
Abstract. — We prove regularity results for minimizers of integral functionals of the type

\[ \int_\Omega f(Xu) \, dx \]

where \( f \) satisfies a nonstandard growth condition and \( Xu \) stands for the horizontal gradient of \( u \). More precisely, we obtain regularity in the scale of Campanato spaces without assuming any restriction on the growth exponents and, under a suitable assumption on them, we get the local boundedness as well as an higher integrability result for the gradient.

Key words: Nonstandard growth conditions, Carnot Carathéodory spaces, regularity.

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1. Introduction

Let \( \Omega \) be a bounded subset in \( \mathbb{R}^n \) and \( X = (X_1, \ldots, X_k) \) be a family of vector fields defined in a neighbourhood of \( \Omega \), with real, \( C^\infty \) smooth and globally Lipschitz coefficients satisfying the Hörmander condition. For \( u: \Omega \to \mathbb{R} \), we consider the integral functional

\[ \mathcal{F}(u) = \int_\Omega F(Xu) \, dx \]

where the integrand \( F: \mathbb{R}^k \to \mathbb{R} \) is a continuous function satisfying

\[ c\{A(|\xi|) - 1\} \leq F(\xi) \leq C\{A(|\xi|) + 1\} \]

\[ |F(\xi + \eta)| \leq C_0[F(\xi) + F(\eta)] \]

where \( A: [0, \infty) \to [0, \infty) \) is an \( N \)-function, that is \( A \) is a continuous, strictly increasing and convex function satisfying

\[ A(0) = 0 \quad \lim_{t \to 0} \frac{A(t)}{t} = 0 \quad \lim_{t \to \infty} \frac{A(t)}{t} = +\infty \]
We shall assume that there exist $1 < p \leq q$ such that
\begin{equation}
\frac{A(t)}{t^p} \rightarrow \frac{A(t)}{t^q} \quad \text{as} \quad t \rightarrow 0.
\end{equation}

**Definition 1.1.** A function $u \in W^{1,A}_X(\Omega)$, is a local minimizer of the integral (1.1) if
\begin{equation}
\int_{\text{supp}(u-v)} F(Xu) \, dx \leq \int_{\text{supp}(u-v)} F(Xv) \, dx \quad \forall v \in W^{1,A}_X(\Omega), \text{supp}(u - v) \subseteq \Omega.
\end{equation}

Combining assumptions in (1.2) and (1.5) we have that the integrand $f$ satisfies the following bounds
\begin{equation}
c(|\xi|^p - 1) \leq f(\xi) \leq c(|\xi|^q + 1)
\end{equation}

Variational integrals whose integrand satisfies growth conditions of the type (1.6) are called functionals “with non standard growth conditions” and were introduced in the Euclidean setting by Marcellini in [17]. From the very beginning, it has been clear that minimizers of functionals satisfying (1.6) can be not only irregular but also unbounded if $q$ is too large with respect to $p$, see [16]. The study of the regularity of minimizers of such integrals has a long history in the Euclidean setting, see for example [1], [6], [18] and [2]. In [18], Moscariello and Nania, assuming that $A$ and its conjugate satisfy the so called $\Delta_2$-condition, proved that any bounded local minimizer of (1.1) is Hölder continuous in $\Omega$. It is worth pointing out that this result was proven without any further condition on $p$ and $q$. In the same paper the local boundedness of minimizers is also proved for exponents $p$ and $q$ opportune close.

Here, without any assumptions on $p$ and $q$, we obtain that minimizers of the integral (1.1) belong to a Campanato space and have the horizontal gradients belonging to a Morrey space. More precisely we get

**Theorem 1.1.** Let $u$ be a local minimizer of the integral functional (1.1). Then there exist $\sigma = \sigma(p,q,C_d)$ and $\tau = \tau(p,q,C_d)$ such that $u \in L^{p,\sigma}_X(\Omega)$ and $Xu \in L^{p,\tau}_X(\Omega)$.

With the additional assumption $p > Qq/(Q+q)$, where $Q$ is a homogeneous dimension relative to $\Omega$, we establish the following higher integrability result for the horizontal gradient of minimizers (Theorem 1.2) and we prove the local boundedness of the minimizers themselves (Theorem 1.3).

**Theorem 1.2.** Let $A$ be an $N$-function satisfying assumptions in (1.5) with $p > \frac{Qq}{Q+q}$ and $u \in W^{1,A}_X(\Omega)$ be a local minimizer for the functional $F(u)$. There exist positive constants $c$ and $\delta = \delta(p,q,C_d)$ such that, for any balls $B_R \subseteq B_{2R} \subseteq \Omega$,
\begin{equation}
\int_{B_R} A^{1+\delta}(|Xu|) \, dx \leq c \left( \int_{B_{2R}} A(|Xu|) \, dx \right)^{1+\delta} + c
\end{equation}
Theorem 1.2 is the analogous of a result contained in [6] concerning the Euclidean setting. Obviously, we need some changes due the fact that we are working in a homogeneous space.

