois group of algebraic numbers Gal($\overline{\mathbb{Q}}$) on these objects. One of the advantages of Harvey's review of Grothendieck's theory is that it is done by means of the classical tools of the theory of Teichmüller space, that is, the techniques introduced by Ahlfors and Bers.

— A study of Teichmüller disks in Teichmüller space, of their stabilizer groups and of their images in moduli space. In particular, the author addresses the question of the existence and the description of Teichmüller disks with large stabilizer groups, and he presents a construction of such disks that arises from hyperbolic tessellations of surfaces, defined by Hecke triangle groups and their subgroups.

We recall that a Teichmüller disk is a holomorphic and isometric embedding of the unit disk in the complex plane equipped with its hyperbolic metric into Teichmüller space equipped with Teichmüller's metric. Teichmüller disks were already studied by Teichmüller himself, and their study was revived by works of Thurston, Veech, McMullen and others.

As a matter of fact, Harvey describes several relations between Grothendieck's theory and the study of Teichmüller disks. The first relation is through a classical construction, due to Thurston, of pseudo-Anosov maps as products of Dehn twists along simple closed curves whose union fills the surface. This construction provides both a dessin d'enfant and a Teichmüller disk. Harvey then describes a general procedure for constructing Teichmüller curves in moduli space and he gives a characterization of all the curves of a given genus that are definable over $\overline{\mathbb{Q}}$. He presents a theorem stating that for any point in moduli space that corresponds to a curve defined over a number field (that is, a finite extension of $\mathbb{Q}$), there exists a Teichmüller disk of an arithmetic nature passing through it.

Chpater 5 also contains background material on the Ahlfors–Bers deformation theory of complex structures on surfaces as solutions of Beltrami partial differential equations, and an overview of some elements of the theory of universal Teichmüller space. The author also mentions works by C. McMullen, by P. Hubert, S. Lelièvre and T. A. Schmidt on Veech disk, and by P. B. Cohen and J. Wolfart on deformations of algebraic curves.

In Chapter 6, Frank Herrlich and Gabriela Schmithüsen study, among others, the following three questions:

— When is the image of a Teichmüller disk in moduli space an algebraic curve?

— What is the limiting behaviour of such a disk at the boundary of Teichmüller space and in Schottky space and its compactification?

— What are the stabilizer groups of Teichmüller disks in the mapping class group, and what is their relation with discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$ such as the Veech groups?

The authors present three different approaches to Teichmüller disks. First, such a disk can be considered as a complex geodesic in Teichmüller space, and in particular as a collection of (real) geodesic rays. The second approach starts with a (singular) flat structure that is induced on a Riemann surface by a holomorphic quadratic differential; a Teichmüller disk is then obtained by an affine deformation of this structure. Finally, a Teichmüller disk can also be obtained by varying the Beltrami differential of the quasiconformal mapping corresponding to a point in Teichmüller space. The authors show that all three approaches lead to the same object.

If the image of a Teichmüller disk in moduli space is an algebraic curve, it is called a Teichmüller curve. Whether this happens or not can be seen from the Veech group, which is introduced in this chapter geometrically as a Fuchsian group and equivalently as a subgroup of the mapping class group.

Chapter 6 also contains a report on W. Abikoff's construction of the augmented Teichmüller space and on an alternative point of view on that space, due to V. Braungardt, using a general complex-analytic device. As a matter of fact, Braungardt's construction is done in the category of locally complex ringed spaces.

A central aspect of Chapter 6 is the investigation of limit points of Teichmüller curves in the Abikoff–Braungardt boundary. The main result is that these points can be described as endpoints of Strebel rays corresponding to parabolic elements in the Veech group. The techniques build on Masur's analysis of Strebel rays.

Finally, a partial compactification of Schottky space is introduced in two ways. The relation between Teichmüller space, Schottky space, moduli space, their boundaries and the groups acting on them are worked out. Using the boundary of Schottky space, properties of the image of a Teichmüller disk in Schottky space are obtained.

