A moving boundary problem for periodic Stokesian Hele–Shaw flows

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This paper is concerned with the motion of an incompressible, viscous fluid in a Hele–Shaw cell. The free surface is moving under the influence of gravity and the fluid is modelled using a modified Darcy law for Stokesian fluids.

We combine results from the theory of quasilinear elliptic equations, analytic semigroups and Fourier multipliers to prove existence of a unique classical solution to the corresponding moving boundary problem.

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1. Introduction

Starting from a non-Newtonian Darcy law as presented in [9], we derive a mathematical model for the flow of a Stokesian fluid located between the plates of a vertical Hele–Shaw cell. The pressure on the bottom of the cell is assumed to be constant. The corresponding mathematical setting is a fully nonlinear coupled system consisting of a quasilinear elliptic Dirichlet problem for the velocity potential and an evolution equation for the free boundary, i.e. the interface separating the fluid from the air. The contact angle problem is avoided by considering periodic flows only. The Newtonian case, studied in [3]–[7] in various contexts, is also included in this model. Our setting is general enough to embrace shear thinning fluids, like Oldroyd-B or power law fluids, as well as shear thickening fluids.

We shall attack this problem by transforming it into a problem on a fixed manifold $S^1 \times (0, 1)$. This will be done in Section 1. In Section 2 we identify the new setting with an abstract Cauchy problem on the unit circle $S^1$:

$$\partial_t f + \Phi(f) = 0, \quad f(0) = f_0.$$

Our analysis shows that $\Phi$ is a pseudodifferential operator of first order with a symbol depending nonlinearly on the function $f$ modelling the free boundary. Moreover, the operator $f \mapsto \Phi(f)$ is fully nonlinear, in the sense that its nonlinear part is of first order as well. Nevertheless, we prove that given any positive constant $c$, the Fréchet derivative $-\partial \Phi(c)$ generates a strongly continuous

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1 In a Stokesian fluid the stress tensor is a continuous function of the deformation. A Newtonian fluid is a linear Stokesian fluid. In particular, the viscosity $\mu$ is constant in this case.
analytic semigroup in $\mathcal{L}(\ell^{1+\sigma}(\mathbb{S}^1))$ with dense domain $h^{2+\sigma}(\mathbb{S}^1)$. Working with small Hölder spaces $h^{m+\sigma}(\mathbb{S}^1)$, $m \in \mathbb{N}$ and $\alpha \in (0, 1)$, is a significant advantage, because $h^{m_1+\alpha_1}(\mathbb{S}^1)$ is dense and compactly embedded in $h^{m_2+\alpha_2}(\mathbb{S}^1)$ provided $m_1 + \alpha_1 > m_2 + \alpha_2$. It is known that this property does not hold for the usual Hölder spaces.

The main result, a well-posedness result for the full flow, is proved in Section 3 and is based on a multiplier theorem for periodic Besov spaces. This theorem generalizes a result of Arendt and Bu presented in [2]. As in [2], our multiplier theorem is also based on Marcinkiewicz type conditions.

1.1 The mathematical model

Given a positive function $f \in C^1(\mathbb{R})$, which is bounded away from 0, we define the set

$$\widetilde{\Omega}_f := \{(x, y) \in \mathbb{R}^2 : 0 < y < f(x)\},$$

and denote the components of its boundary by

$$\widehat{\Gamma}_f := \{(x, f(x)) : x \in \mathbb{R}\}, \quad \widehat{\Gamma}_0 := \mathbb{R} \times \{0\}.$$

The domain $\widetilde{\Omega}_f$ consists of a Stokesian fluid at pressure $p$ and we denote by $v$ the velocity field inside the fluid’s body. The motion of the fluid is governed by the following modified version of Darcy’s law:

$$v = -\frac{Du}{\mu(|Du|^2)}$$

(cf. [9]), where

$$u(x, y) = \frac{p(x, y)}{g \cdot \rho} + y, \quad (x, y) \in \widetilde{\Omega}_f,$$

is the so-called velocity potential or piezometric heat, $g$ is the gravity acceleration, $\rho$ is the density of the fluid and $Du = (\partial_1 u, \partial_2 u)$ is the gradient of $u$. The effective viscosity $\mu$ is defined (see [9]) by

$$\frac{1}{\mu(r)} := c_\mu \int_{-1}^1 \frac{s^2}{\mu(r^2)} \, ds$$

for all $r \geq 0$, where $c_\mu$ is a positive constant. Denoting by $\mu \in C^\infty([0, \infty), (0, \infty))$ the viscosity of the fluid, we have assumed that the mapping $r \mapsto h(r) := r \mu^2(r)$ is invertible. This is true for example if $\mu(r) + 2r \mu'(r) > 0$ for all $r \geq 0$. The mapping $\tilde{\mu}$ is defined by $\tilde{\mu} := \mu \circ h^{-1}$.

We assume the fluid is incompressible ($\text{div} \, v = 0$), thus we get

$$\text{div} \left( \frac{Du}{\mu(|Du|^2)} \right) = 0 \quad \text{in} \quad \widetilde{\Omega}_f. \quad (1)$$
On the boundary component $\tilde{\Gamma}_0$ the velocity potential is known, namely

$$u(x, 0) = \frac{p(x, 0)}{g - \rho} = b(x), \quad x \in \mathbb{R}. \quad (2)$$

Moreover, we assume that the fluid is surrounded by air at atmospheric pressure, normalized to be zero. Then $p(x, f(x)) = 0$ for $x \in \mathbb{R}$, and so

$$u(x, f(x)) = f(x), \quad x \in \mathbb{R}. \quad (3)$$

Set $F(t, z) = y - f(t, x)$ for $z = (x, y) \in \mathbb{R}$ and $t \geq 0$. Then the interface $\tilde{\Gamma}_f$ can be described by the conservative property that $F$ is identically equal to zero on $\tilde{\Gamma}_f$.

Differentiating with respect to the time variable $t$ we get

$$\frac{d}{dt} F(t, z) = -\partial_t f(t, x) + (f_x, 1) \cdot z'. \quad (4)$$

Replacing $z'$ by $-Du/\mu(|Du|^2)$, we obtain

$$\partial_t f + \sqrt{1 + \partial_x f^2} \partial_u u = 0 \quad \text{on} \ \tilde{\Gamma}_f,$$

where $\nu$ denoting the outer normal of $\tilde{\Gamma}_f$. Finally, we set

$$f(0, \cdot) = f_0. \quad (5)$$

where $f_0$ corresponds to the initial surface. We shall make the following periodicity requirement on $f$ and $u$:

$$f(t, x + 2\pi) = f(t, x), \quad \forall x \in \mathbb{R}, \ t \geq 0,$$

$$u(x + 2\pi, y) = u(x, y), \quad \forall (x, y) \in \tilde{\Omega}_f(t), \ t \geq 0.$$

Thus, instead of (1)–(5) we study

$$\text{div} \left( \frac{Du}{\mu(|Du|^2)} \right) = 0 \quad \text{in} \ \Omega_f(t), \ t \geq 0,$$

$$u = b \quad \text{on} \ \Gamma_0, \ t \geq 0,$$

$$u = f \quad \text{on} \ \Gamma_f(t), \ t \geq 0,$$

$$\partial_t f(t, \cdot) + \sqrt{1 + \partial_x f^2(t, \cdot)} \partial_u u(\cdot, f(t, \cdot)) = 0 \quad \text{on} \ \mathbb{S}^1, \ t > 0,$$

$$f(0, \cdot) = f_0 \quad \text{on} \ \mathbb{S}^1, \quad (6)$$

where

$$\Omega_f(t) := \{(x, y) \in \mathbb{S}^1 \times \mathbb{R} : 0 < y < f(t, x) \},$$

$$\Gamma_f(t) := \{(x, f(t, x)) : x \in \mathbb{S}^1 \}, \quad \Gamma_0 = \mathbb{S}^1 \times \{0\}.$$
for \( t \geq 0 \), and \( S^1 \) is the unit circle. For the sake of simplicity, we identify periodic functions on \( \mathbb{R} \) with functions on \( S^1 \), and periodic functions in the \( x \) variable on \( \Omega_f \) with functions on \( \Omega_f \), for positive functions \( f \) on \( S^1 \).