More precisely, an extension of the Maximal Theorem to the context of Orlicz spaces reveals a key tool in the proof of both results above. Moreover, a Poincaré inequality and a Caccioppoli type inequality in the setting of Orlicz-Sobolev spaces are crucial in order to prove Theorem 1.1 and Theorem 1.2 respectively. Carnot-Carathéodory spaces associated with a system of vector fields satisfying the Hörmander condition support a Poincaré inequality in Lebesgue spaces (see Proposition 2.3) and a so called \( \mathcal{A} \)-Poincaré inequality, that is

\[
\int_B \frac{|u - u_B|}{R} \, dx \leq CA^{-1} \left( \int_B A(|Xu|) \, dx \right)
\]

As far as we know, even it should be possible to deduce a \((\mathcal{A}, \mathcal{A})\)-Poincaré inequality (see Proposition 3.1) from a \( \mathcal{A} \)-Poincaré inequality, there is not any explicit proof of it. Inspired by [6], we prove a \((\mathcal{A}, \mathcal{A})\)-Poincaré inequality using the Poincaré inequality in Lebesgue spaces.

In Section 3 we prove all the useful tools mentioned above.

**Theorem 1.3.** Let \( A \) be an \( N \)-function satisfying assumptions in (1.5) with \( p > \frac{Q}{Q+q} \). Let \( B_R \subset \Omega \) be a ball and \( u \in W^{1,A}_{\mathcal{X}}(\Omega) \) be a local minimizer for the functional \( \mathcal{F}(u) \) assuming the value \( u_0 \) on \( \partial B_R \). If \( u_0 \in L^\infty(\partial B_R) \), then \( u \) is locally bounded.

In the proof we follow an idea by Stampacchia [20] as suggested by Boccardo, Marcellini and Sbordone in the Euclidean setting, [1].

It is worth mentioning that regularity results for minimizers of integral functionals under standard growth conditions (i.e. \( p = q \) in (1.6)) have been established for example in [7, 8, 3].

**2. Notation and preliminary results**

**Carnot-Carathéodory spaces** Let \( X_1, \ldots, X_k \) be vector fields defined in \( \mathbb{R}^n \), with real, \( C^\infty \) smooth coefficients. We say that they satisfy the Hörmander’s condition if there exists an integer \( m \) such that the family of commutators of \( X_1, \ldots, X_k \) up to length \( m \)

\[
X_1, \ldots, X_k, [X_{i_1}, X_{i_2}], \ldots, [X_{i_1}, [X_{i_2}, \ldots X_{i_m}]] \ldots, \quad \forall i_j = 1, 2, \ldots, k
\]

spans the tangent space \( T_x \mathbb{R}^n \) at every point \( x \in \mathbb{R}^n \).

For any real valued Lipschitz continuous function \( u \) we define

\[
X_j u(x) = \langle X_j(x), \nabla u(x) \rangle \quad j = 1, 2, \ldots, k
\]
and we call the horizontal gradient of $u$ the vector $Xu = (X_1u, \ldots, X_ku)$ whose length is given by

$$|Xu| = \left(\sum_{j=1}^{k} (X_ju)^2\right)^{1/2}$$

Let $\Omega \subset \mathbb{R}^n$ be an open set. For a function $u \in L^1_{\text{loc}}(\Omega)$, its distributional derivative along the vector fields $X_j$ is defined by the identity

$$(2.1) \quad \langle X_j u, \Phi \rangle = \int_{\Omega} uX_j^* \Phi \, dx \quad \forall \Phi \in C^\infty_0(\Omega)$$

where $X_j^*$ denotes the formal adjoint of $X_j$. Throughout the paper, if $u$ is a nonsmooth function, $X_ju$ will be meant in the distributional sense.

An absolutely continuous curve $\gamma : [a, b] \to \mathbb{R}^n$ is said to be admissible, if there exist functions $c_j : [a, b] \to \mathbb{R}$, $j = 1, \ldots, k$ such that

$$\dot{\gamma}(t) = \sum_{j=1}^{k} c_j(t)X_j(\gamma(t)) \quad \text{and} \quad \sum_{j=1}^{k} c_j(t)^2 \leq 1$$

Observe that $X_j$ do not need to be linearly independent and therefore functions $c_j$ do not need to be unique. Define the distance function $\rho$ as

$$\rho(x, y) = \inf\{T > 0 : \exists \gamma : [0, T] \to \mathbb{R}^n \text{ admissible}, \gamma(0) = x, \gamma(T) = y\}$$

If there is not any such a curve, we set $\rho(x, y) = \infty$. The function $\rho$ is called Carnot-Carathéodory distance and, since it is not clear whether one can connect any two points of $\mathbb{R}^n$ by an admissible curve, it’s not clear whether $\rho$ is a metric. The assumption for which the vector fields $X_1, \ldots, X_k$ satisfy the Hörmander condition ensures that $\rho$ is a metric and in this case $(\mathbb{R}^n, \rho)$ is said to be a Carnot-Carathéodory space.

The following theorem, due to Nagel, Stein and Wainger [19], shows that the metric $\rho$ is locally Hölder continuous with respect to the Euclidean metric.

**Theorem 2.1.** Let $X_1, \ldots, X_k$ be as above. Then for every bounded open set $\Omega \subset \mathbb{R}^n$ there are constants $c_1, c_2$ and $\lambda \in (0, 1)$ such that

$$(2.2) \quad c_1|x - y| \leq \rho(x, y) \leq c_2|x - y|^\lambda$$

for every $x, y \in \Omega$.

