Can Dehn surgery yield three connected summands?

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Abstract. A consequence of the Cabling Conjecture of Gonzalez-Acuña and Short is that Dehn surgery on a knot in $S^3$ cannot produce a manifold with more than two connected summands. In the event that some Dehn surgery produces a manifold with three or more connected summands, then the surgery parameter is bounded in terms of the bridge number by a result of Sayari. Here this bound is sharpened, providing further evidence in favour of the Cabling Conjecture.

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1. Introduction

The Cabling Conjecture of Gonzalez-Acuña and Short [4] asserts that Dehn surgery on a knot in $S^3$ can produce a reducible 3-manifold only if the knot is a cable knot and the surgery slope is that of the cabling annulus.

The Cabling Conjecture is known to hold in many special cases [2], [8], [9], [12], [13], [17], [19].

If $k$ is the $(p, q)$-cable on a knot $K$, then the cabling annulus on $k$ has slope $pq$, and the corresponding surgery manifold $M(k, pq)$ splits as a connected sum

$$M(K, p/q) \# L(p, q);$$

see [5]. (Here $L(p, q)$ is a lens space.) In particular both connected summands are prime [5]. Thus the Cabling Conjecture implies the weaker conjecture below:

Conjecture 1 (Two summands conjecture). Let $k$ be a knot in $S^3$ and $r \in \mathbb{Q} \cup \{\infty\}$ a slope. Then the Dehn surgery manifold $M(k, r)$ cannot be expressed as a connected sum of three non-trivial manifolds.

Since any knot group has weight 1 (in other words, is the normal closure of a single element), the same is true for any homomorphic image of a knot group. Thus the two summands conjecture would follow from the group-theoretic conjecture below, which remains an open problem.
Conjecture 2. A free product of three non-trivial groups has weight at least 2.

The best known upper bound for the number of connected summands in $M(k, r)$ is 3, obtained by combining results of Sayari [14], Valdez Sánchez [18] and the author [11]. These results also show that, should some $M(k, r)$ have three connected summands, then two of these must be lens spaces (necessarily with fundamental groups of coprime orders) and the third must be a $\mathbb{Z}$-homology sphere. (See [11] for details.)

Suppose that $k$ is a knot in $S^3$ with bridge number $b$, and that the 3-manifold $M$ obtained by performing Dehn surgery on $k$ with surgery parameter $r$ has more than two connected summands. It is known from the work of Gordon and Luecke [5] that $r$ must be an integer.

If $\ell_1, \ell_2$ are the orders of the fundamental groups of the lens spaces, then Sayari [15] has proved that $|r| = \ell_1 \ell_2 \leq (b - 1)(b - 2)$.

In this paper we shall prove the following inequality.

**Theorem 1.** Let $k$ be a knot in $S^3$ with bridge-number $b$. Suppose that $r$ is a slope on $k$ such that $M = M(k, r) = M_1 \# M_2 \# M_3$ where $M_1, M_2$ are lens spaces and $M_3$ is a homology sphere but not a homotopy sphere. Then

$$|\pi_1(M_1)| + |\pi_1(M_2)| \leq b + 1.$$  

As an immediate consequence, we obtain a sharpening of Sayari’s inequality.

**Corollary 2.** Under the hypotheses of Theorem 1 we have

$$|r| = |\pi_1(M_1)| \cdot |\pi_1(M_2)| \leq b(b + 2)/4.$$  

We use the standard techniques of intersection graphs developed by Scharlemann [16] and by Gordon and Luecke [1], [6], [7]. In §2 below, we recall the construction of the intersection graphs in the particular context of this problem. A key feature of these is the existence of Scharlemann cycles, which correspond in a well-understood way to the lens space summands. In §3 we show that, should the inequality $\ell_1 + \ell_2 \leq b + 1$ fail, then we can find, trapped between two Scharlemann cycles, a sandwiched disk (see Definition 3.3). We then show in §4 that sandwiched disks are impossible, which completes our proof.

2. The graphs

Throughout the remainder of the paper, we assume that the manifold $M = M(k, r)$ obtained by $r$-Dehn surgery on $k \subset S^3$ is a connected sum of three factors $M_1, M_2, M_3$, where $M_1$ and $M_2$ are lens spaces while $M_3$ is a (prime) integer homology sphere. Note that, since $\pi_1(M)$ has weight 1, the orders $\ell_1, \ell_2$ of $\pi_1(M_1)$ and
$\pi_1(M_2)$ are necessarily coprime. It follows that the factors $M_1$, $M_2$, $M_3$ are pairwise non-homeomorphic.

An essential embedded sphere $\Sigma \subset M$ necessarily separates, with one component of $M \sim \Sigma$ homeomorphic to a punctured $M_s$ and the other to a punctured $M_t \neq M_u$, where $\{s, t, u\} = \{1, 2, 3\}$. We will say that such a $\Sigma$ separates $M_s$ and $M_t$ (and also separates $M_s$ and $M_u$).