Given \( m \in \mathbb{N} \) and \( \alpha \in (0, 1) \), we define the so-called little Hölder space \( h^{m+\alpha}(S^1) \) as the closure of \( C^\infty(S^1) \) in \( C^{m+\alpha}(S^1) \). If \( f \) is a positive function in \( C(S^1) \), then we denote by \( \text{buc}^{m+\alpha}(\Omega_f) \) the closure of \( \text{BUC}^\infty(\Omega_f) \) in the Hölder space \( \text{BUC}^{m+\alpha}(\Omega_f) \). The notation \( \text{BUC}^{m+\alpha}(\Omega_f) \) stands for the space of all maps from \( \Omega_f \) to \( \mathbb{R} \) which have bounded and uniformly continuous derivatives up to order \( m \), and in addition uniformly \( \alpha \)-Hölder continuous derivatives of order \( m \).

Throughout this paper we fix \( \alpha \in (0, 1) \) and we define

\[
\mathcal{U} := \{ f \in C^{2+\alpha}(S^1) : \min_{x \in S^1} f(x) > 0 \}, \quad \mathcal{V} := \mathcal{U} \cap h^{2+\alpha}(S^1).
\]

A pair \((u, f)\) is called a classical Hölder solution of (6) on \([0, T] \), \( T > 0 \), if

\[
f \in C([0, T], \mathcal{V}) \cap C^1([0, T], h^{1+\alpha}(S^1)),
\]

\[
u(\cdot, t) \in \text{buc}^{2+\alpha}(\Omega_f(t)), \quad t \in [0, T],
\]

and \((u, f)\) satisfies the equations in (6) pointwise. Suppose there exist two positive constants \( m_\mu \) and \( M_\mu \) such that

\[
(A_1) \quad m_\mu \leq \overline{\mu}(r) \leq M_\mu, \quad \forall r \geq 0,
\]

\[
(A_2) \quad m_\mu \leq \overline{\mu}(r) - 2r \overline{\mu}'(r) \leq M_\mu, \quad \forall r \geq 0.
\]

Our main result reads as follows.

**Theorem 1.1** Assume \((A_1)\) and \((A_2)\) hold true. Then we have:

(a) Let \( c \) and \( b \) be two positive constants. There exists an open neighbourhood \( \mathcal{O} \) of \( c \) in \( \mathcal{V} \) such that, for each \( f_0 \in \mathcal{O} \), problem (6) has a classical Hölder solution \((u, f)\) on an interval \([0, T] \), \( T > 0 \). Moreover, there exists a constant \( \gamma \in (0, 1) \) such that \( f \in C_\gamma^\infty((0, T], h^{2+\alpha}(S^1)) \).

(b) Let \((u_1, f_1)\) and \((u_2, f_2)\) be solutions of (6) with \( f_1 \in C_\gamma^\infty((0, T], h^{2+\alpha}(S^1)) \), \( \gamma \in (0, 1) \), and \( f_2 \in C_\delta^\infty((0, T], h^{2+\alpha}(S^1)) \), \( \delta \in (0, 1) \). If \( f_1([0, T]) \subset \mathcal{O} \) and \( f_2([0, T]) \subset \mathcal{O} \), then \((u_1, f_1) = (u_2, f_2)\).

For the definition of the weighted Hölder spaces \( C_\gamma^\infty((0, T], h^{2+\alpha}(S^1)) \), \( \gamma \in (0, 1) \) see \([10]\). If the viscosity \( \mu \) is decreasing then the Stokesian fluid is called shear thinning. If \( \mu \) is increasing then the fluid is called shear thickening. Notice that, if \( \mu \) is constant, then \( \overline{\mu} \) is also constant. Moreover, if \( \mu \) is a strictly decreasing or strictly increasing function of its argument, then so is \( \overline{\mu} \). The conditions \((A_1)\) and \((A_2)\) ensure that at great velocities the fluid behaves like a Newtonian fluid.

We now look for conditions on \( \mu \) which imply \((A_1)\) and \((A_2)\). We remark that \((A_1)\) and \((A_2)\) are satisfied if there exist positive constants \( c \) and \( C \) with

\[
c \leq \frac{1}{\overline{\mu}(r)} \leq C, \quad \forall r \geq 0,
\]

\[
c \leq \frac{1}{\overline{\mu}(r)} + 2r \left( \frac{1}{\mu} \right)'(r) \leq C, \quad \forall r \geq 0.
\]
Using the definition of $\mu$ we compute
\[
\frac{1}{\mu(r)} + 2r \left( \frac{1}{\mu} \right)'(r) = c_\mu \int_{-1}^{1} s^2 \left[ \frac{1}{\mu(rs^2)} + 2(rs^2) \left( \frac{1}{\mu} \right)'(rs^2) \right] ds,
\]
hence $(A_1)$ and $(A_2)$ are satisfied if there exist positive constants $\tilde{c}$ and $\tilde{C}$ with
\[
\tilde{c} \leq \frac{1}{\mu(r)} \leq \tilde{C}, \quad \forall r \geq 0,
\]
\[
\tilde{c} \leq \frac{1}{\mu(r)} + 2r \left( \frac{1}{\mu} \right)'(r) \leq \tilde{C}, \quad \forall r \geq 0.
\]
Further we compute
\[
\frac{1}{\mu(r)} + 2r \left( \frac{1}{\mu} \right)'(r) = \frac{1}{\mu^2(h^{-1}(r))} \left( \mu(h^{-1}(r)) - 2r \mu'(h^{-1}(r))(h^{-1})'(r) \right)
\]
\[
= \frac{1}{\mu^2(s)} \left( \mu(s) - 2h(s)\mu'(s) \frac{1}{h'(s)} \right)
\]
\[
= \frac{1}{\mu^2(s)} \left( \mu(s) - 2s\mu^2(s)\mu'(s) \frac{1}{\mu^2(s) + 2s\mu(s)\mu'(s)} \right)
\]
\[
= \frac{1}{\mu(s) + 2s\mu'(s)},
\]
thus, $(A_1)$ and $(A_2)$ hold if there exist positive constants $\overline{\tau}$ and $\overline{C}$ such that
\[
(V_1) \quad \overline{\tau} \leq \mu(r) \leq \overline{C},
\]
\[
(V_2) \quad \overline{\tau} \leq \mu(r) + 2r\mu'(r) \leq \overline{C},
\]
for all $r \geq 0$. The class of fluids with viscosity satisfying $(V_1)$ and $(V_2)$ is quite large.

For Oldroyd-B fluids, e.g. blood, the viscosity is given by
\[
\mu(r) = v_\infty + (v_0 - v_\infty) \frac{1 + \ln(1 + \lambda r)}{1 + \lambda r}, \quad r \geq 0,
\]
where $\lambda > 0$ is a material constant and $v_0 > v_\infty > 0$. The conditions $(V_1)$ and $(V_2)$ hold if $(e^2 + 1)v_\infty > v_0$. Also, various variants of power law fluids belong to this class:
\[
\mu(r) = v_\infty + v_0(1 + r^s)^{s/4} \quad \text{or} \quad \mu(r) = v_\infty + v_0(1 + r)^{s/2},
\]
for all $r \geq 0$, where $v_0$ and $v_\infty$ are positive and $s \leq 0$. In this case $(V_1)$ and $(V_2)$ hold if $-1 \leq s \leq 0$. Notice that the above examples are all shear thinning fluids. We now give an example of a shear thickening fluid which can be considered in our model. If
\[
\mu(r) = \mu_0 \frac{\gamma r + r_0}{r + r_0}, \quad \forall r \geq 0,
\]
with $r_0 > 0$, $\gamma \geq 1$ and $\mu_0 > 0$, then $(V_1)$ and $(V_2)$ hold for any choice of the parameters $r_0$, $\mu_0$ and $\gamma$. 
1.2 The transformed problem

For simplification we introduce first the operator $Q : C^2(\Omega_f) \to C(\Omega_f)$ with

$$Qu := \text{div}\left(\frac{Du}{F(|Du|^2)}\right), \quad u \in C^2(\Omega_f).$$

In order to solve the problem we transfer it onto a fixed reference manifold. Let $\Omega := S^1 \times (0, 1)$.