It follows that the space $(\mathbb{R}^n, \rho)$ is homeomorphic with the Euclidean space $\mathbb{R}^n$ and therefore bounded sets in the Euclidean metric are bounded sets in the metric $\rho$. The inverse is not always true but it is certainly valid if $X_1, \ldots, X_k$ have globally Lipschitz coefficients (see [10]). In the sequel all the distances will be respect to the
metric \( r \), in particular all the balls will be balls with respect to the Carnot-Carathéodory metric. We shall consider in \((\mathbb{R}^n, r)\) the Lebesgue measure which locally satisfies the following doubling condition (see for example [19]):

**Proposition 2.2.** Let \( \Omega \) be an open, bounded subset of \( \mathbb{R}^n \). There exists a constant \( C_d \geq 1 \), called doubling constant, such that

\[
|B(x_0, 2R)| \leq C_d |B(x_0, R)|
\]

provided \( x_0 \in \Omega \) and \( R \leq 5 \text{diam} \Omega \).

Let \( Y \) be a metric space and \( \mu \) a Borel measure in \( Y \). Assume \( \mu \) finite on bounded sets and satisfying the doubling condition on every open, bounded subset \( W \) of \( Y \). If there exists a positive constant \( C \) such that

\[
\frac{\mu(B)}{\mu(B_0)} \geq C \left( \frac{R}{R_0} \right)^Q
\]

for any ball \( B_0 \) having center in \( \Omega \) and radius \( R_0 < \text{diam} \Omega \) and any ball \( B \) centered in \( x \in B_0 \) and having radius \( R \leq R_0 \), we say that \( Q \) is a homogeneous dimension relative to \( \Omega \).

It is well known that doubling property implies the existence of such a \( Q \). However, \( Q \) is not unique and it may change with \( \Omega \). Obviously any \( Q' \geq Q \) is also a homogeneous dimension.

For a bounded open set \( \Omega \) containing a family of vector fields satisfying the Hörmander condition, the Carnot-Carathéodory space \((\Omega, r)\) with the Lebesgue measure has the homogeneous dimension \( Q = \log_2 C_d \).

Recall that the Sobolev space \( W^{1,p}_X(\Omega) \) is defined as

\[
W^{1,p}_X(\Omega) = \{ u \in L^p(\Omega) : X_j u \in L^p(\Omega) \ j = 1, \ldots, k \}
\]

and that \( W^{1,p}_{X,0}(\Omega) \) denotes the closure of \( C^\infty_{X,0}(\Omega) \) in \( W^{1,p}_X(\Omega) \). The following versions of Sobolev and Poincaré type inequalities hold (see for example [5], [10]).

**Proposition 2.3.** Let \( X_1, \ldots, X_k \) be as before. Let \( Q \) be a homogeneous dimension relative to \( \Omega \). There exist constants \( C_1, C_2 > 0 \) such that, for every ball \( B_R \) centered in \( \Omega \) and having radius \( R \leq \text{diam} \Omega \), the following inequalities hold

\[
\left( \int_{B_R} |u - u_R|^p^* \, dx \right)^{1/p^*} \leq C_1 R \left( \int_{B_R} |Xu|^p \, dx \right)^{1/p}
\]

for \( 1 \leq p < Q \) and \( p^* = \frac{Qp}{Q-p} \) and

\[
\int_{B_R} |u - u_R|^p \, dx \leq C_2 R^p \int_{B_R} |Xu|^p \, dx
\]

for \( 1 \leq p < \infty \).
We have denoted by \( u_R \) the average of the function \( u \) on \( B_R \).

The following imbedding property holds under the previous assumptions on the vector fields \( X_1, \ldots, X_k \).

**Proposition 2.4.** Let \( \Omega \subset \mathbb{R}^n \) be an open set with sufficiently smooth boundary and \( Q \) a homogeneous dimension relative to \( \Omega \). Let \( u \in W^{1,p}_{X,0}(\Omega) \) with \( 1 \leq p < Q \). Then there exists a constant \( c > 0 \) such that

\[
|u|_{L^p(\Omega)} \leq c\|u\|_{W^{1,p}_{X,0}(\Omega)}
\]

For the proof the reader can refer to [11] and [12].

For \( 0 < a \leq 1 \) we say that a continuous function on \( \Omega \) belongs to the Hölder class \( C^{0,a}(\Omega) \) if

\[
\sup_{x \neq y, x, y \in \Omega} \frac{|u(x) - u(y)|}{\rho(x, y)^a} < \infty
\]

Finally \( u \in L^p(\Omega) \) is said to belong to the Campanato space \( \mathcal{L}_{X,\gamma}^{p,\sigma}(\Omega) \) if

\[
\frac{1}{|B_R|} \int_{B_R} |u - u_R|^p \, dx \leq c R^\sigma
\]

and to the Morrey space \( L_{X,\gamma}^{p,\tau}(\Omega) \) if

\[
\int_{B_R} |u|^p \, dx \leq c R^\tau
\]

for every ball \( B_R \) centered in \( \Omega \) and having radius \( R < \text{diam} \Omega \).

Note that the following Theorem holds (see for example [14])

**Theorem 2.5.** If \( \gamma < 0 \), the Campanato space \( \mathcal{L}_{X,\gamma}^{p,\gamma}(\Omega) \) is isomorphic to the Morrey space \( L_{X,\gamma}^{p,\gamma}(\Omega) \). If \( 0 < \gamma < p \), the Campanato space \( \mathcal{L}_{X,\gamma}^{p,\gamma}(\Omega) \) is isomorphic to \( C^{0,\frac{\gamma}{p}}(\Omega) \) with \( \alpha = \frac{\gamma}{p} \).