2.2 The group theory, 1

In Chapter 7, Shigeyuki Morita studies several classes of groups which either generalize mapping class groups of surfaces, or are subgroups of mapping class groups. The classes of groups that are involved in this study include the following:

— Diffeotopy groups (that is, groups of isotopy classes of diffeomorphisms) of arbitrary smooth manifolds. When the manifold is a surface, then the diffeotopy group is the extended mapping class of this surface (that is, the group of isotopy classes of diffeomorphisms which do not necessarily preserve an orientation).

— Groups of isotopy classes of diffeomorphisms of smooth manifolds that preserve a volume form, respectively a symplectic form.

— The Torelli group, that is the subgroup of the mapping class group consisting of isotopy classes which act trivially on homology. Morita discusses the action of the mapping class group on the homology of the surface, and he presents properties of the Torelli group, including the fundamental work of D. Johnson on that group.

— The outer automorphism group of a free group. The exposition starts with a description of work done by Jacob Nielsen on this outer automorphism group. The outer automorphism group of a free group acts on a space introduced in the 1980s by M. Culler and K. Vogtmann, called *Outer space*. This space, with respect to its action by the outer automorphism group, is considered to play a role analogous to that of Teichmüller space, with respect to its action by the mapping class group. Morita explains important parallels between the two theories.

In this chapter, Morita presents important aspects of the algebraic structure of all these groups, in particular on the interplay between their cohomology groups, and he discusses some related open problems.

Chapter 8 by Lee Mosher expounds on the dynamics of subgroups of the mapping class group from the point of view of their action on Teichmüller space compactified by its Thurston boundary. The author presents important dynamical, geometric and algebraic properties of this action, with the aim of highlighting analogies between these actions and the actions of discrete groups of isometries of hyperbolic 3-space $\mathbb{H}^3$ on the union of that space with its boundary at infinity. From the purely dynamical point of view, the analogy starts with the notions of limit set and of domain of discontinuity. From the geometric point of view, the analogy is based on the notions of convex–compact subgroups and of Schottky subgroups. Teichmüller disks intervene in this study as isometric images of hyperbolic planes in Teichmüller space, their stabilizers being analogous to Fuchsian subgroups which in the classical case are isometry groups of $\mathbb{H}^3$ that stabilize isometrically embedded 2-dimensional hyperbolic spaces. From the algebraic point of view, the author presents the Tits alternative for subgroups of the mapping class group (proved by J. McCarthy) and the Leininger–Ried combination theorem which provides a method for building closed surface subgroups of mapping class groups. This last theorem is reminiscent of Maskit's combination theorem that is used in the construction of discrete subgroups of isometries of hyperbolic spaces by combining simplest subgroups. The chapter also contains a survey of Thurston's classification theorem of mapping classes.

Chapter 9 by Albert Marden concerns the deformation theory of Kleinian groups, specifically the interplay between the deformation space of a hyperbolic 3-manifold and the Teichmüller space of its boundary surfaces. In his report, Marden recalls several deep results related to this theory, including Thurston's hyperbolization theorem, Mostow rigidity theorem and a theorem of Sullivan on the relation between the conformal structure of a simply connected region in the Riemann sphere and the geometry of a certain surface embedded in $\mathbb{H}^3$. Sullivan's result says that under some mild conditions, the boundary of the convex hull in $\mathbb{H}^3$ of the complement of an open subset U of the Riemann sphere (seen as the boundary at infinity of $\mathbb{H}^3$) is quasiconformally equivalent to U, with a quasiconformality constant being a universal constant. Chapter 9 includes a report on Bers' embedding of Teichmüller space into the Banach space of quadratic differentials, which in Kleinian groups theory leads to the simultaneous uniformization theorem. It also includes a report on the augmented space of representation which, when quotiented by the mapping class group action, gives a compactification of moduli space. The discussion on the representation space includes a review of the various notions of convergence in the study of the representation variety of the 3-manifold group in $\mathrm{PSL}(2, \mathbb{C})$.

Chapter 9 also contains a review of earthquakes and of their generalization to complex earthquakes, and the recent activity on that subject that stems from Sullivan's theorem and that has been carried out by D. Epstein, A. Marden and V. Markovic.