For $f \in \mathcal{U}$ we define $\phi_f \in \text{Diff}^{2+\alpha}(\Omega, \Omega_f)$ by

$$\phi_f(x, y) = (x, (1 - y)f(x)), \quad (x, y) \in \Omega.$$

Defining the push-forward and pull-back operators induced by $\phi_f$,

$$\phi_f^* : \text{BUC}(\Omega_f) \to \text{BUC}(\Omega), \quad u \mapsto u \circ \phi_f,$$

$$\phi_f^* : \text{BUC}(\Omega) \to \text{BUC}(\Omega_f), \quad v \mapsto v \circ \phi_f^{-1},$$

we introduce the transformed operators $A(f)$ and $B$, acting on $\text{BUC}^2(\Omega)$ and $\mathcal{U} \times \text{BUC}^{2+\alpha}(\Omega)$ respectively by

$$A(f) := \phi_f^* \circ Q \circ \phi_f^*,$$

$$B(f, v)(x) := \frac{D(\phi_f^* v)}{|D(\phi_f^* v)|^2}(x, f(x)) \cdot n(x), \quad x \in S^1,$$

with $n(x) := (-f'(x), 1)$, $x \in S^1$.

Transformation of (6) to $\Omega$ yields

$$A(f)v = 0 \quad \text{in} \quad \Omega \times [0, \infty),$$

$$v = f \quad \text{on} \quad \Gamma_0 \times [0, \infty),$$

$$v = b \quad \text{on} \quad \Gamma_1 \times [0, \infty),$$

$$\partial_t f + B(f, v) = 0 \quad \text{on} \quad \Gamma_0 \times (0, \infty),$$

$$f(0) = f_0,$$

where $v := \phi_f^* u$. A pair $(v, f)$ is called a classical Hölder solution of (7) on $[0, T], T > 0$, if

$$f \in C([0, T], \mathcal{V}) \cap C^1([0, T], h^{1+\alpha}(S^1)),$$

$$v(\cdot, t) \in \text{buc}^{2+\alpha}(\Omega), \quad t \in [0, T],$$

and $(v, f)$ satisfies the equations in (7) pointwise.

**Lemma 1.2** Let $f_0 \in \mathcal{V}$ and $b \in h^{2+\alpha}(S^1)$ be given.

(a) If $(u, f)$ is a classical Hölder solution of (6), then $(\phi_f^* u, f)$ is a classical Hölder solution of (7).

(b) If $(v, f)$ is a classical Hölder solution of (7), then $(\phi_f^* v, f)$ is a classical Hölder solution of (6).
Proof. The main difficulty is to show that \( \phi^f_1(buc^a(\Omega)) = buc^a(\Omega_f) \) for each \( f \in \mathcal{V} \). We show just the inclusion \( \phi^f_1(buc^a(\Omega)) \subset buc^a(\Omega_f) \). The proof of \( \phi^f_1(buc^a(\Omega_f)) \subset buc^a(\Omega) \) is similar.

Let \( f \in \mathcal{V} \) and \( v \in buc^a(\Omega) \). We find two sequences \( (f_m) \subset C^\infty(\mathbb{S}^1) \) and \( (v_n) \subset \text{BUC}^\infty(\Omega) \) such that \( f_m \searrow f \) in \( C^a(\mathbb{S}^1) \) and \( v_n \to v \) in \( \text{BUC}^a(\Omega) \). Let \( u := \phi^f_1 v \). We show that each neighbourhood of \( u \) in \( \text{BUC}^a(\Omega_f) \) contains a function \( u_{n,m}, n, m \in \mathbb{N} \), where

\[
u_{n,m}(x, y) = v_n(\phi^{-1}_{f_m}(x, y)) = v_n(x, 1 - \frac{y}{f_m(x)}), \quad (x, y) \in \Omega_f.
\]

are smooth functions on \( \Omega_f \). The functions \( u_{n,m}, n, m \in \mathbb{N} \), are well-defined because \( f_m \geq f \) for all \( m \in \mathbb{N} \). First we have

\[
|u_{n,m}(x, y) - u(x, y)| = |v_n(\phi^{-1}_{f_m}(x, y)) - v(\phi^{-1}_f(x, y))| \leq \|\partial v_n\|_0 \frac{\|f_m - f\|_0}{\min f} + \|v_n - v\|_0
\]

for all \( (x, y) \in \Omega_f \). Let now \( (x, y) \) and \( (x', y') \) be two different points in \( \Omega_f \). We have

\[
|(u_{n,m} - u)(x, y) - (u_{n,m} - u)(x', y')| = |v_n(\phi^{-1}_{f_m}(x, y)) - v(\phi^{-1}_f(x, y)) - v_n(\phi^{-1}_{f_m}(x', y')) + v(\phi^{-1}_f(x', y'))| \\
\leq \|v_n - v\|_{\text{BUC}^a(\Omega)} \cdot \|\phi^{-1}_f(x, y) - \phi^{-1}_f(x', y')\|_0^a \\
+ |v_n(\phi^{-1}_{f_m}(x, y)) - v_n(\phi^{-1}_{f_m}(x', y'))| + |v(\phi^{-1}_f(x, y)) - v(\phi^{-1}_f(x', y'))|
\]

Since

\[
\frac{|\phi^{-1}_f(x, y) - \phi^{-1}_f(x', y')|}{|(x, y) - (x', y')|} \leq 1 + \frac{|y'/f(x') - y/f(x)|}{|(x, y) - (x', y')|} \leq 1 + \frac{1}{\min f} + \frac{\|f\|_0 \cdot \|f'\|_0}{\min f^2},
\]

it remains to estimate the second term on the right hand side. Using the mean value theorem we obtain

\[
|v_n(\phi^{-1}_{f_m}(x, y)) - v_n(\phi^{-1}_{f_m}(x', y'))| + |v(\phi^{-1}_f(x, y)) - v(\phi^{-1}_f(x', y'))| \\
\leq \|\partial v_n\|_0 \left| \frac{y'}{f_m(x')} - \frac{y'}{f_m(x)} \right| + \frac{1}{f(x')} + \frac{1}{f(x)} - \frac{1}{f(x')} + \frac{1}{f(x)} \|\phi^{-1}_f(x, y) - \phi^{-1}_f(x', y')\|
\]

\[
\leq \|\partial v_n\|_0 \left| \frac{y'}{f_m(x')} - \frac{y'}{f_m(x)} \right| + \frac{1}{f(x')} + \frac{1}{f(x)} - \frac{1}{f(x')} + \frac{1}{f(x)} \|\phi^{-1}_f(x, y) - \phi^{-1}_f(x', y')\|
\]

\[
\leq \|\partial v_n\|_0 \left| \frac{y'}{f_m(x')} - \frac{y'}{f_m(x)} \right| + \frac{1}{f(x')} + \frac{1}{f(x)} - \frac{1}{f(x')} + \frac{1}{f(x)} \|\phi^{-1}_f(x, y) - \phi^{-1}_f(x', y')\|
\]

\[
+ \|\partial^2 v_n\|_0 \left| \frac{1}{f_m(x')} - \frac{1}{f(x')} \right| + \frac{1}{f(x')} + \frac{1}{f(x)} - \frac{1}{f(x')} + \frac{1}{f(x)} \|\phi^{-1}_f(x, y) - \phi^{-1}_f(x', y')\|
\]

\[
+ \|\partial^2 v_n\|_0 \left| \frac{1}{f_m(x')} - \frac{1}{f(x')} \right| + \frac{1}{f(x')} + \frac{1}{f(x)} - \frac{1}{f(x')} + \frac{1}{f(x)} \|\phi^{-1}_f(x, y) - \phi^{-1}_f(x', y')\|.
\]
Using the estimates

\[
\left| \frac{y}{f_{m}(x)} - \frac{y'}{f(x')} + \frac{y}{f(x)} \right| \leq \left\| f \right\|_{0}^{1-\alpha} \cdot \left\| \frac{f_{m} - f}{\min f} \right\|_{0} + \left\| f \right\|_{0} \cdot \left\| \frac{f_{m} - f}{C^\alpha(S)} \right\|_{0} \cdot \left\| f \right\|_{C^\alpha(S)} \cdot \left\| f_{m} \right\|_{0} \cdot \left\| f \right\|_{C^\alpha(S)} \cdot \left\| f_{m} \right\|_{0},
\]

we obtain the desired conclusion.