**Orlicz and Orlicz-Sobolev spaces** Let \( A : [0, \infty) \to [0, \infty) \) be a continuous, strictly increasing and convex function satisfying (1.4). We shall assume that there exist \( 1 < p \leq q \) such that

\[
pA(t) \leq tA'(t) \leq qA(t) \quad \forall t \geq 0
\]

It is easy to verify that the second inequality in (2.6) is equivalent to say that there exists a constant \( k > 1 \) such that

\[
A(2t) \leq k A(t) \quad \forall t \geq 0
\]

that is the so called \( \Delta_2 \)-condition on \( A \), while both the inequalities in (2.6) are equivalent to
and to the following conditions

\begin{equation}
\frac{A^*(t)}{t^{p'}} \nearrow \frac{A^*(t)}{t^{q'}}
\end{equation}

where \( p' \) and \( q' \) denote the Hölder conjugate exponents of \( p \) and \( q \) respectively and \( A^* \) is the conjugate N-function of \( A \) defined by

\begin{equation}
A^*(s) = \sup_{t \geq 0} \{ st - A(t) \}
\end{equation}

It follows that

\begin{equation}
c_1(t^p - 1) \leq A(t) \leq c_2(t^q + 1)
\end{equation}

for some constants \( c_1, c_2 \).

Note that (2.9) implies that \( A^* \) also satisfies a \( \Delta_2 \)-condition.

There are many functions \( A \) which behave as above. For example, it is easy to verify that the function

\[ A(t) = t^p \log(1 + t) \quad p > 1 \]

satisfies conditions in (2.8) with \( p = p - \varepsilon \) and \( q = p + \varepsilon \) for all \( \varepsilon > 0 \).

Let \( \Omega \subset \mathbb{R}^n \) be an open set, the Orlicz class \( L^A(\Omega) \) defined by

\[ L^A(\Omega) = \left\{ u : \Omega \to \mathbb{R} : u \text{ is measurable, } \int_{\Omega} A(|u(x)|) \, dx < \infty \right\} \]

is a Banach space equipped with the Luxemburg norm

\[ \|u\|_{L^A(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} A\left(\frac{|u(x)|}{\lambda}\right) \, dx \leq 1 \right\} \]

The space generated by \( A^* \) is the dual of \( L^A \) and the two following fundamental inequalities hold

\begin{equation}
A^*\left(\frac{A(t)}{t}\right) \leq A(t)
\end{equation}

\begin{equation}
st \leq A(t) + A^*(s) \quad \text{(Young’s inequality)}
\end{equation}

The Orlicz-Sobolev space \( W^{1,A}_X(\Omega) \) is the subspace of \( L^A(\Omega) \) of functions \( u \) such that the horizontal gradient \( Xu \) belongs to \( L^A(\Omega) \). It is equipped with the norm

\[ \|u\|_{W^{1,A}_X(\Omega)} = \|u\|_{L^A(\Omega)} + \|Xu\|_{L^A(\Omega)} \]
Maximal functions  For \( f \in L^1_{\text{loc}}(\Omega) \), we define the maximal function by

\[
M^\Omega_R f(x) := \sup_{0 < r < R, B(x, r) \subset \Omega} \int_{B(x, r)} |f| \, dy
\]

In order to simplify the notations we will write \( M_R \) and \( M_{2R} \) in place of \( M^B_R \) and \( M^{2R}_{2R} \) respectively. The following proposition contains a metric version of a “weak type” inequality for the maximal function whose proof can be found in [12].

**Proposition 2.6.** Assume \( X \) be a metric space equipped with a doubling measure \( \mu \) on an open set \( \Omega \subset X \). Let \( h \) be a locally integrable function in \( \Omega \). Then

\[
\mu(\{ x \in \Omega : M^\Omega_R h(x) > t \}) \leq \frac{c}{t} \int_\Omega |h| \, d\mu
\]

for \( t > 0 \), where the constant \( c \) depends only on the doubling constant \( C_d \).

From now on we shall denote by \( \Omega \) a bounded open set in \( \mathbb{R}^n \) and by \( Q \) a homogeneous dimension relative to \( \Omega \). Let us conclude this section with a useful inequality due to Hajlasz and Strzelecki [13].

**Proposition 2.7.** Let \( u \in W^{1,p}_X(\Omega) \). Then

\[
\frac{|u(x) - u_R|}{R} \leq c M^\Omega_{2R} |Xu|(x)
\]

for almost every \( x \in \Omega' \) with \( \Omega' \subset \subset \Omega \).

3. Crucial inequalities

In this section we prove some propositions that reveal crucial in the sequel. We start with the following \((A, A)\)-Poincaré inequality

**Proposition 3.1.** Let \( A \) be an \( N \)-function satisfying (1.5) and \( Q \) a homogeneous dimension relative to \( \Omega \). If \( u \in W^{1,A}_X(\Omega) \), then there exists a positive constant \( C = C(p, q, Q) \) such that

\[
\int_{B_R} A\left(\frac{|u - u_R|}{R}\right) \, dx \leq C \int_{B_R} A(|Xu|) \, dx
\]

for each ball \( B_R \) well contained in \( \Omega \).