Chapter 10, by Ursula Hamenstädt, concerns the geometry of the complex of curves of a surface and its relation to the geometry of Teichmüller space. The natural action on this complex by the mapping class group of the surface has been thoroughly investigated in the last two decades by several people. The bulk of Chapter 10 is about the coarse geometry of this complex. Hamenstädt gives a new proof of a theorem due to H. Masur and Y. Minsky stating that the complex of curves is Gromov hyperbolic, and she reports on a result due to E. Klarreich describing the Gromov boundary of this complex. She describes a map from Teichmüller space to the complex of curves which is coarsely equivariant with respect to the action of the mapping class group on both spaces, with the property that the image under this map of a Teichmüller geodesic is a quasi-geodesic in the complex of curves.

Chapter 10 also contains an exposition of some basic facts on surface topology, namely, a description of the Hausdorff topology of the space of (not necessarily measured) laminations and of train track coordinates for this space.

2.3 Surfaces with singularities and discrete Riemann surfaces

Singular metrics on surfaces of constant curvature with conical singularities have been studied by several authors, including A. D. Alexandrov, W. P. Thurston, W. A. Veech, I. Rivin and B. Bowditch. Such structures naturally appear in the study of cellullar decompositions of Teichmüller space and of moduli space, but also in other geometric contexts such as the study of patterns of circles. The existence and uniqueness results of M. Troyanov and his work on the analytic theory of spaces of singular flat metrics are often quoted as basic references on that subject.

In Chapter 11 of this volume, Marc Troyanov describes classical and new results on the deformation theory of flat metrics with cone singularities on surfaces. The chapter starts with a general introduction to the deformation theory of geometric structures on compact manifolds with boundary, based on the notions of developing map and holonomy homomorphism. The chapter also contains an introduction to the representation theory of a finitely presented group into a real algebraic group. This theory is applied to the special case of flat metrics with cone singularities on surfaces, in which the representation group is the group $\mathrm{SE}(2)$ of rigid motions of the Euclidean plane. The developing map and holonomy homomorphism of such structures are used to define an explicit structure on Teichmüller space that makes it a real algebraic variety, and a corresponding orbifold structure on the quotient moduli space. In the special case where the surface has genus zero, this structure has already been studied by Deligne and Mostow, using techniques of algebraic geometry, and by Thurston. The techniques used by Troyanov are close to those of Thurston.

In Chapter 12, Charalampos Charitos and Ioannis Papadoperakis consider generalizations of hyperbolic structures (that is, metrics of constant curvature -1) on surfaces. This chapter is divided into two distinct parts that concern respectively hyperbolic structures with cone singularities on surfaces, and hyperbolic structures on simplicial 2-complexes.

The hyperbolic structures with cone singularities that are considered here have their cone angles $\geq 2\pi$. With this condition, the surfaces satisfy the CAT(-1) condition (the so-called Cartan–Alexandrov–Toponogov negative curvature condition), and the techniques of the theory of CAT(-1)-spaces can therefore be used in this study. The authors work out in detail coordinates for the moduli space of these structures in the case where the surface is a pair of pants with a unique cone point. In this case, the parameter space is homeomorphic to $\mathbb{R}^6$.

Let us note that a pair of pants equipped with a hyperbolic metric with a unique cone point is the simplest surface that one can study in this context. In some sense, this case is a building block for the general theory of hyperbolic structures with conical singular points and with cone angles $\geq 2\pi$. Indeed, it is conceivable that the space of hyperbolic metrics with cone singularities can be parametrized by decomposing the surface into pairs of pants, in analogy to the case of non-singular hyperbolic structures, except that in this singular setting one has to deal with several kinds of degeneration of pairs of pants.

The other class of singular hyperbolic structures studied in Chapter 12 is the class of complete length metrics on 2-dimensional simplicial complexes in which each 2-simplex with its vertices deleted is isometric to a hyperbolic ideal triangle. Such a metric space is called a 2-dimensional ideal simplicial complex, and it also satisfies the CAT(-1)-condition. The authors describe parameters for the moduli space of ideal simplicial complexes, using lengths of closed geodesics, in analogy to Thurston's parametrization of the Teichmüller space of a surface by embedding it in the space $\mathbb{R}_+^{\mathcal{S}}$ of functions on the set $\mathcal{S}$ of homotopy classes of essential simple closed curves on the surface.