\[\square\]

2. The abstract Cauchy problem

We have already noticed that the conditions \((A_1)\) and \((A_2)\) on \(\Pi\) imply the existence of two positive constants \(c\) and \(C\) such that

\[
c \leq \frac{1}{\mu(r)} \leq C, \quad \forall r \geq 0, \tag{8}
\]

\[
c \leq \frac{1}{\mu(r)} - \frac{2r^2}{\mu(r)} \leq C, \quad \forall r \geq 0. \tag{9}
\]

Under these assumptions the quasilinear operator \(Q\) is uniformly elliptic in \(\mathbb{R}^2\). For \(u \in C^2(\Omega_f)\) we compute

\[
Qu = \langle a_{ij}(Du)u_{ij},
\]

and the coefficients \((a_{ij})_{1 \leq i, j \leq 2}\) are

\[
a_{ij}(p) = \delta_{ij} \frac{\mu(|p|^2)}{\mu(|p|^2)} - \frac{2p_i p_j \|p\|^2}{\mu(|p|^2)}, \quad p = (p_1, p_2) \in \mathbb{R}^2.
\]

Actually, the eigenvalues of \((a_{ij})_{1 \leq i, j \leq 2}\) are

\[
\lambda_1(p) = \frac{1}{\mu(|p|^2)}, \quad \lambda_2(p) = \frac{1}{\mu(|p|^2)} - \frac{2|p|^2 \mu(|p|^2)}{\mu(|p|^2)},
\]

and we have

\[
c|\xi|^2 \leq a_{ij}(p)\xi_i\xi_j \leq C|\xi|^2, \quad \forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^2, \quad p \in \mathbb{R}^2.
\]

**LEMMA 2.1** Given \(f \in \mathcal{U}\), we have

\[
A(f)v = h_{ij}(y, f, Dv)v_{ij} + h(y, f, Dv)v_2 \quad \text{for } v \in BUC^2(\Omega),
\]

where, using the notation

\[
Dfv := \left( v_1 + \frac{(1 - y)f'}{f}v_2, -\frac{1}{f}v_2 \right) \quad \text{for } f \in \mathcal{U}, v \in BUC^2(\Omega) \text{ and } y \in [0, 1],
\]

we have
\[ b_{11}(y, f, Dv) = a_{11}(D_f v), \]
\[ b_{12}(y, f, Dv) = b_{21}(y, f, Dv) = \frac{(1 - y)f'}{f} a_{11}(D_f v) - \frac{1}{f} a_{12}(D_f v), \]
\[ b_{22}(y, f, Dv) = \frac{(1 - y)^2 f'^2}{f^2} a_{11}(D_f v) - 2 \frac{(1 - y)f'}{f^2} a_{12}(D_f v) + \frac{1}{f^2} a_{22}(D_f v), \]
\[ b(y, f, Dv) = (1 - y) \left( \frac{f''}{f} - \frac{2 f'^2}{f^2} \right) a_{11}(D_f v) + \frac{2 f'}{f^2} a_{12}(D_f v). \]

**Proof.** This follows by direct computation. \( \square \)

Given \( f \in \mathcal{U} \), the quasilinear operator \( A(f) \) is uniformly elliptic. Indeed, for \( (y, p) \in [0, 1] \times \mathbb{R}^2 \) and \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \) we have
\[ b_{ij}(y, f, p) \xi_i \xi_j = a_{11} \left( p_1 + \frac{(1 - y)f'}{f} p_2, -\frac{1}{f} p_2 \right) \left( \xi_1 + \frac{(1 - y)f'}{f} \xi_2 \right)^2 \]
\[ + 2 a_{12} \left( p_1 + \frac{(1 - y)f'}{f} p_2, -\frac{1}{f} p_2 \right) \left( \xi_1 + \frac{(1 - y)f'}{f} \xi_2 \right) \left( -\frac{\xi_2}{f} \right) \]
\[ + a_{22} \left( p_1 + \frac{(1 - y)f'}{f} p_2, -\frac{1}{f} p_2 \right) \left( -\frac{\xi_2}{f} \right)^2, \]
and the assertion follows from (8) and (9) upon taking also into account that \( \phi_f \) is a diffeomorphism.

Using maximum principle arguments and Morrey and De Giorgi–Nash type estimates as in [8] one can show that, given
\[ f \in \mathcal{U}, \quad q_1, q_2, q_3, g, b \in C^{2+\alpha}(S^1), \quad \sigma \in [0, 1], \]
there exist constants \( \delta > 0, \beta \in (0, 1) \) and \( M > 0 \) such that every solution \( v \in BUC^2(\Omega) \) of the Dirichlet problem
\[ A(f + q_1) v = 0 \quad \text{in} \ \Omega, \]
\[ v = \sigma g + q_2 \quad \text{on} \ \Gamma_0, \]
\[ v = \sigma b + q_3 \quad \text{on} \ \Gamma_1 \]
(10)
satisfies the estimate
\[ \| v \|_{BUC^{1+\beta}(\Omega)} \leq M \]
provided \( \| q_i \|_{C^{2+\alpha}(S^1)} \leq \delta \) for \( i \in \{1, 2, 3\} \). This a priori estimate allows an application of the techniques developed in Chapter 10 of [8] to derive the following existence, uniqueness and regularity result.

**Lemma 2.2** Let \( f \in \mathcal{V} \) and \( b \in H^{2+\alpha}(S^1) \). Then there exists a unique solution \( T(f) \in BUC^{2+\alpha}(\Omega) \) of the Dirichlet problem
\[ A(f) u = 0 \quad \text{in} \ \Omega, \]
\[ u = f \quad \text{on} \ \Gamma_0, \]
\[ u = b \quad \text{on} \ \Gamma_1. \]
(11)
The mapping \( \mathcal{V} \ni f \mapsto T(f) \in BUC^{2+\alpha}(\Omega) \) is smooth. \( \square \)
We fix $b \in H^{2+\alpha}(S^1)$. Replacing $v$ in the fourth equation of (11), we reduce the full problem (7) into an abstract Cauchy problem over $S^1$,\[ \partial_t f + \Phi(f) = 0, \quad f(0) = f_0, \tag{12} \]where $\Phi(f) := B(f, T(f))$. The operator $\Phi$ is a pseudodifferential operator of the first order, with a symbol depending nonlinearly on the variable $f$. Further we show that $\Phi \in C^\infty(h^{2+\alpha}(S^1), h^{1+\alpha}(S^1))$ and compute the derivative $\partial \Phi(c)$ in the special case $c, b \in \mathbb{R}_{>0}$.

The restriction of the operator $B$ defined in Section 1 to the set $V \times buc^{2+\alpha}(S^1)$ satisfies
\[ B(f, v) = -\frac{1}{\mu(\gamma_0 D_f v^2)} \left( f'\gamma_0 v_1 + f (1 + f^2)\gamma_0 v_2 \right) \]for $(f, v) \in V \times buc^{2+\alpha}(\Omega)$, where $\gamma_0$ is the trace operator on $T_0$. Together with the relation
\[ |\gamma_0 D_f v|^2 = \gamma_0 v_1^2 + 2f f' \gamma_0 v_1 v_2 + \frac{1 + f^2}{f^2} \gamma_0 v_2^2 \]we conclude that the operator $B$ defined above is smooth. More precisely, we have:

**Lemma 2.3** The mapping $B : V \times buc^{2+\alpha}(\Omega) \to h^{1+\alpha}(S^1)$ is smooth. The Fréchet derivative of $B$ at $(f, v) \in V \times buc^{2+\alpha}(\Omega)$ is given by
\[
\partial B(f, v)[h, u] = -\frac{1}{\mu(\gamma_0 D_f v^2)} \left( f'\gamma_0 u_1 + h'\gamma_0 v_1 + \frac{1}{f}(1 + f^2)\gamma_0 u_2 \right.
\]
\[ - \left( \frac{h}{f^2} - \frac{2f'h'}{f} + \frac{hf'^2}{f^2} \right) \gamma_0 u_2 \]
\[ - 2 \left( \frac{1}{\mu} \right)'(\gamma_0 D_f v^2) \left( f'\gamma_0 v_1 + \frac{f}{1 + f^2}\gamma_0 v_2 \right) \left[ \gamma_0 v_1 u_1 + \frac{h'}{f}\gamma_0 v_1 v_2 \right. \]
\[ + \frac{f'}{f}\gamma_0 v_1 u_2 + \frac{f'}{f}\gamma_0 v_1 v_2 - \frac{f'h}{f^2}\gamma_0 v_1 v_2 + \frac{f'h'}{f^2}\gamma_0 v_2^2 + \frac{hf'^2}{f^2} \gamma_0 v_2 u_2 \]
\[ - \frac{hf'^2}{f^2} \gamma_0 v_2^2 + \frac{1}{f^2} \gamma_0 v_2 u_2 - \frac{h}{f^2} \gamma_0 v_2^2 \right]
for all $[h, u] \in h^{2+\alpha}(S^1) \times buc^{2+\alpha}(\Omega)$.