**Proof.** Define the function

\[
K(t) = \int_0^t A(s^{1/q})/s \, ds
\]
It is obviously increasing and, in virtue of (2.6), it is easy to verify that is concave and satisfies the following inequalities

\[ A(t^{1/q}) \leq K(t) \leq \frac{q}{p} A(t^{1/q}) \]  

(3.3)

Denoting by \( H(t) = A(t^{1/q}) \), we have

\[
\int_{B_R} A\left(\frac{|u-u_R|}{R}\right) dx = \int_{B_R} H\left(\frac{|u-u_R|}{R^q}\right) dx
\]

\[
\leq \int_{B_R} K\left(\frac{|u-u_R|}{R^q}\right) dx \leq K\left(\int_{B_R} \frac{|u-u_R|}{R^q} dx\right)
\]

\[
\leq K\left(c \int_{B_R} |Xu|^q dx\right) \leq \frac{q}{p} A\left[\left(\int_{B_R} |Xu|^q dx\right)^{1/q}\right]
\]

\[
\leq cA\left[\left(\int_{B_R} |Xu|^q dx\right)^{1/q}\right]
\]

where we used (3.3), Jensen’s inequality, Poincaré inequality in (2.4) and \( \Delta_2 \)-condition.

Now denoting by \( \Psi(t) = \int_0^t \frac{A(\sigma)}{\sigma} d\sigma \), a simple change of variable gives

\[ \Psi(t) = q \int_0^{t^{1/q}} \frac{A(s^q)}{s} ds =: \tilde{\Psi}(t^{1/q}) \]  

(3.5)

Using conditions in (2.6), we can easily prove that the function \( \Psi(t) \) is convex and that the function \( \Psi(t^{1/q}) \) defined in (3.5) satisfies the following inequalities

\[
\frac{1}{q} A(t^{1/q}) \leq \tilde{\Psi}(t^{1/q}) \leq A(t^{1/q})
\]  

(3.6)

Therefore, by Jensen’s inequality,

\[
cA\left[\left(\int_{B_R} |Xu|^q dx\right)^{1/q}\right] \leq c\tilde{\Psi}\left[\left(\int_{B_R} |Xu|^q dx\right)^{1/q}\right]
\]

\[
= c\Psi\left(\int_{B_R} |Xu|^q dx\right) \leq c \int_{B_R} \Psi(|Xu|) dx
\]

\[
= c \int_{B_R} A(|Xu|) dx \leq c \int_{B_R} A(|Xu|) dx
\]

The conclusion follows combining inequalities in (3.4) and (3.7). \( \square \)
Next Proposition contains an extension of the Maximal Theorem to the context of Orlicz spaces.

**Proposition 3.2.** Let $B_R$ be a ball well contained in $\Omega$. If $A$ is an $N$-function satisfying conditions (1.5) and $f \in L^A(B_R)$ is a non negative function, then there exists a positive constant $c$ depending on $p$, $q$ and on the doubling constant $C_d$, such that

\begin{equation}
\int_{B_R} A(M_{R}f) \, dx \leq c \int_{B_R} A(f) \, dx
\end{equation}

**Proof.** Defining, for any $t > 0$,

\begin{equation}
\lambda(t) = |\{x \in B_R : M_{R}f(x) > t\}|
\end{equation}

we have

\begin{equation}
\int_{B_R} A(M_{R}f) \, dx = \int_{B_R} dx \int_{0}^{M_{R}f(t)} A'(t) \, dt = \int_{0}^{\infty} A'(t) \lambda(t) \, dt
\end{equation}

Now choosing

\[h(x) = \begin{cases} f(x) & \text{if } f(x) > \frac{t}{2} \\ 0 & \text{otherwise} \end{cases}\]

we have that $M_{R}f \leq \frac{t}{2} + M_{R}h$ and therefore

\[\{x \in B_R : M_{R}f(x) > t\} \subseteq \left\{ x \in B_R : M_{R}h(x) > \frac{t}{2} \right\}\]

It follows that

\begin{equation}
\int_{B_R} A(M_{R}f) \, dx \leq \int_{0}^{\infty} A'(t) \left| \left\{ x \in B_R : M_{R}h(x) > \frac{t}{2} \right\} \right| \, dt 
\end{equation}

\[\leq c(C_d) \int_{0}^{\infty} \frac{A'(t)}{t} \, dt \int_{f > t/2} f \, dx.\]

where, in the last inequality, we have used Proposition 2.6. By Fubini’s theorem, integration by parts and assumptions on $A$ we get
\[ (3.12) \quad \int_{B_R} A(M_R f) \, dx \leq c(C_d) \int_{B_R} f(x) \, dx \int_0^{2f(x)} [A'(t)/t] \, dt \]
\[ = c(C_d) \int_{B_R} A(2f(x))/2 \, dx \]
\[ + c(C_d) \int_{B_R} f(x) \, dx \int_0^{2f(x)} [A(t)/t^2] \, dt \]
\[ \leq \frac{1}{2} c(k, C_d) \int_{B_R} A(f(x)) \, dx \]
\[ + c(C_d) \int_{B_R} f(x) \, dx \int_0^{2} \frac{A(sf(x))/s^2f(x)}{A(f(x))} \, ds \]
\[ = \frac{1}{2} c(k, C_d) \int_{B_R} A(f(x)) \, dx \]
\[ + c(C_d) \int_{B_R} dx \int_0^{2} \frac{A(sf(x))/s^2}{A(f(x))} \, ds \]