For $i = 1, 2$ let $P_i$ be a planar surface in the exterior $X(k)$ of $k$ (the complement of an open regular neighbourhood of $k$ in $S^3$) that extends to an essential sphere $\hat{P}_i \subset M$ such that $\hat{P}_i$ separates $M_i$ and $M_3$. Assume also that $P_i$ has the smallest possible number of boundary components amongst all such planar surfaces.

A standard argument ensures that we may also choose $P_1$, $P_2$ to be disjoint (without increasing the number of boundary components of either).

Following Gabai [3], Section 4 (A), we put $k$ in thin position, find a level surface $Q$ for $k$ and isotope $P := P_1 \cup P_2$ such that $P$ meets $Q$ transversely, and such that no component of $Q \cap P$ is an arc that is boundary-parallel in $P$. (The minimality condition in the definition of $P_1$ and $P_2$ ensures also that no component of $Q \cap P$ is a boundary-parallel arc in $Q$.)

The number $q$ of boundary components of $Q$ is necessarily even, and is bounded above by twice the bridge number, $q \leq 2b$. We can complete $Q$ to a sphere $\hat{Q} \subset S^3$ by attaching $q$ meridional disks.

We denote the intersection graph of $P_i$ and $Q$ in $\hat{Q}$ by $G_i$ for $i = 1, 2$. The (fat) vertices of $G_i$ are the meridional disks $\hat{Q} \sim Q$, and the edges are the components of $P_i \cap Q$ (some of which may be closed curves rather than arcs). Each fat vertex contains precisely one point of intersection of $k$ with $\hat{Q}$, so a choice of orientation for $k$ and for $Q$ induces an orientation on the collection of fat vertices – that is, a partition of fat vertices into two types, which we call positive and negative. There are precisely $q/2$ vertices of each type.

Note that the graphs $G_1$ and $G_2$ have the same vertex set but disjoint edges sets. Let $G_Q$ denote their union: $G_Q := G_1 \cup G_2$.

Similarly, we denote the intersection graph of $P$ and $Q$ in $\hat{P} = \hat{P}_1 \cup \hat{P}_2$ by $G_P$ (noting that this graph is the union of two disjoint non-empty subgraphs $G_{P_i} := G_P \cap \hat{P}_i$, $i = 1, 2$, and hence is not connected).

The edges incident at a vertex $v$ of $G_Q$ are labelled by the boundary components of $P$. These labels always occur in the same cyclic order around $v$ (subject to change of orientation). We choose a numbering $1, \ldots, p$ of $\pi_0(\partial P)$ in such a way that the labels $1, \ldots, p$ always occur in that cyclic order around each vertex of $G_Q$ (without loss of generality, clockwise for positive vertices and anti-clockwise for negative vertices).

The corner at a vertex $v$ of $G_Q$ between the edges labelled $x$ and $x + 1$ (modulo $p$) is also given a label: $g_x$ if $v$ is positively oriented, and $g_x^{-1}$ if $v$ is negatively oriented. Note that corners are arcs in $\partial X(k)$ with endpoints in $P$. In the usual set-up for intersection disks, $P$ is connected, and one can interpret the labels $g_x^\pm_1$ as elements of $\pi_1(M)$ (relative to a base-point on $P$). In our context it is more natural
to interpret \( g_x^{\pm 1} \) as an element of the path-groupoid \( \Pi = \pi(M, P) \), whose elements are (free) homotopy classes of maps of pairs from \( ([0, 1], [0, 1]) \) to \( (M, P) \). Thus \( \Pi \) is a connected 2-vertex groupoid whose vertex groups are isomorphic to \( \pi_1(M) \).

Let \( T \subset M \) denote the Dehn-filling solid torus, and \( k' \subset T \) its core (a knot in \( M \)).

A Scharlemann cycle in \( G_i \) is a cycle \( C \) bounding a disk-component \( \Delta \) of \( \hat{\Theta} \sim G_i \) (which we call a Scharlemann disk), such that each edge of \( C \), regarded as an arc in \( P_t \), joins two fixed components of \( \partial P_t \) (\( x \) and \( y \), say). Thus each edge of \( C \) has label \( x \) at one end, and \( y \) at the other. Since \( x \), \( y \) are consecutive edges of \( G_i \) at each vertex of \( C \), the edges of \( G_Q \cap \Delta \) between \( x \) and \( y \) at \( v \) belong to \( G_{3-i} \) and correspond to intersection points of \( k' \) with \( P_{3-i} \). Since \( P_{3-i} \) is separating, it follows that \( x - y \) is odd, and hence from the parity rule (see for example [6], p. 386) that all vertices of \( C \) have the same orientation.

It is well known (see for example [1], [6]) that any Scharlemann cycle in \( G_i \) corresponds to a lens-space summand of \( M \). We have set things up in such a way that this summand is necessarily isotopic to \( M_i \), which leads to the following observation.

