Maximal Characterization of Locally Summable Functions

F. Andreano and R. Grande

Abstract. We prove a characterization of locally summable functions with bounded Stepanoff norm through the maximal function

\[ M_\phi f(x) = \sup_{t > 0} |(f \ast \phi_t)(x)|, \]

where \( \phi \) is a suitable function in the class of Schwartz.

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1. Introduction

Given \( \phi \) in the Schwartz class \( \mathcal{S} \), the maximal function \( M_\phi \) of a distribution \( f \) is

\[ M_\phi f(x) = \sup_{t > 0} |(f \ast \phi_t)(x)|, \]

where \( \phi_t(x) = \frac{1}{t} \phi\left(\frac{x}{t}\right) \). The following maximal characterization for \( L^p(\mathbb{R}) \) is well known (cfr. [5]).

Theorem 1.1. Let \( 1 < p \leq +\infty \). If \( f \) is a distribution, then:

\[ f \in L^p(\mathbb{R}) \iff \exists \phi \in \mathcal{S}, \text{ with } \int \phi \, dx \neq 0, \text{ so that } M_\phi f \in L^p(\mathbb{R}). \]

It is interesting to consider maximal characterizations of spaces of functions which are only locally summable, that is, “big” at infinity. This problem has been suggested by A. Pankov in a seminar given at the Department of Metodi e Modelli Matematici per le Scienze Applicate at Università di Roma “La Sapienza”.

First we need to introduce an appropriate Banach space structure on locally \( L^p \) functions. Let us define the spaces \( BS^p(\mathbb{R}) \) of Stepanoff bounded functions.

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Definition 1.2. Let $1 \leq p < +\infty$. Then $f \in \text{BS}^p(\mathbb{R})$ if

1. $f \in L^p_{\text{loc}}(\mathbb{R})$;
2. $\sup_{x \in \mathbb{R}} \int_{x}^{x+1} |f(t)|^p dt \leq c$, for $c \in \mathbb{R}$.

Note 1.3.

1. $\|f\|_{S^p} = \sup_{x \in \mathbb{R}} \left( \int_{x}^{x+1} |f(t)|^p dt \right)^{\frac{1}{p}}$ is a norm; such norm is equivalent to the norm
   \[ \|f\|_{S^p} = \sup_{x \in \mathbb{R}} \left( \frac{1}{l} \int_{x}^{x+l} |f(t)|^p dt \right)^{\frac{1}{p}} \]
   where $l \in \mathbb{R}_+$ (cfr. [1]).

2. The space $\text{BS}^p(\mathbb{R})$ contains the space of Stepanoff almost-periodic functions $S^p(\mathbb{R})$, i.e., the space of functions that can be approximated by trigonometric polynomials in the Stepanoff norm $\| \cdot \|_{S^p}$ defined above (cfr. [1] and [4]).

We prove the following maximal characterization of $\text{BS}^p(\mathbb{R})$:

Theorem 1.4. Let $1 < p < +\infty$. If $f$ is a distribution, then:

\[ f \in \text{BS}^p(\mathbb{R}) \iff \exists \phi \in \mathcal{S}, \text{ with } \int \phi \, dx \neq 0, \text{ so that } M_{\phi} f \in \text{BS}^p(\mathbb{R}). \]

For this we need to prove an analogue, for $\text{BS}^p(\mathbb{R})$, of the Hardy-Littlewood maximal theorem. As already observed, a similar result is known for $L^p$. The proof on $L^p$, however, does not readily extend to this more general framework, because, in the case of $L^p(\mathbb{R})$, one uses the weak compactness of $L^p(\mathbb{R})$ to prove the sufficient part of the maximal characterization. The dual space of $\text{BS}^p(\mathbb{R})$ is not known and therefore we do not have a weak convergence result in such spaces.

2. The maximal characterization for $\text{BS}^p(\mathbb{R})$

In this section we prove Theorem 1.4. The necessary part follows from the Hardy-Littlewood maximal theorem for $\text{BS}^p(\mathbb{R})$, that we prove separately.

Let $Mf$ be the maximal function of $f$, defined by

\[ Mf(x) = \sup_{x \in I} \frac{1}{m(I)} \int_{I} |f(t)| \, dt, \]

where the supremum is taken over all intervals $I$ containing $x$. Here $m$ denotes the Lebesgue measure.

Note 2.1. $M_{\phi} f(x) \leq c Mf(x)$, where $c$ is a constant (for the proof of this inequality see [5, Chapter 2, Section 2.1]).
Let $l > 0$ and $M_\nu f(x)$ be the maximal function of $f$ restricted to $[\nu, \nu + l]$, $\nu \in \mathbb{R}$, i.e., let

$$M_\nu f(x) = \sup_{x \in I \subseteq [\nu, \nu + l]} \frac{1}{m(I)} \int_I |f(t)| \, dt,$$

if $x \in [\nu, \nu + l]$, and $M_\nu f(x) = 0$, if $x \notin [\nu, \nu + l]$.