Note that in the last equality we have used the change of variable \( t = sf(x) \). Splitting the last integral, by using (1.5) and the \( \Delta_2 \)-condition, we have

\[ (3.13) \quad \int_0^{2} \frac{A(sf(x))/s^2}{A(f(x))} \, ds = \int_0^{1} \frac{A(sf(x))/s^2}{A(f(x))} \, ds + \int_1^{2} \frac{A(sf(x))/s^2}{A(f(x))} \, ds \]
\[ \leq A(f(x)) \int_0^{1} s^{p-2} \, ds + kA(f(x)) \int_1^{2} s^{-2} \, ds \]

Inserting (3.13) in (3.12) we conclude that

\[ \int_{B_R} A(M_R f) \, dx \leq c(C_d, p, q) \int_{B_R} A(f(x)) \, dx \]

Arguing as in [6], we can easily deduce from Proposition 3.2 the following extension of Gehring’s lemma for \( N \)-functions \( A \) satisfying (1.5).

**Proposition 3.3.** Let \( A \) be an \( N \)-function satisfying conditions (1.5), and let \( f \in L^1_{\text{loc}}(\Omega) \) a non negative function such that, for any ball \( B_R \subseteq \Omega \),

\[ (3.14) \quad \int_{B_{R/2}} A(f) \, dx \leq b_1 A\left(\int_{B_R} f \right) + b_2. \]

Then there exist \( c_1, c_2, \delta > 0 \) depending on \( b_1, b_2, p, q, Q \) such that

\[ (3.15) \quad \int_{B_{R/2}} A^{1+\delta}(f) \, dx \leq c_1 A^{1+\delta}\left(\int_{B_R} f \right) + c_2. \]
Let us conclude this section with the following Caccioppoli type inequality

**Theorem 3.4.** Let $A$ be an $N$-function satisfying conditions (1.5) and $u \in W^{1,A}_X(\Omega)$ be a minimizer for the functional $F(u)$. Then, for $R \leq s < t \leq 2R$,

$$
(3.16) \quad \int_{B_s} A(|Xu|) \, dx \leq c \left[ \int_{B_s \setminus B_t} A \left( \frac{|u-u_R|}{t-s} \right) \, dx + \int_{B_t \setminus B_s} A(|Xu|) \, dx + R^Q \right]
$$

where $c$ is a constant depending on $q$ and on the $\Delta_2$-constant of $A$.

**Proof.** Let $\eta \in C_0^\infty(B_t)$ be a cut-off function such that $\eta \equiv 1$ on $B_s$, $|X\eta| \leq \frac{c}{t-s}$. The proof of the existence of a such function can be found, for example, in [4]. Since $\varphi = (u-u_R)\eta$ belongs to the space $W^{1,A}_X(B_t)$, it can be used as a test function in Definition (1.1). The assumption on (1.2) and (1.3), the monotonicity of the function $A$ and the $\Delta_2$-condition give us

$$
\int_{B_t} F(Xu) \, dx \leq \int_{B_t} F(X(u-\varphi)) \, dx
$$

$$
= \int_{B_t} F((1-\eta)Xu - X\eta(u-u_R)) \, dx
$$

$$
\leq c \left[ \int_{B_t \setminus B_s} A(|1-\eta| |Xu|) \, dx + \int_{B_t \setminus B_s} A(|X\eta| |u-u_R|) \, dx + R^Q \right]
$$

$$
\leq c \left[ \int_{B_t \setminus B_s} A(|1-\eta| |Xu|) \, dx + \int_{B_t \setminus B_s} A \left( \frac{|u-u_R|}{t-s} \right) \, dx + R^Q \right]
$$

Therefore assumption on (1.2) and the monotonicity of $A$ imply

$$
\int_{B_t} A(|Xu|) \, dx \leq c \left[ \int_{B_t} F(Xu) \, dx + R^Q \right]
$$

$$
\leq c \left[ \int_{B_t \setminus B_s} A(|Xu|) \, dx + \int_{B_t \setminus B_s} A \left( \frac{|u-u_R|}{t-s} \right) \, dx + R^Q \right]
$$

hence the conclusion.

\[ \square \]

4. **The regularity result**

This section is devoted to the proof of Theorem 1.1.

**Proof.** Let $B_{2R}$ be a ball in $\Omega$. Combining the Maximal inequality proved in Proposition 3.2, the Caccioppoli type inequality in (3.16) for $s = R$ and $t = 2R$ and the pointwise inequality $|Xu| \leq M_{2R}(|Xu|)$, we easily get
\[
\int_{B_R} A(M_{2R}(|Xu|)) \, dx \\
\leq \int_{B_{2R}} A(M_{2R}(|Xu|)) \, dx \leq c \int_{B_{2R}} A(|Xu|) \, dx \\
\leq c \left[ \int_{B_{2R} \setminus B_R} A(M_{2R}(|Xu|)) \, dx + \int_{B_{2R} \setminus B_R} A\left(\frac{|u - u_R|}{R}\right) \, dx + R^Q \right]
\]

Now, since \( A \) is increasing and Proposition 2.7 holds, we get

\[
\int_{B_R} A(M_{2R}(|Xu|)) \, dx \leq c \left[ \int_{B_{2R} \setminus B_R} A(M_{2R}(|Xu|)) \, dx + R^Q \right]
\]