(Compare also [10], Lemma 2.1, which states a similar conclusion under slightly different hypotheses.)

**Lemma 2.1.** Any Scharlemann cycle in \( G_i \) has length \( \ell_i := |\pi_1(M_i)| \).

**Proof.** Without loss of generality, we may assume that \( i = 1 \). Let \( C \) be a Scharlemann cycle in \( G_1 \), and \( \Delta \) the corresponding Scharlemann disk. Assume that \( x \), \( y \) are the labels on the edges of \( C \).

Following [1], [6], we construct a twice punctured lens space in \( M \) as follows. The fat vertices of \( G_{P_1} \) can be regarded as meridional slices of the filling solid torus \( T \). The fat vertices \( x \) and \( y \) divide \( T \) into two 1-handles, one of which – \( H \), say – satisfies \( \partial \Delta \subset P_1 \cup \partial H \).

Then a regular neighbourhood \( L \) of \( \hat{P}_1 \cup H \cup \Delta \) is a twice-punctured lens space, with \( \pi_1(L) \cong \mathbb{Z}_\ell \), where \( \ell \) is the length of \( C \).

One component of \( \partial L \) is \( \hat{P}_1 \). The second component \( \Sigma \) has precisely two fewer points of intersection with \( k' \) than \( \hat{P}_1 \).

By the uniqueness of the prime decomposition \( M = M_1 \# M_2 \# M_3 \), \( L \) is homeomorphic to a twice-punctured copy of \( M_1 \) or of \( M_2 \). In the latter case, \( \Sigma \) also separates \( M_1 \) from \( M_3 \), which contradicts the minimality hypothesis on \( P_1 \). Hence \( L \) is homeomorphic to a twice-punctured copy of \( M_1 \), whence \( \ell = \ell_1 \) as claimed.

More generally, we have the following essentially well-known result, which is an important tool in our proof.

Define the 2-complex \( K \) as follows. \( K \) has two vertices, labelled 1 and 2, and \( p \) edges, labelled \( g_1, \ldots, g_p \). The initial (resp. terminal) vertex of \( g_i \) is 1 or 2 depending on whether the vertex \( i \) (resp. \( i + 1 \)) of \( G_P \) is contained in \( P_1 \) or in \( P_2 \). The 2-cells of \( K \) are in one-to-one correspondence with the disk-regions of \( G_Q \); the attaching map for a 2-cell being read off from the corner-labels of the corresponding region of \( G_Q \).
Lemma 2.2. Let $K_0$ be a subcomplex of $K$ with $H^1(K_0, \mathbb{Z}) = \{0\}$. If $K_0$ is connected then $M$ has a connected summand with fundamental group isomorphic to $\pi_1(K_0)$. If $K_0$ is disconnected, then $M$ has a connected summand with fundamental group isomorphic to $\pi_1(K_0, 1) * \pi_1(K_0, 2)$.

Proof. The intersection of $\hat{P}$ with the filling solid torus $T$ is precisely the set of fat vertices of $G_P$, each of which is a meridional disk in $T$. These disks divide $T$ into 1-handles $H_1, \ldots, H_p$, where $H_i$ is the section of $T$ between the fat vertices $i$ and $i + 1$ (modulo $p$).

Suppose first that $K_0$ is connected. Define $K'$ to be the union of the following subsets of $M$:

1. $P_1$ if $K_0$ contains the vertex 1 of $K$;
2. $P_2$ if $K_0$ contains the vertex 2 of $K$;
3. the one-handle $H_i$ for each edge $g_i \in K_0$;
4. the disk-region of $G_Q$ corresponding to each 2-cell of $K_0$.

It is easy to check that $K'$ is connected, and that $\pi_1(K') \cong \pi_1(K_0)$. Let $N$ be a regular neighbourhood of $K'$ in $M$.

Then $\hat{N}$ is a compact, connected, orientable 3-manifold with $\pi_1(N) \cong \pi_1(K_0)$ and hence $H^1(N, \mathbb{Z}) = \{0\}$. It follows that $\partial N$ consists entirely of spheres, by Poincaré duality.

Capping off each boundary component of $N$ by a ball yields a closed manifold $\hat{N}$ with $\pi_1(\hat{N}) \cong \pi_1(N) \cong \pi_1(K_0)$, and $\hat{N}$ is a connected summand of $M$ since $\hat{N} \subset M$.

Next suppose that $K_0$ is disconnected. Then $K_0$ contains both vertices 1, 2 of $K$, but no edge from 1 to 2. Choose an edge $g_z$ of $K$ joining 1 to 2, and define $K_1 = K_0 \cup \{g_z\}$. Then $K_1$ is connected and $\pi_1(K_1) \cong \pi_1(K_0, 1) * \pi_1(K_0, 2)$. Replacing $K_0$ by $K_1$ in the above gives the result. □

Corollary 2.3. No subcomplex of $K$ has fundamental group which is a free product of three or more finite cyclic groups.