Combining Lemmas 2.2 and 2.3 we conclude that $\Phi \in C^\infty(V, h^{1+\alpha}(S^1))$. Since $\Phi(f) = B \circ [f \mapsto (f, T(f))]$, the chain rule implies that $\partial \Phi(f) = \partial B(f, T(f)) \circ (\text{id}_{h^{2+\alpha}(S^1)}, \partial T(f))$ for $f \in V$. We are thus left with the task of computing the derivative $\partial T(f)$.

**Lemma 2.4** Given $f \in V$ and $h \in h^{2+\alpha}(S^1)$ the mapping $\partial T[f]h$ is the unique solution of the linear Dirichlet problem
where $w = h_1 w_1 + b w_2 + D_f w \left[ u_{11} \partial a_{11}(D_f u) + 2 u_{12} \left( \frac{(1-y) f'}{f} \partial a_{11}(D_f u) - \frac{1}{f} \partial a_{12}(D_f u) \right) \right] + u_{22} \left( \frac{(1-y)^2 f'^2}{f^2} \partial a_{11}(D_f u) - 2 \frac{(1-y) f'}{f} \partial a_{12}(D_f u) + \frac{1}{f^2} \partial a_{22}(D_f u) \right) + u_2 \left( (1-y) \left( \frac{f''}{f} - 2 \frac{f'^2}{f^2} \right) \partial a_{11}(D_f u) + 2 \frac{f'}{f} \partial a_{12}(D_f u) \right) \] = - u_2 \left( (1-y) \frac{f h' - f' h}{f^2} + \frac{h}{f^2} \right) \left[ u_{11} \partial a_{11}(D_f u) + 2 u_{12} \left( \frac{(1-y) f'}{f} \partial a_{11}(D_f u) - \frac{1}{f} \partial a_{12}(D_f u) \right) \right] + u_{22} \left( \frac{(1-y)^2 f'^2}{f^2} \partial a_{11}(D_f u) - 2 \frac{(1-y) f'}{f} \partial a_{12}(D_f u) + \frac{1}{f^2} \partial a_{22}(D_f u) \right) + u_2 \left( (1-y) \left( \frac{f''}{f} - 2 \frac{f'^2}{f^2} \right) \partial a_{11}(D_f u) + 2 \frac{f'}{f} \partial a_{12}(D_f u) \right) \] \[ - 2 u_{12} \left( (1-y) \frac{f h' - f' h}{f^2} \partial a_{11}(D_f u) + \frac{h}{f^2} \partial a_{12}(D_f u) \right) \] \[ - 2 u_{22} \left( \frac{(1-y)^2 (f h' - f' h)}{f^2} \partial a_{11}(D_f u) - (1-y) \frac{f h' - 2 f' h}{f^2} \partial a_{12}(D_f u) + \frac{h}{f^2} \partial a_{22}(D_f u) \right) \] \[ - u_2 \left( (1-y) \frac{f h'' - 2 f' h}{f^2} - 4 \frac{f f'h' - 2 f' h}{f^3} \right) \partial a_{11}(D_f u) + 2 \frac{f h' - 2 f' h}{f^2} \partial a_{12}(D_f u) \right) \] in $\Omega$, \[ w = h \quad \text{on } \Gamma_0, \] \[ w = 0 \quad \text{on } \Gamma_1, \] where $u := T(f)$ and $b_{ij} = b_{ij}(y, f, D_u), b = b(y, f, D_u)$ are the coefficients of $A(f)$.

Our next goal is to compute $\partial \Phi(c)$ when $c$ and $b$ are positive constant functions. More precisely, we would like to know how it acts on Fourier series. The solution $T(c)$ of the Dirichlet problem \[ (11) \] is \[ T(c)(x, y) = (1-y)c + yb, \quad (x, y) \in \Omega. \] Given $(h, u) \in H^{2+\alpha}(\Omega) \times buc^{2+\alpha}(\Omega)$, we therefore get \[ \partial B(c, T(c))[h, u] = - \frac{1}{c} \zeta \gamma_0 u_2 + \frac{b - c}{c^2} \zeta h, \] where \[ \zeta := \frac{1}{\lambda} \left( \left( \frac{b - c}{c} \right)^2 + 2 \left( \frac{b - c}{c} \right)^2 \left( \frac{1}{\mu} \right) \right)^{\frac{1}{2}} > 0. \] Consequently, \[ \partial \Phi(c)[h] = - \frac{1}{\zeta} \zeta \gamma_0 u_2 + \frac{b - c}{c^2} \zeta h, \] where $w := \partial T(c)[h] \in buc^{2+\alpha}(\Omega)$ denotes the solution of the linear Dirichlet problem \[ w_{11} + \beta_2 w_{22} = \frac{c - b}{c} (1-y) h'' \quad \text{in } \Omega, \] \[ w = h \quad \text{on } \Gamma_0, \] \[ w = 0 \quad \text{on } \Gamma_1, \] (13)
and where
\[ \beta^2 := \frac{1}{c^2} \left( 1 - 2 \left( \frac{c - b}{c} \right)^2 \frac{\mu'((\frac{c-b}{c})^2)}{\mu((\frac{c-b}{c})^2)} \right) > 0. \]

We now expand \( h \) and \( w \) in the following way:
\[ h(x) = \sum_{k \in \mathbb{Z}} c_k e^{ikx}, \quad w(x,y) = \sum_{k \in \mathbb{Z}} C_k(y) e^{ikx}. \]

Substituting these expressions into equations (13) and comparing the coefficients of \( e^{ikx} \) for every \( k \), we get the following equations for \( C_k(y) \):
\[ \beta^2 C_k'' - k^2 C_k = \frac{b-c}{c} k^2 e_k(1-y), \quad 0 < y < 1, \quad C_k(0) = c_k, \quad C_k(1) = 0, \quad (14) \]
for \( k \in \mathbb{Z} \setminus \{0\} \), and
\[ C_0'' = 0, \quad 0 < y < 1, \quad C_0(0) = c_0, \quad C_0(1) = 0. \quad (15) \]

One can easily verify that the solution of (15) is \( C_0(y) = (1-y)c_0 \). The solutions of (14) are given by
\[ C_k(y) = c_k d_k(y) \]
with
\[ d_k(y) = \frac{c-b}{c} (1-y) + \frac{b}{c} \left( \frac{e^{ky/\beta} + e^{-ky/\beta}}{1 - e^{2k/\beta}} \right). \]

Thus we obtain
\[ w(x,y) = (1-y)c_0 + \sum_{k \in \mathbb{Z} \setminus \{0\}} d_k(y) c_k e^{ikx}, \quad \forall (x,y) \in \Omega, \quad (16) \]
and
\[ \partial \Phi(c) \left[ \sum_{k \in \mathbb{Z}} c_k e^{ikx} \right] = \sum_{k \in \mathbb{Z}} \lambda_k c_k e^{ikx} \quad (17) \]
for all \( h = \sum_{k \in \mathbb{Z}} c_k e^{ikx} \in H^{2+\alpha}(S^1) \), with
\[ \lambda_0 := \frac{b c}{c^2}, \quad \lambda_k = \frac{b \zeta}{\beta c^2} \left( k e^{2k/\beta} + 1 \right), \quad k \neq 0. \quad (18) \]