Now we fill the hole adding \( c \int_{B_R} A(M_{2R}(|Xu|)) \, dx \) to both sides of the obtained inequality having

\[
\int_{B_R} A(M_{2R}(|Xu|)) \, dx \leq \theta \int_{B_{2R}} A(M_{2R}(|Xu|)) \, dx + cR^Q
\]

for \( \theta \in (0, 1) \). A standard iteration argument implies the existence of a constant \( \tau \) such that the following decay estimate holds

\[
(4.1) \quad \int_{B_R} A(M_{2R}(|Xu|)) \, dx \leq cR^\tau
\]

and observing that

\[
(4.2) \quad \int_{B_R} A(|Xu|) \, dx \leq \int_{B_R} A(M_{2R}(|Xu|)) \, dx \leq \int_{B_R} A(M_{2R}(|Xu|)) \, dx
\]

we get

\[
(4.3) \quad \int_{B_R} |Xu|^p \, dx \leq cR^\tau
\]

that means \( Xu \in L_{X_{\infty}}^{p, \tau}(\Omega) \).

Moreover, applying the Poincaré inequality of Proposition 3.1 to the left hand side of (4.2) and using (4.1), we have

\[
\int_{B_R} A\left(\frac{|u - u_R|}{R}\right) \, dx \leq \int_{B_R} A(|Xu|) \, dx \\
\leq \int_{B_R} A(M_{2R}(|Xu|)) \, dx \leq cR^\tau
\]

and then
\[
\frac{1}{|B_R|} \int_{B_R} \frac{|u-u_R|^p}{R^p} \, dx \leq c R^{\tau-Q}
\]
that is \( u \in \mathcal{L}^{p,p+\tau-Q}(\Omega) \), i.e. the conclusion.

\begin{remark}
Since \( \mathcal{L}^{p,p+\tau-Q}(\Omega) \) is isomorphic to \( C^{0,\tau}(\Omega) \) for \( \tau = 1 + \frac{\tau-Q}{p} \), provided \( \tau > Q-p \), in the particular case \( p = Q \) the minimizers of the integral (1.1) belong to \( C^{0,\tau}(\Omega) \) for \( \tau = \frac{\tau}{Q} \).
\end{remark}

5. The higher integrability

The Caccioppoli type inequality in (3.16) combined with the Gehring’s lemma 3.3 will give Theorem 1.2.

**Proof** (of Theorem 1.2). Fix \( B_{2R} \) an arbitrary ball well contained in \( \Omega \) and \( R < s < t \leq 2R \). By Theorem 3.4 we have

\[
\int_{B_s} A(|Xu|) \, dx \leq c \left[ \int_{B_s \setminus B_{t}} \frac{|u-u_R|}{t-s} \, dx + \int_{B_{t} \setminus B_s} A(|Xu|) \, dx + R^Q \right]
\]
and therefore, filling the hole adding to both sides of the inequality the integral \( c \int_{B_s} A(|Xu|) \, dx \), we get

\[
(5.1) \quad \int_{B_s} A(|Xu|) \, dx \leq \theta \int_{B_s} A(|Xu|) \, dx + c \left[ \int_{B_s} A\left(\frac{|u-u_R|}{t-s}\right) \, dx + R^Q \right]
\]
for \( \theta \in (0,1) \). It follows that

\[
(5.2) \quad \int_{B_R} A(|Xu|) \, dx \leq c \left[ \int_{B_{2R}} A\left(\frac{|u-u_R|}{R}\right) \, dx + R^Q \right]
\]
hence by Hölder’s inequality, we deduce that

\[
\int_{B_R} A(|Xu|) \, dx
\]
\[
\leq c \int_{B_{2R}} \frac{A\left(\frac{|u-u_R|}{R}\right)}{|(u-u_R)/R|^Q} \frac{|u-u_R|^Q/Q}{R^Q} \, dx + c
\]
\[
\leq c \left[ \int_{B_{2R}} A^{Q/q}\left(\frac{|u-u_R|}{R}\right) \, dx \right]^{q/(Q+q)} \left[ \int_{B_{2R}} \frac{|u-u_R|^q}{R^q} \, dx \right]^{Q/(Q+q)} + c
\]
Define

\[
(5.3) \quad K(t) = \int_0^t \frac{A(s^{1/q})/s}{s^{1/(Q+q)}} \, ds, \quad H(t) = \frac{[A(t^{1/q})/(Q+q)]}{t^{Q/q}}
\]
In virtue of (2.6), it is possible to prove that $K(t)$ is concave and that there exists a constant $c$ such that

$$H(t) \leq K(t) \leq cH(t) \quad \forall t > 0$$

Therefore, for $q_* = \frac{Qq}{Q+q}$, using Proposition 2.3, we have

$$\int_{B_R} A(|Xu|) \, dx \leq c \left[ \int_{B_{2R}} K\left((u-u_R)/R\right)^{q/(Q+q)} \right]^{q/(Q+q)} \int_{B_{2R}} |Xu|^{q_*} \, dx + c$$

$$\leq cK^{q/(Q+q)} \left( \int_{B_{2R}} |(u-u_R)/R|^{q} \, dx \right) \int_{B_{2R}} |Xu|^{q_*} \, dx + c$$

$$\leq cH^{q/(Q+q)} \left( \left[ \int_{B_{2R}} |Xu|^{q_*} \, dx \right]^{q/q_*} \right) \int_{B_{2R}} |Xu|^{q_*} \, dx + c$$

$$= c \frac{A\left( \left[ \int_{B_{2R}} |Xu|^{q_*} \, dx \right]^{1/q_*} \right)}{\left( \int_{B_{2R}} |Xu|^{q_*} \, dx \right)} \int_{B_{2R}} |Xu|^{q_*} \, dx + c$$

$$= cA \left( \left[ \int_{B_{2R}} |Xu|^{q_*} \, dx \right]^{1/q_*} \right) + c$$

