Partial Differential Equations — Double Ball Property for non-divergence horizontally elliptic operators on step two Carnot groups, by Giulio Tralli, communicated on 20 April 2012.

Abstract. — Let \( L \) be a linear second order horizontally elliptic operator on a Carnot group of step two. We assume \( L \) in non-divergence form and with measurable coefficients. Then, we prove the Double Ball Property for the nonnegative sub-solutions of \( L \). With our result, in order to solve the Harnack inequality problem for this kind of operators, it becomes sufficient to prove the so called \( \epsilon \)-Critical Density.

Key words: Degenerate elliptic equation, invariant Harnack inequality, double ball property.

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1. Introduction

As it is well known, in the theory of fully nonlinear elliptic equations a crucial role is played by the Krilov-Safonov's Harnack inequality for nonnegative solutions to the linear equations in non-divergence form and measurable coefficients.

However, in several research areas, such as Complex or CR Geometry, there are fully nonlinear equations characterized by an underlying sub-Riemannian structure which are not elliptic at any point, see e.g. [8], [11], [9], [10], [2], [3], [7]. The existence theory for viscosity solutions to such equations is quite well settled, mainly thanks to the papers [11], [9], [3]. On the contrary, the problem of the solutions regularity is still widely open. This is mainly due to the lack of pointwise estimates for solutions to linear sub-elliptic equations with rough coefficients. In this context, a long standing open problem is the invariant Harnack inequality for positive solutions to horizontally elliptic equations on Lie groups, in non-divergence form and rough coefficients.

Di Fazio, Gutiérrez and Lanconelli in [4] found an axiomatic procedure to establish the scale invariant Harnack inequality in very general settings like doubling Hölder quasimetric spaces. Homogeneous Lie groups and, more generally, Carnot-Carathéodory spaces are remarkable examples of settings where their procedure applies. Di Fazio, Gutiérrez and Lanconelli proved that the double-ball property and the \( \epsilon \)-critical density are sufficient conditions for the Harnack inequality to hold. Recently, this general approach has been used by Gutiérrez and Tournier to prove the Harnack inequality for a class of horizontally elliptic operators with measurable coefficients in the Heisenberg group.
In this paper, we establish the double ball property for non-divergence linear second order operators which are elliptic with respect to the generators of a step two Carnot group. To be precise, let us fix some definitions. Let \((\mathbb{R}^N, \cdot, \delta_2)\) be an homogeneous Carnot group of step two and \(X_1, \ldots, X_m\) be the vector fields generating the Lie algebra. The dilations \(\{\delta_l\}_{l>0}\) are defined by \(\delta_l(x, t) = (\lambda x, \lambda^2 t)\) for all \((x, t) \in \mathbb{R}^{m+n} = \mathbb{R}^N\). Consider the second order differential operator

\[
\mathcal{L} = \sum_{i,j=1}^m a_{ij}(x, t)X_iX_j
\]

where \(A(x, t) = (a_{ij}(x, t))_{i, j \leq m}\) is a \(m \times m\) symmetric matrix with measurable entries. We say that \(\mathcal{L}\) is horizontally elliptic on \(\mathbb{R}^N\) if there exist \(\Lambda > \lambda > 0\) such that

\[
\lambda \|v\|^2 \leq \langle A(x, t)v, v \rangle \leq \Lambda \|v\|^2
\]

for all \(v \in \mathbb{R}^m\) and for all \((x, t) \in \mathbb{R}^N\).

Here \(\| \cdot \|\) stands for the Euclidean norm on \(\mathbb{R}^m\), but we shall use the same notation for all the Euclidean norms. Moreover, we denote with \(B_R(p_0)\) the homogeneous open ball of radius \(R\) centered at \(p_0\), i.e.

\[
B_R(p_0) = \{p_0 \ast (x, t) \in \mathbb{R}^N : \|x\|^4 + \|t\|^2 < R^4\}.
\]

Following Di Fazio, Gutiérrez and Lanconelli, in the present context we can state the double ball property as follows.

**Double Ball Property 1.1.** Let \(R\) be a positive constant and \(p_0 \in \mathbb{R}^N\). We set

\[
K = \{u \in C^2(B_{3R}(p_0)) : u \geq 0 \text{ and } \mathcal{L}u \leq 0 \text{ on } B_{3R}(p_0), u \geq 1 \text{ on } B_R(p_0)\}.
\]

We say that \(\mathcal{L}\) satisfies the Double Ball Property on \(B_{3R}(p_0)\) if there exists a positive constant \(\gamma\) depending only on the ellipticity constants \(\lambda, \Lambda\) such that

\[
u \geq \gamma \quad \text{on } B_{2\gamma}(p_0)
\]

for all \(u \in K\).