Chapter 13, by Christian Mercat, is an introduction to the theory of discrete Riemann surfaces. Here, a discrete Riemann surface is a topological surface equipped with a cell-decomposition whose faces are quadrilaterals, with positive weights assigned to the diagonals in such a way that for each quadrilateral the product of the weights of the two diagonals is equal to one. A discrete holomorphic map on a discrete Riemann surface is a function on the set of vertices satisfying a discrete version of the Cauchy–Riemann equations. These definitions lead to the notions of differential forms, holomorphic forms, wedge products, Dirichlet energy, a Hodge-star operator, harmonicity and period matrices for discrete Riemann surfaces. Using these notions, there are discrete analogs of a number of classical theorems in Riemann surface theory, e.g. Hodge decomposition theorems, existence theorems for holomorphic forms with prescribed holonomies, and several approximation theorems. The results that are presented in Chapter 13 include a convergence theorem for the period matrices as a smooth Riemann surface is approximated by continuous ones, and results connecting a discrete version of the exponential function to the Bäcklund (or Darboux) transform for discrete holomorphic maps. The author also mentions a relation between discrete analytic functions and circle patterns.

2.4 The quantum theory of Teichmüller space, 1

In Chapter 14, Leonid Chekhov and Robert Penner present details of the Chekhov–Fock version of the quantization theory of the Teichmüller space of a punctured surface. They furthermore present results towards the quantization of its Thurston boundary, whose elements are projective measured foliations of compact support, and they show that the required operatorial limits exist weakly in the special case of the once-punctured torus. They introduce two classes of quantum operators, defined on the set of homotopy classes of essential simple closed curves on the surface, namely, quantum versions of the geodesic length operators associated to hyperbolic structures and quantum versions of the intersection function operators associated to measured foliations of compact support. Relating the quantization of Teichmüller space to the quantization of Thurston's boundary is realized by showing that if a sequence of hyperbolic structures (g_n) converges to a projective class of a measured foliation λ, then the sequence of quantum operators associated to (g_n) converges weakly to the quantum operator associated to λ. This chapter also contains some basic material on Teichmüller theory, including an introduction to measured foliations, to train tracks and to shear coordinates associated to ideal triangulations.

Chapter 15 by Vladimir Fock and Alexander Goncharov is a report on mostly introductory material related to the geometry of surfaces with boundary and with distinguished points on their boundaries (which the authors call *ciliated surfaces*), and of the corresponding spaces of measured laminations. The authors give detailed descriptions of coordinates for these spaces, with formulae for the Poisson and the symplectic structures on Teichmüller and lamination spaces, of pairing between lamination and Teichmüller spaces, and of the action of the mapping class group in these coordinates. Their exposition also contains an interpretation of lamination spaces as tropical limits of Teichmüller spaces.

Chapter 16 by Jörg Teschner contains a construction of a modular functor out of quantum Teichmüller space. A modular functor is a functor from the category of Riemann surfaces with isotopy classes of embeddings as morphisms, to the category of vector spaces with linear maps as morphisms. In this work, the vector spaces are infinite-dimensional. Modular functors are natural generalizations of representations of mapping class groups. In his work, Teschner extends the set of operators corresponding to morphisms of a surface into itself (and arising from the quantum

Teichmüller space) to the morphisms between different surfaces in a compatible way. This chapter also contains a review of the coordinates for Teichmüller space that were used by Penner, by Fock and by Kashaev, as well as a self-contained presentation of the quantization theory of Teichmüller space.

Chapter 17 by Rinat Kashaev is a review of his quantization theory of the moduli space of irreducible flat $\mathrm{PSL}(2,\mathbb{R})$-connections on a punctured surface, or, equivalently, of the space of conjugacy classes of irreducible representations of the fundamental group of the surface in $\mathrm{PSL}(2,\mathbb{R})$ with parabolicity conditions at the punctures. Teichmüller space embeds as a connected component of this space. The moduli space of irreducible flat $\mathrm{PSL}(2,\mathbb{R})$-connections is equipped with a symplectic form defined by Goldman, which restricts to the Weil–Petersson form on Teichmüller space. This quantization theory leads to an infinite dimensional projective unitary representation of the mapping class group.