Notice that equations (14) and (15) have been obtained formally by differentiating \( w \) with respect to the variables \( x \) and \( y \). Thus, it remains to show that the mapping \( w \), given by (16), is the solution of the Dirichlet problem (13). Since \( h \in H^{2+\alpha}(S^1) \), there is a positive constant \( L \) such that
\[ |c_k| \leq \frac{L}{k^2}, \quad \forall k \in \mathbb{Z} \setminus \{0\}. \]
The functions $d_k$, $k \in \mathbb{Z} \setminus \{0\}$, are uniformly bounded on $[0, 1]$, i.e.

$$M := \sup_{k \in \mathbb{Z} \setminus \{0\}} \max_{0 \leq y \leq 1} |d_k| < \infty.$$ 

Therefore $w \in BUC(\Omega)$. Let $\overline{w}$ denote the solution of (13). Pick further a sequence $(h_p)_p \subset C^\infty(S^1)$ which converges to $h$ in $C^{2+\alpha}(S^1)$, and denote by $w_p \in BUC^\infty(\Omega)$ the solution of (13) which corresponds to $h_p$. Then

$$w_p \to \overline{w} \quad \text{in} \quad BUC^{2+\alpha}(\Omega). \quad (19)$$

Using the Fourier expansions

$$h_p = \sum_{k \in \mathbb{Z}} c_{p,k} e^{ikx},$$

we find for each $l \in \mathbb{N}$ a constant $L_{p,l} > 0$ such that

$$|k|^l |c_{p,k}| \leq L_{p,l}, \quad \forall k \in \mathbb{Z},$$

and, as before, we obtain

$$w_p(x, y) = (1 - y)c_{p,0} + \sum_{k \in \mathbb{Z} \setminus \{0\}} d_k(y)c_{p,k} e^{ikx}, \quad \forall (x, y) \in \Omega.$$ 

Notice that these Fourier series are smooth for all $p$. Fix now $y \in [0, 1]$. Given $p \in \mathbb{N}$, we have

$$w_p(x, y) - w(x, y) = (1 - y)(c_{p,0} - c_0) + \sum_{k \in \mathbb{Z} \setminus \{0\}} d_k(y)(c_{p,k} - c_k)e^{ikx},$$

and so

$$\|w_p(\cdot, y) - w(\cdot, y)\|_{L^2(S^1)}^2 = (1 - y)^2 (c_{p,0} - c_0)^2 + \sum_{k \in \mathbb{Z} \setminus \{0\}} d_k^2(y)|c_{p,k} - c_k|^2 \leq M^2 \sum_{k \in \mathbb{Z}} |c_{p,k} - c_k|^2 = M^2 \|h_p - h\|_{L^2(G)}^2,$$ 

Observing $h_p \to h$ in $C^{2+\alpha}(S^1)$ and invoking (19), we see that the previous inequality implies that

$$w(\cdot, y) = \overline{w}(\cdot, y) \quad \text{in} \quad L^2(S^1)$$

for all $y \in [0, 1]$. Using the continuity of $w$ and $\overline{w}$, we conclude that $w = \overline{w}$, and formula (17) is proved.

3. The proof of the main result

In this section we regard the spaces $h^{m+\alpha}(S^1)$, $m = 1, 2$, as Banach spaces over the complex numbers. In order to prove Theorem 1.1 we have to show that the complexification of $-\partial \Phi(c)$, which we also denote by $-\partial \Phi(c)$, considered as an operator in $h^{1+\alpha}(S^1)$ with domain $h^{2+\alpha}(S^1)$, generates a strongly continuous analytic semigroup in $\mathcal{L}(h^{1+\alpha}(S^1))$, i.e. $\partial \Phi(c) \in \mathcal{H}(h^{2+\alpha}(S^1), h^{1+\alpha}(S^1))$. 

Using the same notations as in \[1\], we have $h^{2+\alpha}(S^1) \leftrightarrow h^{1+\alpha}(S^1)$ and, given $\kappa \geq 1$ and $\omega > 0$, we write
\[
\partial \Phi(c) \in \mathcal{H}(h^{2+\alpha}(S^1), h^{1+\alpha}(S^1), \kappa, \omega)
\]
if $\omega + \partial \Phi(c) \in \text{Lis}(h^{2+\alpha}(S^1), h^{1+\alpha}(S^1))$ and
\[
\kappa^{-1} \leq \frac{\| (\lambda + \partial \Phi(c)) h \|_{h^{1+\alpha}(S^1)}}{\| \lambda \| \| h \|_{h^{1+\alpha}(S^1)}} \leq \kappa, \quad h \in h^{2+\alpha}(S^1) \setminus \{ 0 \}, \quad \Re \lambda \geq \omega.
\]
Since
\[
\mathcal{H}(h^{2+\alpha}(S^1), h^{1+\alpha}(S^1)) = \bigcup_{\kappa \geq 1 \atop \omega > 0} \mathcal{H}(h^{2+\alpha}(S^1), h^{1+\alpha}(S^1), \kappa, \omega),
\]
it is sufficient to show that $\partial \Phi(c) \in \text{Lis}(h^{2+\alpha}(S^1), h^{1+\alpha}(S^1), \kappa, \omega)$ for some $\kappa \geq 1$ and $\omega > 0$. In fact, it is enough to find $\kappa \geq 1$ and $\omega > 0$ such that
\[
\lambda + \partial \Phi(c) \in \text{Lis}(h^{2+\alpha}(S^1), h^{1+\alpha}(S^1)),
\]
\[
|\lambda| \cdot \| R(\lambda, -\partial \Phi(c)) \|_{\mathcal{L}(h^{1+\alpha}(S^1))} \leq \kappa, \quad (20)
\]
for all $\Re \lambda \geq \omega$.

3.1 \textit{Sobolev spaces over the unit circle}

Let us recall that the Fréchet derivative $\partial \Phi(c) \in \mathcal{L}(h^{2+\alpha}(S^1), h^{1+\alpha}(S^1))$ is defined by
\[
\partial \Phi(c) = \sum_{k \in \mathbb{Z}} \hat{h}(k)e^{ikx} = \sum_{k \in \mathbb{Z}} \hat{h}(k)e^{ikx}
\]
for all $h = \sum_{k \in \mathbb{Z}} \hat{h}(k)e^{ikx} \in h^{2+\alpha}(S^1)$, with $(\hat{h}_k)_{k \in \mathbb{Z}}$ given by \[15\]. We denote here by $\hat{h}(k)$ the $k$-th Fourier coefficient of $h \in h^{2+\alpha}(S^1)$. For $r \geq 0$ we introduce the Sobolev space
\[
H^r(S^1) := \left\{ f \in L^2(S^1) : \sum_{k \in \mathbb{Z}} (1 + k^2)^r |\hat{f}(k)|^2 < \infty \right\},
\]
equipped with the scalar product $(f, g) := \sum_{k \in \mathbb{Z}} (1 + k^2)^r |\hat{f}(k)|^2 |\hat{g}(k)|^2$. The smooth functions are dense in $H^r(S^1)$, and the Sobolev embedding
\[
H^{m+\sigma}(S^1) \hookrightarrow C^m(S^1)
\]
holds for all $m \in \mathbb{N}$ provided $\sigma > 1/2$.

\textbf{PROPOSITION 3.1}

\[
H^{m+s}(S^1) \overset{d}{\hookrightarrow} h^{m+\alpha}(S^1)
\]
for all $m \in \mathbb{N}$, $\alpha \in [0, 1]$ and $s > 3/2$.

\textbf{Proof.} Given $m \in \mathbb{N}$, $\alpha \in [0, 1]$ and $s > 3/2$ we have the embeddings
\[
C^\infty(S^1) \subset H^{m+s}(S^1) \hookrightarrow C^{m+\alpha}(S^1),
\]
thus
\[
h^{m+\alpha}(S^1) = C^\infty(S^1)^{1:1} \hookrightarrow H^{m+s}(S^1)^{1:1} \hookrightarrow C^{m+\alpha}(S^1),
\]
\[
\text{PROPOSITION 3.1}
\]

\[
H^{m+s}(S^1) \overset{d}{\hookrightarrow} h^{m+\alpha}(S^1)
\]
for all $m \in \mathbb{N}$, $\alpha \in [0, 1]$ and $s > 3/2$. 