In [5] Gutiérrez and Tournier proved this property for the Heisenberg group. In this paper we prove that it holds for a general Carnot group of step two. We first recognize that, via weak Maximum Principle, the double ball property is a consequence of a kind of solvability of the Dirichlet problem for \(\mathcal{L}\) in the exterior of any homogeneous ball \(B_R(p_0)\). As a matter of fact, our main tool is the existence of suitable barrier functions in the interior of \(B_R(p_0)\) at any point of the boundary. In Section 2 we show that, indeed, the existence of such kind of barriers implies the double-ball property. In Section 3 we find explicit barriers at every boundary point of the homogeneous balls. At the non-characteristic points
(i.e. where the horizontal gradient does not vanish) we use some standard arguments, whereas at the characteristic points our construction requires the explicit knowledge of the vector fields $X_1, \ldots, X_m$ and of the composition law for groups of step two.

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2. Interior barriers

We start by recalling the weak maximum principle for the operator $\mathcal{L}$ in (1).

**Weak Maximum Principle** 2.1. Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$ and $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that $u \geq v$ on $\partial \Omega$ and $\mathcal{L}u \leq \mathcal{L}v$ in $\Omega$. Then, $u \geq v$ in $\Omega$.

A proof of this principle for this kind of operators can be found in [6], Corollary 1.3.

We now give the definition of interior $\mathcal{L}$-barrier function.

**Definition** 2.2. Let $\Omega$ be an open set of $\mathbb{R}^N$ with non-empty boundary. Fix $p \in \partial \Omega$. A function $h$ is an interior $\mathcal{L}$-barrier function for $\Omega$ at $p$ if

- $h$ is a $C^2$ function defined on an open bounded neighborhood $U$ of $p$,
- $h$ and $U$ depend only on $\Lambda, \lambda$ and the vector fields $X_j$’s,
- $\mathcal{L}h \leq 0$ on $U$,
- $h(p) = 0$,
- $\{(x, t) \in U : h \leq 0\} \setminus \{p\} \subseteq \Omega$.

We are going to prove some lemmas.

**Lemma** 2.3. Let $T$ be a compact subset of an open set $O \subset \mathbb{R}^N$. There exists $v_0 > 1$ such that

$$\delta_v T \subseteq O$$

for all $v \in [1, v_0]$.

**Proof.** The sets $T$ and $\mathbb{R}^N \setminus O$ are close and disjoint. Therefore, their distance $d$ is a positive number. If $(x, t) \in T$ and $\lambda > 0$, we have

$$\text{dist}(\delta_\lambda (x, t), T) \leq \text{dist}(\delta_\lambda (x, t), (x, t)) \leq |\lambda - 1| \sqrt{|x|^2 + (\lambda + 1)^2 \|t\|^2}.$$ 

Since $T$ is bounded, it is easy to choose $v_0 > 1$ such that

$$\sup_{(x, t) \in T} \text{dist}(\delta_v (x, t), T) < d$$

for all $v \in [1, v_0]$.  \qed
We set
\[
K_0 = K_0(\mathcal{L}) = \{u \in C^2(B_{3/2}(0)) : u \geq 0 \text{ and } \mathcal{L}u \leq 0 \text{ on } B_{3/2}(0), \\
u \geq 1 \text{ on } B_1(0)\}.
\]

The next lemma is an application of the weak maximum principle for the operator \(\mathcal{L}\).

**Lemma 2.4.** Suppose that, for every \(p \in \partial B_1(0)\), there exists an interior \(\mathcal{L}\)-barrier function for \(B_1(0)\) at \(p\). Then, there exists \(v \in (1, \frac{3}{2})\) (not depending on the coefficients of the matrix \(A\)) such that
\[
u \geq \frac{1}{2} \text{ on } B_v(0)
\]
for all \(u \in K_0\).