Proof. Suppose that $K$ has such a subcomplex. Then by Lemma 2.2 $M$ has a connected summand which is the connected sum of three lens spaces. This contradicts [11], Corollary 5.3. □

Finally, the element $R = g_1 g_2 \ldots g_p \in \pi_1(M)$ is a weight element – that is, its normal closure is the whole of $\pi_1(M)$ – since it is represented by a meridian in $S^3 \smallsetminus k$. This leads to the following observation, which will be useful later.

Lemma 2.4. Let $x \in \{1, \ldots, p\}$. There is at least one integer $i \in \{1, \ldots, (p - 2)/2\}$ such that no 2-gonal region of $G_Q$ has corners $g_{x+i}$ and $g_{x-i}$ (or $g_{x+i}^{-1}$ and $g_{x-i}^{-1}$).
**Proof.** Otherwise we have \( g_{x+i} = g_{x-i}^{-1} \) in \( \pi_1(M) \) for each \( i = 1, \ldots, (p-2)/2 \), and hence the weight element \( W = g_1 \cdots g_p \) is conjugate to a word of the form \( g_x U g_y U^{-1} \) (where \( U = g_{x+1} \cdots g_{x+(p-2)/2} \) and \( y = x + \frac{p}{2} \) modulo \( p \)). Moreover, \( g_x \) is conjugate in \( \pi_1(M) \) to an element of \( \pi_1(M_i) \) for some \( i \in \{1, 2, 3\} \), and a similar statement holds for \( g_y \). Hence \( W \) belongs to the normal closure in \( \pi_1(M) = \pi_1(M_1) \ast \pi_1(M_2) \ast \pi_1(M_3) \) of the free factors containing conjugates of \( g_x \) and \( g_y \). Since all three free factors are non-trivial, this normal subgroup is proper, which contradicts the fact that \( W \) is a weight element. \( \square \)

### 3. Analysis of Scharlemann cycles

By [6], Proposition 2.8.1, there are Scharlemann cycles in \( G_1 \) and in \( G_2 \). In this section we show that, if \( \ell_1 + \ell_2 \) is big enough, then these form a configuration we call a *sandwiched disk* (which we will show in the next section to be impossible). Our next two results should be compared to [15], Lemmas 3.2 and 5.3, and [7], Theorem 2.4, respectively, where the conclusions are similar but the hypotheses slightly different.

**Lemma 3.1.** If \( \Delta \) is a Scharlemann disk in bounded by a Scharlemann cycle in \( G_1 \) (resp. \( G_2 \)) then \( \Delta \) contains no edges of \( G_2 \) (resp. \( G_1 \)).

**Proof.** Suppose that \( \Delta \) is bounded by a Scharlemann cycle \( C \) in \( G_1 \), and that it contains edges of \( G_2 \). By [6], Proposition 2.8.1, we know that there exists a Scharlemann cycle in \( G_2 \cap \Delta \). We will find such a Scharlemann cycle explicitly, and use it to obtain a contradiction.

Recall that \( C \) has length \( \ell_1 \), by Lemma 2.1. Let \( v_1, \ldots, v_{\ell_1} \) denote the vertices of \( C \) in cyclic order. Each edge of \( C \) has labels \( x \) and \( x + 2t + 1 \), say, which correspond to vertices in \( G_{P_1} \), and the intermediate labels \( x + 1, \ldots, x + 2t \) correspond to vertices of \( G_{P_2} \). (Necessarily, these are even in number and alternating in orientation, since they correspond to consecutive intersection points of \( k' \) with \( \hat{P}_2 \) between two consecutive intersection points of \( k' \) with \( \hat{P}_1 \).)

The graph \( Y := G_2 \cap \Delta \) has \( \ell_1 \) vertices, each of valence \( 2t \) and each of the same orientation (which we assume to be positive).

If \( \ell_1 = 2 \), then every edge of \( Y \) joins \( v_1 \) to \( v_2 \). Such an edge has labels \( x + j \) at one end and \( x + 2t + 1 - j \) at the other, for some \( j \). The two edges whose labels are \( x + t \) and \( x + t + 1 \) bound a 2-gonal region, and hence form a Scharlemann cycle of length 2. But then \( \ell_1 = \ell_2 = 2 \), contradicting the fact that \( \ell_1, \ell_2 \) are coprime.