Furthermore let

$$m_f(\alpha, \nu, l) = m(\{ x \in [\nu, \nu + l] : M_\nu f(x) > \alpha \}),$$

with $\alpha, l \in \mathbb{R}_+$, $\nu \in \mathbb{R}$.

Then we have the following

**Lemma 2.2.** For any $\nu \in \mathbb{R}$ and $l > 0$, $\alpha > 0$, and $f \in B^{p}(\mathbb{R})$, $p > 1$,

$$m_f(\alpha, \nu, l) \leq \frac{4}{\alpha} \int_{\{x \in [\nu, \nu + l] : M_\nu f(x) > \alpha \}} |f(t)| \, dt$$

and

$$\frac{1}{l} \int_{\nu}^{\nu + l} |M_\nu f(t)|^p \, dt \leq \frac{2^{p+1} p}{p - 1} \int_{\nu}^{\nu + l} |f(t)|^p \, dt.$$

The proof is analogous to the proof of Theorem 4.3 in [3, Chapter 1].

**Theorem 2.3** (Hardy-Littlewood maximal theorem for $B^{p}(\mathbb{R})$). Let $p > 1$, $f \in B^{p}(\mathbb{R})$, and let $l > 0$. There exists $c > 0$ such that

$$\|Mf\|_{S^p} \leq c\|f\|_{S^p}.$$

**Proof.** Let $\nu \in \mathbb{R}$ and $l > 0$. Set

$$\Pi_1 = \{ I : I \subseteq [\nu - l, (\nu - l) + 3l] \}$$

and

$$\Pi_2 = \{ J : J \cap (\mathbb{R} \setminus [\nu - l, (\nu - l) + 3l]) \neq \emptyset \},$$

where $I, J$ are intervals. Furthermore set

$$N_{\nu - l, 3l} f(x) = \sup_{x \in J \in \Pi_2} \frac{1}{m(J)} \int_J |f(t)| \, dt.$$

Then, for all $x \in \mathbb{R}$, $Mf(x) = \max\{M_{\nu - l, 3l} f(x), N_{\nu - l, 3l} f(x)\}$. By Lemma 2.2, it suffices to prove that $N_{\nu - l, 3l} f(x) \leq c\|f\|_{S^p}$, for $x \in [\nu, \nu + l]$.

Let $J = [a, b] \in \Pi_2$. Since $x \in [\nu, \nu + l]$,

$$a < \nu - l \quad \text{and} \quad b > \nu$$
or else
\[ \nu - l < a < \nu + l \quad \text{and} \quad b > (\nu - l) + 3l \]
and hence, in both cases, we have that \( l' = b - a > l \). We can write
\[
\frac{1}{m(J)} \int_J |f(t)| \, dt = \frac{1}{b - a} \int_a^b |f(t)| \, dt = \frac{1}{b - a} \int_{a+(b-a)}^{a+b} |f(t)| \, dt
\]
\[
= \frac{1}{b - a} \int_a^{a+b'} |f(t)| \, dt.
\]
Since \( l' > l \), we may write \( l' = nl + \vartheta l \), with \( n \in \mathbb{N} \) and \( 0 < \vartheta < 1 \). Hence
\[
\frac{1}{l'} \int_a^{a+l'} |f(t)| \, dt = \frac{1}{nl + \vartheta l} \int_{a+l}^{a+nl+\vartheta l} |f(t)| \, dt
\]
\[
< \frac{1}{nl} \int_a^{a+(n+1)l} |f(t)| \, dt
\]
\[
\leq \frac{1}{nl} \left\{ \int_a^{a+l} |f(t)| \, dt + \cdots + \int_{a+nl}^{a+(n+1)l} |f(t)| \, dt \right\}
\]
\[
\leq \frac{n+1}{n} \sup_{a \in \mathbb{R}} \frac{1}{l} \int_a^{a+l} |f(t)| \, dt
\]
\[
\leq 2 \sup_{a \in \mathbb{R}} \frac{1}{l} \int_a^{a+l} |f(t)| \, dt \leq 2\|f\|_{S^p}
\]
and therefore \( N_{\nu-l,3l}f(x) \leq 2\|f\|_{S^p}, \) for \( x \in [\nu, \nu + l] \), and hence the thesis of the theorem is proved. \( \square \)

In the proof of Theorem 1.4 we use a result due to R. Doss (cfr. [2]). For completeness we state that theorem:

**Theorem 2.4.** Let \( \{\sigma_m(x)\} \) be a sequence of functions summable in every finite interval and verifying the following condition: to every \( \epsilon > 0 \) there corresponds a \( \delta > 0 \) such that, for every set \( E \) of diameter less than or equal to \( \delta \) and of measure less than or equal to \( \delta \),
\[
\int_E |\sigma_m(x)| \, dx \leq \epsilon, \quad \forall m.
\]
Then there exists a summable function \( \sigma(x) \) and a subsequence \( \{\sigma_{m_k}\} \) such that, for every bounded function \( f(x) \) and every finite interval \((a, b)\),
\[
\lim_{k \to \infty} \int_a^b f(x)\sigma_{m_k}(x) \, dx = \int_a^b f(x)\sigma(x) \, dx.
\]
Proof of Theorem 1.4. We first prove that $M_\phi f \in \text{BS}^p$, for all $f \in \text{BS}^p$. Let $f \in \text{BS}^p(\mathbb{R})$. By Theorem 2.3, we have that there exists $c > 0$ such that $\|M f\|_{S^p} \leq c \|f\|_{S^p}$. Let $\phi \in \mathcal{S}$ and $M_\phi f(x) = \sup_{t \geq 0} |(f * \phi_t)(x)|$. Then there exists $c' > 0$ such that $M_\phi f(x) \leq c'Mf(x)$ (see Note 2.1). Hence

$$\|M_\phi f\|_{S^p} \leq c'\|M f\|_{S^p} \leq c'c\|f\|_{S^p}$$

and $M_\phi f \in \text{BS}^p(\mathbb{R})$.

Viceversa, suppose that $M_\phi f \in \text{BS}^p(\mathbb{R})$, with $\phi \in \mathcal{S}$ such that $\int \phi = 1$. We want to show that $f \in \text{BS}^p$. Let us consider the sequence $(f * \phi_\frac{1}{n})(x)$. We have that

$$\|f * \phi_\frac{1}{n}\|_{S^p} = \sup_{x \in \mathbb{R}} \left( \int_x^{x+1} |(f * \phi_\frac{1}{n})|^p dt \right)^{\frac{1}{p}}.$$

Since $M_\phi f \in \text{BS}^p(\mathbb{R})$,

$$\sup_{x \in \mathbb{R}} \int_x^{x+1} |(f * \phi_\frac{1}{n})|^p dt \leq \sup_{x \in \mathbb{R}} \int_x^{x+1} \left( \sup_{s > 0} |(f * \phi_s)(t)| \right)^p dt \leq B^p < +\infty,$$

where $B$ is a constant, and hence $\|f * \phi_\frac{1}{n}\|_{S^p} \leq B < +\infty$, i.e., $f * \phi_\frac{1}{n}$ is a bounded sequence in $\text{BS}^p(\mathbb{R})$.

Set $h_n = f * \phi_\frac{1}{n}$. We want to show that there exists a subsequence ${h_{n_j}}_{j \in \mathbb{N}}$ and a function $f_\circ \in \text{BS}^p(\mathbb{R})$ such that for any measurable and bounded function $\varphi$ and for any bounded interval $(a, b) \subset \mathbb{R}$, one has

$$\lim_{j \to +\infty} \int_a^b \varphi(t)h_{n_j}(t) \, dt = \int_a^b \varphi(t)f_\circ(t) \, dt. \quad (1)$$

We apply Theorem 2.4 (cfr. [2]) in order to get that there exists a function $f_\circ \in L^1_{loc}(\mathbb{R})$ verifying (1) for any measurable bounded function $\varphi$ and for any bounded interval $(a, b)$ in $\mathbb{R}$. In order to do this, we need to prove that, if $E$ is any measurable set such that $m(E) \to 0$, then $\int_E |h_n(t)| \, dt \to 0$ uniformly with respect to $n \in \mathbb{N}$.

Let $E$ be measurable such that $m(E) \to 0$. The diameter of $E$ is therefore less than 1 and hence $E \subset (x, x+1)$, for $x \in \mathbb{R}$ suitably chosen. Therefore

$$\int_E |h_n(t)| \, dt = \int_x^{x+1} \chi_E(t)|h_n(t)| \, dt \leq \left( \int_x^{x+1} |h_n(t)|^p \, dt \right)^{\frac{1}{p}} \left( \int_x^{x+1} \chi_E(t) \, dt \right)^{\frac{1}{q}} \leq \sup_{x \in \mathbb{R}} \left( \int_x^{x+1} |h_n(t)|^p \, dt \right)^{\frac{1}{p}} \left[ m(E) \right]^{\frac{1}{q}} \leq B[m(E)]^{\frac{1}{q}},$$
where \( \frac{1}{p} + \frac{1}{q} = 1 \). Hence \( \int_E |h_n(t)| \, dt \to 0 \), if \( m(E) \to 0 \), uniformly with respect to \( n \in \mathbb{N} \). The hypotheses of Theorem 2.4 are satisfied and therefore (1) holds with \( f_o \in L^1_{\text{loc}}(\mathbb{R}) \).