**Proof.** Fix \(p \in \partial B_1(0)\) and consider the barrier function \(h = h_p\) defined on \(U = U_p\). If we set \(V = (U \cap B_{3/2}(0)) \setminus B_1(0)\), we have that \(h \geq 0\) and \(\mathcal{L}h \leq 0\) on \(V\). Let us now consider the boundary \(\partial V = \Gamma_1 \cup \Gamma_2\), where \(\Gamma_1 = \partial V \cap \partial B_1(0)\) and \(\Gamma_2 = \partial V \setminus \Gamma_1\). The number \(m = \inf_{\Gamma_2} h\) is strictly positive because \(\{(x, t) \in \partial V : h(x, t) = 0\} = \{p\}\). So, the function \(w = 1 - \frac{1}{m} h\) is well defined. We get
\[
\mathcal{L}w = -\frac{1}{m} \mathcal{L}h \geq 0 \text{ on } V, \quad w \leq 1 \text{ on } \Gamma_1 \quad \text{and} \quad w \leq 0 \text{ on } \Gamma_2.
\]
If \(u \in K_0\), we deduce
\[
\mathcal{L}u \leq \mathcal{L}w \text{ on } V, \quad u \geq w \text{ on } \partial V.
\]
By the Weak Maximum Principle for \(\mathcal{L}\), \(u \geq w\) on \(V\). Since \(w(p) = 1\), there exists an open neighborhood \(W_p\) of \(p\) contained in \(U \cap B_{3/2}(0)\) where \(w \geq \frac{1}{2}\). The sets \(W_p\) depend only on the barrier functions and so on the ellipticity constants.

The compact set \(\partial B_1(0)\) is contained in the open set \(O = \bigcup_{p \in \partial B_1(0)} W_p\). By the previous lemma, there exists \(v > 1\) such that \((B_v(0) \setminus B_1(0)) \subset O\). Therefore, we deduce
\[
u \geq \frac{1}{2} \text{ on } B_v(0)
\]
for all \(u \in K_0\). \(\Box\)

We are now ready to prove the double ball property under the assumptions of the previous lemma.

**Proposition 2.5.** Suppose that there exists an interior \(\mathcal{L}\)-barrier function for \(B_1(0)\) at every point of \(\partial B_1(0)\). Then, the Double Ball Property 1.1 on \(B_{3R}(p_0)\) is satisfied for all \(R > 0\) and \(p_0 \in \mathbb{R}^N\).
Proof. Fix $p_0 = 0$ and $R = 1$. If $u \in K$, in particular $u \in K_0$. By the previous lemma, $u \geq \frac{1}{2}$ on $B_{v}(0)$ for a fixed $1 < v < \frac{3}{2}$. Let us consider the function

$$v = 2u \circ \delta_v.$$ 

It is a non-negative function of class $C^2$ defined on $B_{\frac{3}{2}}(0) \supseteq B_{\frac{3}{2}}(0)$ (since $v < 2$). We have that $v \geq 1$ on $B_{1}(0)$. By setting $\mathcal{L} = \sum_{i,j} A_{i,j}(p) X_i X_j$ where $A(p) = A(\delta_v(p))$, we get

$$\mathcal{L} v(p) = 2v^2(\mathcal{L}u)(\delta_v(p)) \leq 0$$

because of the homogeneity of the vector fields. This means that $v \in K_0(\mathcal{L})$, but $A(p)$ have the same ellipticity constants of $A(p)$ and $v$ depends only on these. So $v \geq \frac{1}{2}$ on $B_v(0)$, that implies $u \geq \frac{1}{4}$ on $B_{v^2}(0)$. If $v^2 \geq 2$, we have just proved the statement. If it is not, the argument can be reapplied. Since $v > 1$, there exists an integer $n_0$ such that $v^{n_0} \geq 2$. Therefore, we get

$$u \geq \frac{1}{2^{n_0}} =: \gamma \quad \text{on } B_2(0)$$

for all $u \in K$.

If $p_0$ and $R$ are arbitrary, we can argue in the same way. As a matter of fact, we consider the function

$$\bar{u}(p) = u(p_0 \ast \delta_R(p))$$

for $u \in C^2(B_{3R}(p_0))$. The homogeneity and the left-invariance of $\mathcal{L}$ imply that

$$\sum_{i,j} A_{i,j}(p_0 \ast \delta_R(p)) X_i X_j \bar{u}(p) = R^2 (\mathcal{L}u)(p_0 \ast \delta_R(p)).$$

So, the argument above works with the same constant $\gamma$. \qed

3. Explicit barriers

It is known (see, e.g. [1]) that the $m$ vector fields generating an $N$-dimensional Carnot group of step two are, up to isomorphism, of the form

$$X_i(x,t) = \partial_x + \frac{1}{2} \sum_{k=1}^{n} (B^k x)_i \partial_{t_k},$$

where $B^1, \ldots, B^n$ are suitable $m \times m$ linearly independent skew-symmetric matrices.