Suppose then that \( \ell_1 > 2 \). There must be a vertex \( v_j \) in \( C \) that is joined only to \( v_{j-1} \) and \( v_{j+1} \) (subscripts modulo \( \ell_2 \)) by edges of \( Y \). In particular there are two consecutive vertices of \( C \) that are joined by \( s \geq t \) edges of \( Y \). The resulting \( s \) 2-gonal regions of \( G_2 \cap \Delta \) give rise to relations \( g_{x+j} g_{x+2t-j} = 1 \) for \( 0 \leq j \leq s-1 \) in the path-groupoid \( \Pi = \pi(M, P) \). But all the corners of the Scharlemann disk \( \Delta \) have
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Choose a pair $v_i, v_j$ of vertices of $C$ with $i < j - 1$ with $j - i$ minimal subject to the condition that $v_i, v_j$ are joined by an edge of $Y$. Then each pair $(v_i, v_{i+1}), \ldots, (v_j, v_j)$ is joined by precisely $t$ edges of $Y$, so there is an edge joining $v_i$ and $v_j$ that has labels $x + t$ and $x + t + 1$, and this forms part of a Scharlemann cycle of length $j + 1 - i$ in $G_2$. (See Figure 1.)

Since $g_{x+t}$ has order $\ell_1$ in $\Pi$, we deduce that $\ell_1 = \ell_2$, which again contradicts the fact that $\ell_1, \ell_2$ are coprime.

In particular, if $C$ is a Scharlemann cycle in $G_1$ or $G_2$, then the two labels appearing on the edges of $C$ are consecutive (modulo $p$): say $x, x + 1$. We call $x$ the label of $C$. Note that all the corners of the corresponding Scharlemann disk have the same label $g_x$ or $g_x^{-1}$.

**Corollary 3.2.** Any two Scharlemann cycles in $G_1$ (respectively in $G_2$) have the same label.

**Proof.** Let $C, C'$ be Scharlemann cycles in $G_1$, bounding Scharlemann disks $\Delta, \Delta'$ respectively. By Lemma 3.1, $\Delta$ and $\Delta'$ contain no edges of $G_2$, so are Scharlemann disks of $G_Q$. By Lemma 2.1 each of $C, C'$ has length $\ell_1$. Suppose that $C$ has label $x$ and $C'$ has label $y \neq x$. Then $K$ has a subcomplex $K_0$ with one vertex 1, two edges $g_x, g_y$ and two 2-cells $\Delta, \Delta'$, so that

$$\pi_1(K_0) = \langle g_x, g_y \mid g_x^{\ell_1} = g_y^{\ell_1} = 1 \rangle \cong \mathbb{Z}_{\ell_1} \ast \mathbb{Z}_{\ell_1}.$$  

In particular, $\pi_1(K_0)$ has weight 2, so cannot be isomorphic to a free factor of $\pi_1(M)$, which contradicts Lemma 2.2.

**Definition 3.3.** A sandwiched disk in $\tilde{Q}$ is a disk $D \subset \tilde{Q}$ such that

(a) $\partial D$ is the union of a subpath $a_1$ of a Scharlemann cycle $C_1 \subset G_1$ and a subpath $a_2$ of a Scharlemann cycle $C_2 \subset G_2$, with $a_1 \cap a_2 = \partial a_1 = \partial a_2$;
(b) there are no vertices of $G_Q$ in the interior of $D$.

**Lemma 3.4.** If $|\pi_1(M_1)| + |\pi_1(M_2)| > (q + 2)/2$, then there exists a sandwiched disk $D \subset \hat{Q}$.

*Proof.* As observed in [10], p. 551, and [15], Lemma 6.1, we know that there are at least two Scharlemann cycles in $G_1$—necessarily with disjoint sets of vertices, since they have the same label (Corollary 3.2). Similarly there are at least two Scharlemann cycles in $G_2$—again with the same label and hence with disjoint sets of vertices.

By hypothesis, at least one of $\ell_1 = |\pi_1(M_1)|$, $\ell_2 := |\pi_1(M_2)|$ is greater than $q/4$. Without loss of generality, assume that $\ell_1 > q/4$. If $G_1$ contained two Scharlemann cycles with the same (say, positive) orientation, then $G_1$ would have at least $2\ell_1 > q/2$ positive vertices, contradicting the fact that $G_1$ has precisely $q/2$ vertices of each orientation.

Hence $G_1$ must contain precisely two Scharlemann cycles, one of each possible orientation. Let us call them $C_1^+$ and $C_1^-$, and let $\Delta_1^\pm$ denote the Scharlemann disks bounded by $C_1^\pm$.

Now let $C_2$, $C_2'$ denote two disjoint Scharlemann cycles in $G_2$, and $\Delta_2$, $\Delta_2'$ the corresponding Scharlemann disks. Since $\ell_1 + \ell_2 > q/2$, $C_2$ must intersect $C_1^+$ (if the vertices of $C_2$ are positive) or $C_1^-$ (if the vertices of $C_2$ are negative). On the other hand, consideration of vertex orientations shows that $C_2$ cannot intersect both $C_1^+$ and $C_1^-$. Similar remarks apply to $C_2'$.

Now $(C_1^+ \cup C_1^-) \cap (C_2 \cup C_2')$ consists only of some set $V$ (of cardinality $t$, say) of vertices.