Setting $\Phi(t) = A(t^{1/q_*})$, we have

$$\Phi(2t) \leq k\Phi(t) \quad \text{and} \quad \Phi'(t) \geq \frac{p}{q_*} \frac{\Phi(t)}{t}$$

where, by assumption, $\frac{p}{q_*} > 1$. Hence inequality (5.5) can be written as

$$\int_{B_R} \Phi(|Xu|^{q_*}) \, dx \leq c\Phi\left( \int_{B_{2R}} |Xu|^{q_*} \, dx \right) + c$$

Using now Proposition 3.3, we deduce that there exists $\delta > 0$ such that

$$\int_{B_R} \Phi^{1+\delta}(|Xu|^{q_*}) \, dx \leq c\Phi^{1+\delta}\left( \int_{B_{2R}} |Xu|^{q_*} \, dx \right) + c$$

that is

$$\int_{B_R} A^{1+\delta}(|Xu|) \, dx \leq cA^{1+\delta}\left( \left[ \int_{B_{2R}} |Xu|^{q_*} \, dx \right]^{1/q_*} \right) + c$$

Setting

$$\Psi(t) = \int_{0}^{t} \frac{A(s)}{s} \, ds,$$
it is easy to prove that

\begin{equation}
\frac{1}{q} A(t) \leq \Psi(t) \leq A(t)
\end{equation}

and that $\Psi(t)$ and $\Psi(t^{1/p})$ are both convex. It follows that

\begin{equation}
\left[ \int_{B_2^R} |Xu|^p \, dx \right]^{1/p} \leq \Psi^{-1} \left( \int_{B_2^R} |Xu| \, dx \right) + c
\end{equation}

see [15]. Finally, since $p > q_*$, we have from (5.8) and (5.9) that

\begin{equation}
\frac{1}{q} A \left( \left[ \int_{B_2^R} |Xu|^{q_*} \, dx \right]^{1/q_*} \right) \leq \frac{1}{q} A \left( \left[ \int_{B_2^R} |Xu|^p \, dx \right]^{1/p} \right) + c
\end{equation}

\begin{equation}
\leq c \int_{B_2^R} A(|Xu|) \, dx
\end{equation}

The conclusion follows from (5.6) and (5.10).

For the case of spherical Quasi-minima, compare with the proof given in [7].

6. The local boundedness

In this section we prove the boundedness of the local minimizers of the functional (1.1) with a fixed boundary value.

Proof (of Theorem 1.3). For a positive constant $\lambda \geq \|u_0\|_{\infty}$, let us consider the function

\begin{equation}
w = \operatorname{sign}(u) \max\{|u| - \lambda, 0\}
\end{equation}

and use $v = u - w$ as test function in Definition (1.1), that is

\begin{equation}
\int_{\operatorname{supp} w} F(Xu) \, dx \leq \int_{\operatorname{supp} w} F(Xv) \, dx
\end{equation}

Since $Xu = Xw$ on the set $E_\lambda = \{x \in B_R : |u(x)| > \lambda\}$, it follows that

\begin{equation}
\int_{E_\lambda} F(Xu) \, dx \leq \int_{E_\lambda} F(0) \, dx
\end{equation}

thus, by assumptions in (1.2), we get

\begin{equation}
\int_{E_\lambda} A(|Xu|) \, dx \leq c |E_\lambda|
\end{equation}
By a simple use of the Sobolev embedding in (2.5) and the hypotheses on \( A \), we have

\[
\left( \int_{B_R} |w|^{p^*} \, dx \right)^{p/p^*} \leq c \int_{B_R} |Xw|^p \, dx \leq c \int_{B_R} A(|Xw|) \, dx
\]

\[
= c \left[ \int_{B_R \setminus E_\delta} A(|Xw|) \, dx + \int_{E_\delta} A(|Xw|) \, dx \right]
\]

\[
= c \int_{E_\delta} A(|Xu|) \, dx
\]

and combining (6.2) and (6.3) we obtain

\[
\left( \int_{B_R} |w|^{p^*} \, dx \right)^{p/p^*} \leq c |E_\delta|
\]

(6.4)

Recalling the definition of the function \( w \), we have for \( \delta > \lambda \),

\[
\int_{B_R} |w|^{p^*} \, dx = \int_{E_\delta} \|u - \lambda\|^{p^*} \, dx \geq \int_{E_\delta} \|u - \lambda\|^{p^*} \, dx
\]

\[
\geq \int_{E_\delta} |\delta - \lambda|^{p^*} \, dx = |\delta - \lambda|^{p^*} |E_\delta|
\]

and therefore, from (6.4) and (6.5), we have

\[
|E_\delta| \leq c \frac{|E_\delta|^{p'/p}}{|\delta - \lambda|^{p^*}}
\]

Applying Lemma 4.1 of [20], we obtain that

\[
|E_\tau| = 0 \quad \text{where } \tau = c |B_R|^{1/Q} = cR
\]

that implies

\[
\sup_{B_R} |u| \leq \|u_0\|_{\infty} + cR
\]

i.e. the conclusion.

\[
\square
\]

References