In order to apply Proposition (2.5), we have to prove that there exists an interior $\mathcal{L}$-barrier function for $B_1(0)$ at every point of $\partial B_1(0)$. We are going to give a sufficient condition for the existence of these barriers.

Lemma 3.1. Let $\Omega$ be a bounded domain defined by

$$\Omega = \{(x,t) \in \mathbb{R}^N : F(x,t) < 0\},$$
where $F$ is a real-valued function. Fix $p = (x_0, t_0) \in \partial \Omega$. Suppose that $F$ is smooth near $p$ and

$$\nabla_X F := (X_1 F, \ldots, X_m F) \neq 0$$

at $p$. Then, there exists an interior $\mathcal{L}$-barrier function for $\Omega$ at $p$.

**Proof.** Let us denote by $B_{c}((x_0, t_0), \beta)$ the euclidean ball centered at $(x_0, t_0)$ with radius $\beta$. We choose

$$(x_0, t_0) = p - \beta \frac{\nabla F(p)}{\|\nabla F(p)\|}$$

and $\beta$ small enough such that $B_{c}((x_0, t_0), \beta)$ is tangent to $\partial \Omega$ at $p$ and contained in $\Omega$. Let us now consider the function

$$h(x, t) = e^{-\alpha \rho^2} - e^{-\alpha((\|x-x_0\|^2 + \|t-t_0\|^2)}.$$

The positive constant $\alpha$ will be fixed later on. This function is strictly positive out of $B_{c}((x_0, t_0), \beta)$ and vanishing on the sphere. An easy computation shows that, for $j = 1, \ldots, m$,

$$X_j h(x, t) = \alpha e^{-\alpha((\|x-x_0\|^2 + \|t-t_0\|^2)} \left(2(x-x_0)_j + \sum_{k=1}^{n} (B^k x)_j (t-t_0)_k \right)$$

$$=: \alpha e^{-\alpha((\|x-x_0\|^2 + \|t-t_0\|^2)} v_j(x, t).$$

We have

$$\mathcal{L} h(x, t) = \alpha e^{-\alpha((\|x-x_0\|^2 + \|t-t_0\|^2)} \sum_{i,j=1}^{m} a_{ij}(x, t)$$

$$\times \left(2\delta_{ij} + \sum_{k=1}^{n} b_{ij}^k (t-t_0)_k + \frac{1}{2} \sum_{k=1}^{n} (B^k x)_j (B^k x)_i - \alpha v_i(x, t) v_j(x, t) \right).$$

The product of a symmetric matrix and a skew-symmetric matrix has zero trace, so $\text{Tr}(A(x, t)B^k) = 0$ and we get

$$\mathcal{L} h(x, t) = \alpha e^{-\alpha((\|x-x_0\|^2 + \|t-t_0\|^2)} \left(2 \text{Tr}(A(x, t)) + \frac{1}{2} \sum_{k=1}^{n} \langle A(x, t)B^k x, B^k x \rangle \right)$$

$$- \alpha \langle A(x, t)v(x, t), v(x, t) \rangle \right)$$

$$\leq \alpha e^{-\alpha((\|x-x_0\|^2 + \|t-t_0\|^2)} \left(2m\Lambda + \frac{\Lambda}{2} \sum_{k=1}^{n} \|B^k x\|^2 - \alpha \|v(x, t)\|^2 \right)$$

$$=: H(x, t).$$
We stress that the function $H$ depends on $\lambda$, $\Lambda$, but it does not depend on the coefficients of the matrix $A$. We also remark that

$$v(p) = \frac{2\beta}{\|\nabla F(p)\|} \nabla_x F(p) \neq 0.$$ 

Therefore, if we choose

$$\alpha > \frac{\Lambda}{\lambda} \left( 2m + \frac{1}{2} \sum_{k=1}^{n} \|B^k x_0\|^2 \right) \frac{\|\nabla F(p)\|^2}{4\beta^2 \|\nabla_x F(p)\|^2},$$

we obtain $H(p) < 0$. Then, there exists an open bounded neighborhood $U$ of $p$ (depending only on the function $H$, namely on $p$, $F$, $\lambda$, $\Lambda$ and on the matrices defining the vector fields) where $\mathcal{L} h \leq H < 0$. The function $h$ has all the properties required to be an interior $\mathcal{L}$-barrier function for $\Omega$ at $p$. \qed