Then $\Delta := \Delta_1^+ \cup \Delta_1^- \cup \Delta_2 \cup \Delta_2'$ has precisely two components, $2\ell_1 + 2\ell_2 - t$ vertices, $2\ell_1 + 2\ell_2$ edges, and four 2-cells. The complement of $\Delta$ in $\hat{Q}$ thus contains $t - 1$ components, one of which is an annulus and $t - 2$ are disks. But $\partial(\hat{Q} \setminus \Delta)$ also contains precisely $q - 2\ell_1 - 2\ell_2 + t$ vertices. Since $2\ell_1 + 2\ell_2 \geq q + 4$, this number is at most $t - 4$. Hence there are at least two disk-components of $\hat{Q} \setminus \Delta$ that contain no vertices of $G_Q$.

Moreover, each vertex of $V$ appears twice in $\partial(\hat{Q} \setminus \Delta)$, so there are $2t$ such occurrences in total. Each occurrence separates an arc of $C_1^+ \cup C_1^-$ from an arc of $C_2 \cup C_2'$ in $\partial(\hat{Q} \setminus \Delta)$, so each component of $\partial(\hat{Q} \setminus \Delta)$ contains an even number of occurrences of vertices from $V$.

The number of boundary components of $\hat{Q} \setminus \Delta$ is precisely $t$. If the vertices in $C_2$ and those in $C_2'$ have the same orientation, then one of $C_1^+$, $C_1^-$ is a boundary component of (the annulus component of) $\hat{Q} \setminus \Delta$ and contains no vertices from $V$. With that exception, each boundary component of $\hat{Q} \setminus \Delta$ contains at least two occurrences of vertices from $V$. Hence at most one boundary component of $\hat{Q} \setminus \Delta$ can contain more than two occurrences of vertices from $V$.

Hence, of the two (or more) disk-components of $\hat{Q} \setminus \Delta$ that contain no vertices of $G_Q$, each contains at least two occurrences of vertices from $V$, while at most one
of these disk-components contains more than two occurrences of vertices from $V$. It follows that there is at least one disk component $D$ of $\hat{Q} \setminus \Delta$ whose boundary contains precisely two occurrences of vertices from $V$ and whose interior contains no vertices of $G_Q$.

Any such $D$ is, by definition, a sandwiched disk.

\section{Analysis of sandwiched disks}

In this section we complete the proof of our upper bound on $\lvert r \rvert$ by showing that sandwiched disks do not exist. This result holds with no assumptions on $\ell_1$ or $\ell_2$, so may have wider applications.

We assume throughout that $G_1, G_2$ contain Scharlemann cycles of length $\ell_1, \ell_2$, respectively, with labels $x_1, x_2$ respectively.

\begin{lemma}
Let $D$ be a sandwiched disk with $\partial D = a_1 \cup a_2$, where $a_1, a_2$ are sub-paths of Scharlemann cycles in $G_1, G_2$, respectively. Then no two consecutive vertices of $a_1$ (or of $a_2$) are joined by $p/2$ edges in $G_Q$.
\end{lemma}

\begin{proof}
Suppose that two vertices of (say) $a_1$ are joined by $p/2$ edges. Then there are 2-gonal regions $D_i$ in $G_Q \cap D$ such that the corner labels of $D_i$ are $g_{x_1+i}$ and $g_{x_1-i}$. This contradicts Lemma 2.4.
\end{proof}

\begin{corollary}
Let $D, a_1, a_2$ be as in Lemma 4.1. If two vertices of $a_1$ (or of $a_2$) are connected by an edge in $G_Q$, then they are consecutive vertices of $a_1$ (respectively of $a_2$).
\end{corollary}

\begin{proof}
Let $w_0, \ldots, w_t$ be the vertices of $a_1$, in order. Suppose that $w_i, w_j$ are joined by an edge in $G_Q$, where $j > i + 1$, and that $j - i$ is minimal for such pairs of vertices. Then $w_{i+1}$ has precisely two neighbours in $G_Q$: $w_i$ and $w_{i+2}$. By Lemma 4.1 it is connected to each by fewer than $p/2$ edges, contradicting the fact that it has valence $p$.
\end{proof}

\begin{corollary}
Let $D, a_1, a_2$ be as in Lemma 4.1. Each of $a_1, a_2$ has length greater than 1, and each interior vertex of $a_1$ (respectively $a_2$) is joined to an interior vertex of $a_2$ (respectively $a_1$) by an edge of $G_Q \cap D$.
\end{corollary}

\begin{proof}
If $a_1, a_2$ both have length 1, then every edge of $G_Q \cap D$ joins the two common endpoints $u, v$ of $a_1$ and $a_2$. Without loss of generality, the edges of $G_Q \cap D$ incident at $u$ have labels $x_1 + 1, x_1 + 2, \ldots, x_2$, while those incident at $v$ have labels $x_2 + 1, x_2 + 2, \ldots, x_1$. Hence $\lvert x_1 - x_2 \rvert = p/2$, and $D$ contains precisely $p/2$ arcs joining $u$ to $v$. But this contradicts Lemma 4.1.