We need to prove that \( f_o \in \text{BS}^p(\mathbb{R}) \). To see this, let \( \delta \in (0, 1) \) and set, for \( n \in \mathbb{N} \),

\[
 f_o^{(\delta)}(x) = \frac{1}{\delta} \int_x^{x+\delta} f_o(t) \, dt \quad \text{and} \quad h_n^{(\delta)}(x) = \frac{1}{\delta} \int_x^{x+\delta} h_n(t) \, dt.
\]

Since the integral function can be derivated a.e., using Lebesgue’s theorem we get

\[
 \lim_{\delta \to 0^+} f_o^{(\delta)}(x) = f_o(x), \quad x \in \mathbb{R} \quad \text{a.e.}
\]

\[
 \lim_{\delta \to 0^+} h_n^{(\delta)}(x) = h_n(x), \quad x \in \mathbb{R} \quad \text{a.e.}
\]

Furthermore, by (1), we get that

\[
 \lim_{j \to +\infty} h_n^{(\delta)}(x) = f_o^{(\delta)}(x), \quad \forall x \in \mathbb{R}, \ \forall \delta \in (0, 1).
\]

Hence, by Fatou’s lemma

\[
 \int_x^{x+1} |f_o^{(\delta)}(t)|^p \, dt \leq \liminf_{j \to +\infty} \int_x^{x+1} |h_n^{(\delta)}(t)|^p \, dt.
\]

In order to prove that \( f_o \in \text{BS}^p(\mathbb{R}) \), we need to show first that \( \int_x^{x+1} |f_o^{(\delta)}(t)|^p \, dt \) is bounded independently of \( x \). To see this, consider

\[
 \int_x^{x+1} |h_n^{(\delta)}(t)|^p \, dt = \int_x^{x+1} \left( \frac{1}{\delta} \int_t^{t+\delta} |h_n(s)| \, ds \right)^p \, dt \\
\leq \int_x^{x+1} \left( \frac{1}{\delta} \int_t^{t+\delta} |h_n(s)| \, ds \right)^p \, dt.
\]

Consider now

\[
 \frac{1}{\delta} \int_t^{t+\delta} |h_n(s)| \, ds \leq \frac{1}{\delta} \left( \int_t^{t+\delta} |h_n(s)|^p \, ds \right)^{\frac{1}{p}} \delta^\frac{1}{p} = \left( \frac{1}{\delta} \int_t^{t+\delta} |h_n(s)|^p \, ds \right)^{\frac{1}{p}}.
\]

Hence

\[
 \liminf_{j \to +\infty} \int_x^{x+1} |h_n^{(\delta)}(t)|^p \, dt \leq \liminf_{j \to +\infty} \int_x^{x+1} \frac{1}{\delta} \int_t^{t+\delta} |h_n(s)|^p \, ds \, dt \\
\leq \liminf_{j \to +\infty} \left( \int_x^{x+1+\delta} |h_n(s)|^p \, ds \right) \left( \frac{1}{\delta} \int_{x+\delta}^{s} \, dt \right) \\
= \liminf_{j \to +\infty} \int_x^{x+1+\delta} |h_n(s)|^p \, ds \\
\leq \liminf_{j \to +\infty} \int_x^{x+2} |h_n(s)|^p \, ds \\
\leq 2B^p < +\infty,
\]
and so \( f_{x}^{x+1}|f_o^{(\delta)}(t)|^p dt \leq 2B^p \). Applying once more Fatou’s lemma, we get

\[
\int_x^{x+1} |f_o(t)|^p dt \leq \liminf_{\delta \to 0} \int_x^{x+1} |f_o^{(\delta)}(t)|^p dt \leq 2B^p.
\]

Hence

\[
\sup_{x \in \mathbb{R}} \left( \int_x^{x+1} |f_o(t)|^p dt \right)^{\frac{1}{p}} \leq 2^{\frac{1}{p}} B < +\infty
\]

and \( f_o \in BS^p(\mathbb{R}) \).

We have shown that there exists \( f_o \in BS^p(\mathbb{R}) \) such that

\[
\lim_{j \to +\infty} \int_a^b \varphi(x)h_{nj}(x) \, dx = \lim_{j \to +\infty} \int_a^b \varphi(x)(f * \phi_{\frac{1}{nj}})(x) \, dx = \int_a^b \varphi(x)f_o(x) \, dx,
\]

for any measurable bounded function \( \varphi \) and for any bounded interval \((a, b) \subset \mathbb{R}\).

On the other hand \( f * \phi_{\frac{1}{nj}} \to f \) as \( j \to +\infty \) in the sense of distributions, and so \( f = f_o \in BS^p(\mathbb{R}) \).

\[\square\]

**References**


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