**Remark 3.2.** If we denote with $N$ the defining function of $B_1(0)$, i.e. $N(x, t) = \|x\|^4 + \|t\|^2 - 1$, we have

$$\nabla_x N(x, t) = 4\|x\|^2 x + \sum_{k=1}^{n} t_k B^k x.$$ 

Since the matrices $B^k$'s are skewsymmetric, the vectors $x$ and $B^k x$ are orthogonal for every $k = 1, \ldots, n$. So, we can state that

$$\nabla_x N(x, t) = 0 \iff x = 0.$$

**Proposition 3.3.** For every $p \in \partial B_1(0)$, there exists an interior $\mathcal{L}$-barrier function for $B_1(0)$ at $p$.

**Proof.** By Lemma 3.1 and the last remark, it remains only to prove the existence of a barrier at the points $(0, t_0) \in \partial B_1(0)$. So, let us fix $t_0 = (t_0^1, \ldots, t_0^n)$ with $\|t_0\| = 1$. Denote with $P$ the orthogonal projector on $\text{Range}(\sum_{k=1}^{n} t_0^k B^k) = \text{Ker}(\sum_{k=1}^{n} t_0^k B^k)^\perp$ and with $Q$ the orthogonal projector on $\text{Ker}(\sum_{k=1}^{n} t_0^k B^k)$. We remark that $x = Px + Qx$ and

$$\left\| \sum_{k=1}^{n} t_0^k B^k x \right\| \geq \sigma \|P x\|, \quad \sigma > 0,$$

for all $x \in \mathbb{R}^m$. Since the matrices $B^k$'s are linearly independent, the matrix $P$ has got a positive rank $N_1$, $0 < N_1 \leq m$. Moreover, we put $M = \max_k \|B^k\|$. For a fixed

$$\gamma > \frac{\Lambda}{\lambda} \left( \frac{5m}{2N_1} + \frac{15 + m - N_1}{N_1} + \frac{5nM^2}{16N_1} \right),$$
(in particular we note that $\gamma > 2$ and $\gamma > \frac{\Lambda m - N_1}{N_1}$), we set
\[
f(x, t) = \|x\|^4 + (\|Qx\|^2 - \gamma\|Px\|^2)^2 + \|t'\|^2 + \langle t, t_0 \rangle,
\]
where $t' = t - \langle t, t_0 \rangle t_0$. Finally, for a positive constant $\beta$ to be fixed later on, we put
\[
h(x, t) = e^{-\beta} - e^{-\beta f(x, t)}.
\]
The function $h$ vanishes at $(0, t_0)$ and it is negative if and only if $f < 1$. So, we have
\[
\{ (x, t) \in \mathbb{R}^N : h(x, t) \leq 0, \langle t, t_0 \rangle > 0 \} \setminus \{ (0, t_0) \} = B_1(0).
\]
A straightforward calculation shows that
\[
X_j h(x, t) = \beta e^{-\beta f(x, t)} \left( 4 \|x\|^2 x_j + 4 (\|Qx\|^2 - \gamma\|Px\|^2)(Qx - \gamma Px)_j 
+ \sum_{k=1}^n t'_k (B^k x)_j + \frac{1}{2} \sum_{k=1}^n t^0_k (B^k x)_j \right) = \beta e^{-\beta f(x, t)} X_j f(x, t).
\]
Then we get
\[
\mathcal{L} h(x, t) = \beta e^{-\beta f(x, t)} \left( 4 \|x\|^2 \text{Tr}(A(x, t)) + 8 \langle A(x, t)x, x \rangle 
+ 4 (\|Qx\|^2 - \gamma\|Px\|^2)(\text{Tr}(A(x, t)Q) - \gamma \text{Tr}(A(x, t)P)) 
+ 8 \langle A(x, t)(Qx - \gamma Px), Qx - \gamma Px \rangle 
+ \frac{1}{2} \sum_{k=1}^n \langle A(x, t)B^k x, B^k x \rangle 
- \langle A(x, t) \sum_{k=1}^n t^0_k B^k x, \sum_{k=1}^n t^0_k B^k x \rangle 
- \beta \langle A(x, t)\nabla_x f(x, t), \nabla_x f(x, t) \rangle \right)
\leq \beta e^{-\beta f(x, t)} \left( 4\Lambda \|x\|^2 (m + 2) + 8\Lambda (\|Qx\|^2 + \gamma^2\|Px\|^2) 
+ 4 (\|Qx\|^2 - \gamma\|Px\|^2)(\text{Tr}(A(x, t)Q) - \gamma \text{Tr}(A(x, t)P)) 
+ \frac{\Lambda}{2} \sum_{k=1}^n \|B^k x\|^2 - \lambda \left\| \sum_{k=1}^n t^0_k B^k x \right\|^2 
- \beta \lambda \|\nabla_x f(x, t)\|^2 \right).
\]
Since $\gamma > \frac{\Lambda m - N_1}{N_1}$, we have
\[
\text{Tr}(A(x, t)Q) - \gamma \text{Tr}(A(x, t)P) \leq (m - N_1)\Lambda - \gamma N_1 \lambda < 0.
\]
If \( \|Px\|^2 \leq \frac{1}{\gamma} \|Qx\|^2 \), then in particular \( \|Qx\|^2 - \gamma \|Px\|^2 \geq \frac{2}{\gamma} \|x\|^2 \) (since \( \gamma > 2 \)) and so we deduce