If $w$ is an interior vertex of (say) $a_1$, then $w$ has two neighbours in $\partial D$. It is joined to each of these by strictly fewer than $p/2$ arcs, by Lemma 4.1, and hence is also
Theorem 4.6. There are no sandwiched disks in $G_Q$. Since all the edges of $G_Q$ incident at $w$ are contained in $D$, this third vertex is also in $\partial D$. By Corollary 4.2 it cannot be a vertex of $a_1$, so it must be an interior vertex of $a_2$. \hfill \Box

Lemma 4.4. Let $D$ be a sandwiched disk in $G_Q$. Then there are no Scharlemann cycles in $G_Q \cap D$.

Proof. Any Scharlemann cycle $C$ in $G_Q \cap D$ is a Scharlemann cycle in $G_1$ or in $G_2$, so has label $x_1$ or $x_2$ by Corollary 3.2. Assume without loss of generality that $C$ has label $x_2$. For any vertex $v$ of $a_2$, the corner labelled $g_{x_2}$ does not lie in $D$, so the vertices of $C$ are interior vertices of $a_1$.

By Corollary 4.2, the vertices of $C$ must be pairwise consecutive vertices of $a_1$, and hence $C$ has length 2. Moreover, if $v_1$, $v_2$ are the vertices of $C$, then $v_1$, $v_2$ are connected by edges labelled $x_1 + 1, \ldots, x_2$ at one end (say the $v_1$ end), and by edges labelled $x_2 + 1, \ldots, x_1$ at the other ($v_2$) end. In particular, they are joined by at least $p/2$ edges, contradicting Lemma 4.1. \hfill \Box

Corollary 4.5. If there is a sandwiched disk $D$ in $G_Q$ such that $\partial D = a_1 \cup a_2$ where $a_i$ is a subpath of a Scharlemann cycle with label $x_i$, then $|x_1 - x_2| = p/2$.

Proof. Let $a_1 \cap a_2 = \{u, v\}$. Without loss of generality, $x_1 = p$ and the edges of $G_Q \cap D$ meeting $u$ are labelled $1, \ldots, x_2$ at $u$, while those meeting $v$ are labelled $x_2 + 1, \ldots, p$ at $v$. If (say) $x_2 < p/2$, then there is a label $y$ with $x_2 < y \leq p$ such that $y$ does not appear as either label of any edge meeting $u$ that is contained in $D$. Consider the subgraph $\Gamma$ of $G_Q \cap D$ that is obtained by removing $u$ and its incident edges. At each vertex of $\Gamma$, the edge labelled $y$ leads to another vertex of $\Gamma$. Since all vertices of $\Gamma$ are positive, it follows that $\Gamma$ contains a great $y$-cycle, and hence a Scharlemann cycle by [1], Lemma 2.6.2. This contradicts Lemma 4.4. \hfill \Box

Theorem 4.6. There are no sandwiched disks in $G_Q$.

Proof. We assume that there is a sandwiched disk $D$ in $G_Q$, and derive a contradiction. Suppose that $\partial D = a_1 \cup a_2$, where $a_i$ is a subpath of a Scharlemann cycle $C_i$. Let $x_i$ be the label of $C_i$. By Corollary 4.5, it follows that $|x_1 - x_2| = p/2$.

Let $u$, $v$ denote the common vertices of $a_1, a_2$. By Corollary 4.2 each of $a_1, a_2$ has length greater than 1. Let $s_1, s_2$ be the vertices of $a_1, a_2$, respectively, which are adjacent to $u$, and let $t_1, t_2$ be the vertices of $a_1, a_2$, respectively, which are adjacent to $v$. (Note that neither of the possibilities $s_1 = t_1, s_2 = t_2$ is excluded at this stage.)

By Corollary 4.2 again, $s_1$ is connected to a vertex of $a_2$ other than $u, v$ by an edge contained in $D$. Similarly, $s_2$ is connected to a vertex of $a_1$ other than $u, v$ by an edge contained in $D$. These edges cannot cross; hence $s_1$ and $s_2$ are joined by an edge. Similarly $t_1$ and $t_2$ are joined by an edge. Hence each of $u, v$ is incident at a triangular region of $G_Q \cap D$: call them $\Delta_u$ and $\Delta_v$. 
Suppose that the edges of \( G_Q \cap D \) that are incident at \( u \) have labels \( x_1 + 1, \ldots, x_2 \) at \( u \), and suppose that \( i \) of these edges (namely those with labels \( x_1 + 1, \ldots, x_1 + i \)) are connected to \( s_1 \). Then these edges have labels \( x_1, x_1 - 1, \ldots, x_1 - i + 1 \) at \( s_1 \), and together they bound \( i - 1 \) \( 2 \)-gonal faces of \( G_Q \), of which the \( j \)’th has corner labels \( g_{x_1 + j} \) and \( g_{x_1 - j} \).