\textbf{Proof.} Given $m \in \mathbb{N}$, $\alpha \in [0, 1]$ and $s > 3/2$ we have the embeddings
\[
C^\infty(S^1) \subset H^{m+s}(S^1) \hookrightarrow C^{m+\alpha}(S^1),
\]
thus
\[
h^{m+\alpha}(S^1) = C^\infty(S^1)^{1:1} \hookrightarrow H^{m+s}(S^1)^{1:1} \hookrightarrow C^{m+\alpha}(S^1).
Fix now \( u \in H^{m+\alpha}(\mathbb{S}^1) \) and choose \( \varepsilon > 0 \). We can find \( u_0 \in H^{m}(\mathbb{S}^1) \) with \( \| u - u_0 \|_{C^{m+\alpha}(\mathbb{S}^1)} \leq \varepsilon/2 \). Due to (23) there is a constant \( C > 0 \) such that

\[
\| v \|_{C^{m+\alpha}(\mathbb{S}^1)} \leq C \| v \|_{H^{m+\alpha}(\mathbb{S}^1)}, \quad \forall v \in H^{m+\alpha}(\mathbb{S}^1).
\]

Let \( u_1 \in C^{\infty}(\mathbb{S}^1) \) be a smooth function with \( \| u_0 - u_1 \|_{H^{m+\alpha}(\mathbb{S}^1)} \leq \varepsilon/2C \). Combining these last inequalities, we get \( \| u - u_1 \|_{C^{m+\alpha}(\mathbb{S}^1)} \leq \varepsilon \) and the proof is complete.

Let us now consider the coefficients \( \lambda_k, k \in \mathbb{Z} \). We notice that \( \lambda_k = \lambda_{-k} \) and that \( \lambda_k \) is positive for every \( k \in \mathbb{Z} \). Moreover,

\[
\lim_{k \to \infty} \frac{\lambda_k}{k} = \frac{b \xi}{\beta c^2}.
\]

We now fix

\[
\omega := 1.
\]

**Proposition 3.2** Given \( r \geq 0 \) and \( \Re \lambda \geq \omega \), we have \( \lambda + \partial \Phi(c) \in \mathcal{L}(H^{r+1}(\mathbb{S}^1), H^r(\mathbb{S}^1)) \).

**Proof.** We first prove that \( \partial \Phi(c) \) is well-defined. Due to (23) there is a constant \( M > 0 \) such that

\[
|\lambda_k| \leq M(1 + k^2)^{1/2}, \quad \forall k \in \mathbb{Z}.
\]

Given \( h = \sum_{k \in \mathbb{Z}} \hat{h}(k) e^{ikx} \in H^{r+1}(\mathbb{S}^1) \), we have

\[
\| \partial \Phi(c) \sum_{k \in \mathbb{Z}} \hat{h}(k) e^{ikx} \|_{H^r(\mathbb{S}^1)} = \sum_{k \in \mathbb{Z}} (1 + k^2)^r |\lambda_k \hat{h}(k)|^2 \leq M^2 \sum_{k \in \mathbb{Z}} (1 + k^2)^{r+1} |\hat{h}(k)|^2
\]

Thus \( \partial \Phi(c) \) is well-defined. For \( \Re \lambda \geq \omega \) we have \( \lambda + \lambda_k \geq 1 \), and therefore \( \lambda + \partial \Phi(c) \) is injective. In order to show that \( \lambda + \partial \Phi(c) \) is onto, we have to show that for \( h = \sum_{k \in \mathbb{Z}} \hat{h}(k) e^{ikx} \in H^r(\mathbb{S}^1) \), the function \( \sum_{k \in \mathbb{Z}} (1/(\lambda + \lambda_k)) \hat{h}(k) e^{ikx} \) is in \( H^{r+1}(\mathbb{S}^1) \). Invoking again (23), we find \( M_\alpha > 0 \) such that

\[
|\lambda + \lambda_k|^2 \geq M_\alpha(1 + k^2), \quad \forall k \in \mathbb{Z}.
\]

Now

\[
\left\| \sum_{k \in \mathbb{Z}} \frac{1}{\lambda + \lambda_k} \hat{h}(k) e^{ikx} \right\|_{H^{r+1}(\mathbb{S}^1)} = \sum_{k \in \mathbb{Z}} (1 + k^2)^{r+1} \left| \frac{\hat{h}(k)}{\lambda + \lambda_k} \right|^2 \leq \frac{1}{M_\alpha} \sum_{k \in \mathbb{Z}} (1 + k^2)^r |\hat{h}(k)|^2
\]

and the proof is complete.

Combining these two propositions we obtain the following result.

**Corollary 3.3** Let \( m \in \{1, 2\} \) and suppose \( R(\lambda, -\partial \Phi(c)) \in \mathcal{L}(C^{1+\alpha}(\mathbb{S}^1), C^{m+\alpha}(\mathbb{S}^1)) \) for some \( \Re \lambda \geq \omega \). Then \( R(\lambda, -\partial \Phi(c)) \in \mathcal{L}(h^{1+\alpha}(\mathbb{S}^1), h^{m+\alpha}(\mathbb{S}^1)) \).
Proof. We prove just the case $m = 2$. The proof in the case $m = 1$ is similar. By assumption, $R(\lambda, -\partial\Phi(c)) \in \mathcal{L}(h^{1+\sigma}(S^1), C^{2+\sigma}(S^1))$. Given $f \in h^{1+\sigma}(S^1)$, Proposition 3.1 ensures the existence of a sequence $(f_n)_n \subset H^r(S^1), r > 3$, such that $f_n \to f$ in $C^{1+\sigma}(S^1)$. Thus
\[
R(\lambda, -\partial\Phi(c))f_n \to R(\lambda, -\partial\Phi(c))f \quad \text{in } C^{2+\sigma}(S^1).
\]
We know that $R(\lambda, -\partial\Phi(c))f_n \in H^{r+1}(S^1)$. Consequently,
\[
R(\lambda, -\partial\Phi(c))f \in H^{r+1}(S^1)_{c,2+\sigma(\cdot)} = h^{2+\sigma}(S^1).
\]

3.2 Periodic Besov spaces

Let $(\phi_j)_{j \geq 0} \subset \mathcal{S}(\mathbb{R})$ be a sequence with the following properties:

(i) $\text{supp } \phi_0 \subset [-2, 2], \quad \text{supp } \phi_j \subset \{x : 2^{j-1} \leq |x| \leq 2^{j+1}\}, \quad j \geq 1,$

(ii) $\sum_{j \in \mathbb{N}} \phi_j = 1 \quad \text{in } \mathbb{R},$

(iii) $\forall k \in \mathbb{N} \exists c_k > 0 : 2^k \|\phi_j^{(k)}\|_0 \leq c_k, \forall j \in \mathbb{N}.$

Further, let $\mathcal{D}'(S^1)$ denote the topological dual of $\mathcal{D}(S^1)$. The Fourier coefficients of $f \in \mathcal{D}'(S^1)$ are
\[
\hat{f}(k) := (2\pi)^{-1} f(e^{-ikt}), \quad k \in \mathbb{Z},
\]
and the series $\sum_{k \in \mathbb{Z}} \hat{f}(k)e^{ikt}$ converges to $f$ in $\mathcal{D}'(S^1)$. The Besov spaces $B^s_{\infty, \infty}(S^1), s \geq 0,$ are defined as follows:
\[
B^s_{\infty, \infty}(S^1) := \left\{ f \in \mathcal{D}'(S^1) : \|f\|_{B^s_{\infty, \infty}(S^1)} := \sup_{j \in \mathbb{N}} \left\| \sum_{k \in \mathbb{Z}} \phi_j(k) \hat{f}(k)e^{ikt} \right\|_{C(S^1)} < \infty \right\}.
\]

If $s > 0$ is not an integer, then $B^s_{\infty, \infty}(S^1) = C^s(S^1)$. For details see e.g. [1]. As one sees from previous computations, the operators $R(\lambda, -\partial\Phi(c))$ are Fourier multiplier operators. In order to prove (20) and (21) we can use, due to former considerations, multiplier theorems for operators between Besov spaces. Using the techniques of [2], it is not difficult to prove the following generalization of a result presented there.