\[
\mathcal{L} h(x, t) \leq \beta e^{-\beta f(x, t)} \|x\|^2 \left( 4m\Lambda + 24\Lambda + \frac{8}{5}(m - N_1)\Lambda - \gamma N_1\lambda + \Lambda n \frac{M^2}{2} \right) < 0
\]

because of our choice of \( \gamma \). Otherwise, if \( \|Px\|^2 > \frac{1}{\gamma} \|Qx\|^2 \), then \( \|Px\|^2 \geq \frac{1}{1 + \gamma^2} \|x\|^2 \) and we have

\[
\|\nabla_X f(x, t)\| \geq \left\| 4\|x\|^2 x + 4(\|Qx\|^2 - \gamma \|Px\|^2)(Qx - \gamma Px) + \frac{1}{2} \sum_{k=1}^{n} t_k^0 B^k x \right\| - \left\| \sum_{k=1}^{n} t_k' B^k x \right\| - \gamma \|x\| \geq \left( \frac{\sigma}{2\sqrt{1 + \gamma^2} - \|t'\|nM} \right) \|x\|.
\]

Here we used the fact that the vector \( \sum_{k=1}^{n} t_k^0 B^k x \) is orthogonal to \( Px \) and \( Qx \). Hence, if in addition \( \|t'\| < \frac{\sigma}{4nM\sqrt{1 + \gamma^2}} \), then

\[
\|\nabla_X f(x, t)\| \geq \frac{\sigma}{4\sqrt{1 + \gamma^2}} \|x\|
\]

and so we deduce

\[
\mathcal{L} h(x, t) \leq \beta e^{-\beta f(x, t)} \|x\|^2 \left( 4\Lambda (m + 2) + 4\gamma (\gamma N_1 \Lambda - (m - N_1)\lambda) + 16\Lambda \gamma^2 + \Lambda n \frac{M^2}{2} - \lambda \frac{\sigma^2}{1 + \gamma^2} - \beta \lambda \frac{\sigma^2}{16(1 + \gamma^2)} \right).
\]

By choosing \( \beta \) big enough, we obtain \( \mathcal{L} h < 0 \). Summing up, the function \( h \) is an interior \( \mathcal{L}' \)-barrier function for \( B_1(0) \) at \( (0, t_0) \) if we consider it on the domain \( \{(x, t) : \langle t, t_0 \rangle > 0, \|t'\| < \frac{\sigma}{4nM\sqrt{1 + \gamma^2}} \} \).

We stress that, if \( m = N_1 \) (that is \( Q = 0 \)), we can choose a simpler barrier like

\[
e^{-\beta} - e^{-\beta \|x\|^2 + \|t'\|^2 + \langle t, t_0 \rangle}.
\]

The condition \( m = N_1 \) for all \( (0, t_0) \in \partial B_1(0) \) means exactly that the group is an H-group in the sense of Metivier (in particular the groups of Heisenberg type satisfy this condition). \( \square \)

**References**


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