The remaining \( (p - 2i)/2 \) edges of \( G_Q \cap D \) incident at \( u \) join \( u \) to \( s_2 \). They have labels \( x_1 + i + 1, \ldots, x_2 \) at \( u \), and \( x_1 - i, \ldots, x_2 + 1 \) at \( s_2 \). Together they bound \( (p - 2i - 2)/2 \) \( 2 \)-gonal regions of \( G_Q \), the \( j \)’th of which has corner labels \( g_{x_2 - j} \) and \( g_{x_2 + j} \). Thus the triangular region \( \Delta_u \) of \( D \cap G_Q \) that is incident at \( u \) has corner labels \( g_y \) at \( u \) and \( g_z \) at each of \( s_1 \) and \( s_2 \), where \( y = x_1 + i \) and \( z = x_1 - i \) (modulo \( p \)).

We can now perform a similar analysis on the edges of \( G_Q \cap D \) that are incident at \( v \). Note, however, that for all \( j \in \{1, \ldots, (p - 2)/2\} \setminus \{i\} \) there is a \( 2 \)-gonal region of \( G_Q \cap D \) with corner labels \( g_{x_1 - j} \) and \( g_{x_1 + j} \). By Lemma 2.4 there cannot be a \( 2 \)-gonal region of \( G_Q \cap D \) with corner labels \( g_{x_1 - i} \) and \( g_{x_1 + i} \). It follows that there are also precisely \( i \) edges joining \( v \) to \( t_1 \), and \( (p - 2i)/2 \) joining \( v \) to \( t_2 \). The triangular region \( \Delta_v \) of \( D \cap G_Q \) that is incident at \( v \) then has corner labels \( g_z \) at \( v \) and \( g_y \) at each of \( t_1, t_2 \), where \( y = x_1 + i \) and \( z = x_1 - i \) as above (see Figure 2).

![Figure 2](image)

Finally, let \( K_0 \) denote the (disconnected) subcomplex of \( K \) with vertices \( \{0, 1\} \), edges \( \{g_{x_1}, g_{x_2}, g_y, g_z\} \) and 2-cells \( \{\Delta_1, \Delta_2, \Delta_u, \Delta_v\} \).

Then by Lemma 2.2, \( M \) has a connected summand with fundamental group

\[
\pi_1(K_0, 1) \ast \pi_1(K_0, 2) \cong \langle g_{x_1}, g_{x_2}, g_y, g_z \mid g_{x_1}^2 = g_{x_2}^2 = g_y g_z^2 = 1 \rangle \cong \mathbb{Z}_{\ell_1} \ast \mathbb{Z}_{\ell_2} \ast \mathbb{Z}_3.
\]

But this contradicts Corollary 2.3, which completes the proof. \( \square \)

**Theorem 4.7** (= Theorem 1). Let \( k \) be a knot in \( S^3 \) with bridge-number \( b \). Suppose that \( r \) is a slope on \( k \) such that \( M = M(k, r) = M_1 \# M_2 \# M_3 \) where \( M_1, M_2 \) are
lens spaces and $M_3$ is a homology sphere but not a homotopy sphere. Then

$$|\pi_1(M_1)| + |\pi_1(M_2)| \leq b + 1.$$  

**Proof.** As discussed in §2, we put $k$ in thin position, and choose a level surface $Q$ and disjoint planar surfaces $P_1, P_2$ such that

- $P_i$ extends to a sphere in $M$ separating $M_i$ from $M_3$, and has fewest boundary components among all such;
- no component of $Q \cap P_i$ is a boundary-parallel arc in $Q$ or $P_i$.

By Gordon and Luecke [5], there are Scharlemann cycles $C_i$ in $G_i$ for $i = 1, 2$. Moreover, the Scharlemann cycle $C_i$ has length $\ell_i := |\pi_1(M_i)|$ and bounds a disk-region $\Delta_i$ of $G_Q$. If $\ell_1 + \ell_2 > b + 1 \geq (q + 2)/2$, then by Lemma 3.4 there is at least one sandwiched disk $D$ in $G_Q$. But this contradicts Theorem 4.6.

Hence $\ell_1 + \ell_2 \leq b + 1$ as claimed. $\square$

**Corollary 4.8** (= Corollary 2). With the hypotheses and notation of Theorem 4.7, we have

$$|r| = |\pi_1(M_1)| \cdot |\pi_1(M_2)| \leq \frac{b(b + 2)}{4}.$$  

**Proof.** Let $\ell_1 = |\pi_1(M_1)|$ and $\ell_2 = |\pi_1(M_2)|$. The equation $|r| = \ell_1 \cdot \ell_2$ comes from computing $|H_1(M, \mathbb{Z})|$ in two different ways.

Given that $\ell_1, \ell_2$ are distinct positive integers, the inequality $\ell_1 \cdot \ell_2 \leq b(b + 2)/4$ follows easily from Theorem 4.7. $\square$

**References**


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