Theorem 3.4. Let $r, s$ be positive constants and let $(M_k)_{k \in \mathbb{Z}} \subset \mathbb{C}$ be a sequence satisfying the following conditions:

(i) $\sup_{k \in \mathbb{Z}, (0)} |k|^{-r} |M_k| < \infty,$

(ii) $\sup_{k \in \mathbb{Z}, (0)} |k|^{-r+1} |M_{k+1} - M_k| < \infty,$

(iii) $\sup_{k \in \mathbb{Z}, (0)} |k|^{-r+2} |M_{k+2} - 2M_{k+1} + M_k| < \infty.$

Then the mapping
\[
\sum_{k \in \mathbb{Z}} \widehat{h}(k)e^{ikt} \mapsto \sum_{k \in \mathbb{Z}} M_k \widehat{h}(k)e^{ikt}
\]
belongs to $\mathcal{L}(B^s_{\infty, \infty}(S^1), B^r_{\infty, \infty}(S^1)).$

Proof. The case $r = s$ is proved in [2]. For $r \neq s$ the proof is similar, with obvious modifications. \qed
COROLLARY 3.5

\[ \{ \lambda \in \mathbb{C} : \text{Re } \lambda \geq \omega \} \subset \rho(-\partial \Phi(c)). \]

**Proof.** Fix \( \lambda \in \mathbb{C} \) with \( \text{Re } \lambda \geq \omega \). Due to Corollary 3.3, it is enough to show that \( R(\lambda, -\partial \Phi(c)) \in \mathcal{L}(C^{1+\alpha}(\mathbb{S}^1), C^{2+\alpha}(\mathbb{S}^1)) \). Here \( R(\lambda, -\partial \Phi(c)) \) denotes the multiplier operator

\[
\sum_{k \in \mathbb{Z}} \hat{h}(k)e^{ikx} \mapsto \sum_{k \in \mathbb{Z}} M_k^\lambda \hat{h}(k)e^{ikx}
\]

with \( M_k^\lambda = 1/(\lambda + \lambda_k) \) for \( k \in \mathbb{Z} \). In order to prove this assertion, we use the previous theorem with \( r := 2 + \alpha \) and \( s := 1 + \alpha \). Using relation (24), we obtain

\[
\lim_{|k| \to \infty} |k| |M_k^\lambda| = \frac{\beta c^2}{b^2},
\]

thus condition (i) in Theorem 3.4 is satisfied. Given \( k \neq 0 \), we have

\[
k^2 |M_{k+1}^\lambda - M_k^\lambda| = \frac{|k|}{|\lambda + \lambda_{k+1}|} \frac{|k|}{|\lambda + \lambda_k|} |\lambda_{k+1} - \lambda_k| \to \frac{\beta c^2}{b^2},
\]

and (ii) is verified. Furthermore, we have

\[
|k|^3 |M_{k+2}^\lambda - 2M_{k+1}^\lambda - M_k^\lambda| = \frac{|k|}{|\lambda + \lambda_{k+2}|} \frac{|k|}{|\lambda + \lambda_{k+1}|} \frac{|k|}{|\lambda + \lambda_k|} |\lambda_{k+2} - 2\lambda_{k+1} + \lambda_k| + \lambda_k |\lambda_{k+1} - \lambda_{k+2}| + \lambda_{k+2} |\lambda_{k+1} - \lambda_k|,
\]

with \( (\lambda_{k+2} - 2\lambda_{k+1} + \lambda_k) \to 0 \) as \( |k| \to \infty \). One can easily verify that

\[
\lambda_k (\lambda_{k+1} - \lambda_{k+2}) + \lambda_{k+2} (\lambda_{k+1} - \lambda_k) \to 2 \left( \frac{b^2}{\beta c^2} \right)^2,
\]

and the proof is complete. \( \square \)

It remains to prove assertion (21). We shall make again use of Theorem 3.4, but now in the special case \( r = s = 1 + \alpha \). Notice that for \( k \in \mathbb{Z} \) and \( \text{Re } \lambda \geq \omega \) we have

\[
\lambda + \lambda_k \geq \max\{1, \lambda, \lambda_k\}. \tag{26}
\]

COROLLARY 3.6 There exists \( \kappa \geq 1 \) such that

\[
|\lambda| \cdot \| R(\lambda, -\partial \Phi(c)) \|_{\mathcal{L}(h^{1+\alpha}(\mathbb{S}^1))} \leq \kappa
\]

for all \( \text{Re } \lambda \geq \omega \).

**Proof.** Let \( \lambda \in \mathbb{C} \) with \( \text{Re } \lambda \geq \omega \). Then \( |\lambda| R(\lambda, -\partial \Phi(c)) \) belongs to \( \mathcal{L}(h^{1+\alpha}(\mathbb{S}^1)) \). We regard \( |\lambda| R(\lambda, -\partial \Phi(c)) \) as a multiplier operator,

\[
\sum_{k \in \mathbb{Z}} \hat{h}(k)e^{ikx} \mapsto \sum_{k \in \mathbb{Z}} M_k^\lambda \hat{h}(k)e^{ikx},
\]
with

\[ M_k^i = \frac{|\lambda|}{\lambda + \lambda_k}, \quad \forall k \in \mathbb{Z}, \]

and we wish to find positive real numbers \( s_1, s_2 \) and \( s_3 \) such that

(i) \( \sup_{k \in \mathbb{Z}} |M_k^i| \leq s_1, \)

(ii) \( \sup_{k \in \mathbb{Z}} |k| |M_k^{i+1} - M_k^i| \leq s_2, \)

(iii) \( \sup_{k \in \mathbb{Z}} |k|^2 |M_{k+2}^i - 2M_{k+1}^i + M_k^i| \leq s_3, \)

for all \( \text{Re} \lambda \geq \omega. \) The existence of such constants is equivalent to the uniform boundedness of the family \( \{ |\lambda| R(\lambda, -\partial \Phi(c)) |_{\text{Re} \lambda \geq \omega} \in \mathcal{L}(h^{1+\alpha}(\mathbb{S}^1)) \}. \) For details see [2]. From relation (26) we obtain

\[ |M_k^i| = \frac{|\lambda|}{|\lambda + \lambda_k|} \leq 1 \]

for all \( k \in \mathbb{Z} \) and \( \text{Re} \lambda \geq \omega. \) We also have

\[ |k| |M_{k+1}^i - M_k^i| = \frac{|\lambda|}{|\lambda + \lambda_{k+1}|} |\lambda_{k+1} - \lambda_k| \leq \frac{|k|}{\lambda_k} |\lambda_{k+1} - \lambda_k|, \]

which, together with (24), implies estimate (ii). Further,

\[ |k|^2 |M_{k+2}^i - 2M_{k+1}^i + M_k^i| \leq \frac{|\lambda|}{|\lambda + \lambda_{k+2}|} |\lambda_{k+1} - \lambda_k| \leq \frac{|k|}{\lambda_k} |\lambda_{k+2} - 2\lambda_{k+1} + \lambda_k| + \lambda_k (\lambda_{k+1} - \lambda_{k+2}) + \lambda_k (\lambda_{k+1} - \lambda_k) \]

\[ \leq \frac{|k|}{\lambda_k} |\lambda_{k+2} - 2\lambda_{k+1} + \lambda_k| + |\lambda_k| |\lambda_{k+1} - \lambda_{k+2}| + |\lambda_k| |\lambda_{k+1} - \lambda_k|. \]

The relation

\[ |k| |\lambda_{k+2} - 2\lambda_{k+1} + \lambda_k| \xrightarrow{|k| \to \infty} 0 \]

completes the proof.

We have proved that for every positive constant \( c \), the complexification of the derivative \( \partial \Phi(c) \) generates a strongly continuous analytic semigroup in \( \mathcal{L}(h^{1+\alpha}(\mathbb{S}^1)) \), i.e. it belongs to \( \mathcal{H}(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1)) \). It is known that \( \mathcal{H}(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1)) \) is an open subset in \( \mathcal{L}(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1)) \) (see [1]), and because \( \partial \Phi \) is continuous, there is a neighbourhood \( \mathcal{O} \) of \( c \) in \( \mathcal{V} \) such that the complexification of \( \partial \Phi(f) \) is an element of \( \mathcal{H}(h^{2+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1)) \) for all \( f_0 \in \mathcal{O} \). The proof of Theorem 1.1 is now similar to the proof of Theorem 8.1.1 in [10].

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