Existence and Multiplicity of Positive Solutions for Singular p–Laplacian Equations

Haishen Lü and Yi Xie

Abstract. Positive solutions are obtained for the boundary value problem

\begin{equation}
\begin{cases}
-\Delta_p u = \lambda(u^\beta + \frac{1}{u^\alpha}) & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where \(\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)\), \(1 < p < N\), \(N \geq 3\), \(\Omega \subset \mathbb{R}^N\) is a bounded domain, \(0 < \alpha < 1\) and \(p - 1 < \beta < p^* - 1\) \(\left(p^* = \frac{Np}{N-p}\right)\) are two constants, \(\lambda > 0\) is a real parameter. We obtain that Problem (\(\ast\)) has two positive weakly solutions if \(\lambda\) is small enough.

Keywords. p-Laplacian, positive solution, critical point theory

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1. Introduction

In this paper we study the singular boundary value problem

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\end{cases}
\end{equation}

where \(\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)\), \(1 < p < N\), \(N \geq 3\), \(\Omega \subset \mathbb{R}^N\) is a bounded domain, \(0 < \alpha < 1\) and \(p - 1 < \beta < p^* - 1\) \(\left(p^* = \frac{Np}{N-p}\right)\) are two constants, \(\lambda > 0\) is a real parameter.

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**Definition 1.1.** A function $u \in W^{1,p}_0(\Omega)$ is called a *positive weakly solution* of Problem (1), if $u(x) > 0$ for $x \in \Omega$ and

$$
\int_{\Omega} (|\nabla u|^{p-2} \nabla u, \nabla \varphi) \, dx = \lambda \int_{\Omega} u^\beta \varphi \, dx + \lambda \int_{\Omega} \frac{\varphi}{u^\alpha} \, dx \quad \text{for all } \varphi \in W^{1,p}_0(\Omega)
$$

holds.

In the pioneering work [1], A. Ambrosetti, H. Brezis and G. Cerami investigated the problem

$$
\begin{cases}
-\Delta u = \lambda u^a + u^b & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
$$

with $0 < a < 1 < b$. In the succeeding work [2], the above problem is extended to the $p$-Laplacian by A. Ambrosetti, J. G. Azorero and I. Peral. Motivated by this, this paper attempt to improve the above results to the singular $p$-Laplacian equation, i.e., $-1 < a < 0$. We must point out that since the functional of (1) fails to be Frechet differentiable in $\Omega$, critical point theory where [1, 2] have used could not be applied to obtain the existence of solutions. So the method in [1, 2] could not be used. So, it is very difficult to find existence and multiplicity of positive solutions for Problem (1).

The existence of solutions to the elliptic equation

$$
\begin{cases}
-\Delta u = \frac{f(x)}{u^\gamma} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
$$

on a smooth domain $\Omega \subset \mathbb{R}^N$ has been extensively studied (cf. [5, 7, 8, 11, 12] and their references). For bounded $\Omega$, in [7] it is shown that Problem (2) with $0 < \gamma < 1$ has a unique positive weakly solution in $H^1_0(\Omega)$ if $p(x)$ is a nonnegative nontrivial function in $L^2(\Omega)$. For the general problem

$$
\begin{cases}
-\Delta u = \frac{\sigma}{u^\gamma} + \lambda u^\beta & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
$$

It is worth mentioning that, in [10] the existence of a unique positive solution in the cases when $\beta = 1$ and $0 < \beta < 1$ (the sub-linear problem) has been proved. On the other hand, in [4], Y. Sun, S. Wu and Y. Long have proved that Problem (3) has at least one positive weakly solution $u \in H^1_0(\Omega)$ for all $\lambda > 0$ and $\sigma \in (0, \sigma^*)$.

Our goal in this paper is to prove that Problem (1) has two positive weakly solutions for all $\lambda$ small enough. In this paper, critical point theory could not be
Positive Solutions

We work on the Sobolev space $W^{1,p}_0(\Omega)$ equipped with the norm $\|u\| = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}$. For $u \in W^{1,p}_0(\Omega)$ we define $I : W^{1,p}_0(\Omega) \to \mathbb{R}$ by

$$I(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx - \frac{\lambda}{\beta + 1} \int_\Omega |u|^\beta + 1 dx - \frac{\lambda}{1 - \alpha} \int_\Omega |u|^{1-\alpha} dx.$$ 

On the other hand, $L^p(\Omega)$ denote Lebesgue’s spaces, the norm in $L^p$ is denoted by $\| \cdot \|_p$; $C_1, C_2, \ldots$ denote (possibly different) positive constants. Our main results is the following:

**Theorem 1.2.** Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$. Let $0 < \alpha < 1$, $p < \beta + 1 < p^*$. Then there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ Problem (1) possesses at least two positive weakly solutions $u_1(\cdot), u_2(\cdot) \in W^{1,p}_0(\Omega)$ and

$$\int_\Omega |\nabla u_i|^{p-2}\nabla u_i \cdot \nabla \varphi dx = \lambda \int_\Omega u_i^\beta \varphi dx + \lambda \int_\Omega \frac{\varphi}{u_i^\alpha} dx \quad \text{for all } \varphi \in W^{1,p}_0(\Omega), \ i = 1, 2.$$ 

Moreover, $u_1$ is a local minimizer of $I$ in $W^{1,p}_0(\Omega)$ with $I(u_1) < 0$; and $u_2$ is a minimizer of $I$ on $\Lambda_-$ ($\Lambda_-$ is defined behind) with $I(u_2) \geq 0$.

**Remark 1.3.** The conclusion of Theorem 1.2 can be extended to the case of the more general problem

$$\begin{cases}
-\Delta_p u = \mu \left( \frac{f(x)}{u^{q-2}} + g(x) u^\tau \right) & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $f, g : \Omega \to \mathbb{R}$ are two given non-negative and non-trivial function in $L^p(\Omega)$.

**Remark 1.4.** When $N = 1$, the type of equations has been studied by Agarwal and O’Regan [9] who proved that the equation

$$\begin{cases}
-((|u'|^{q-2}u')') = \varsigma \left( \frac{1}{u^{\alpha_1}} + u^{\beta_1} + 1 \right) & \text{for } 0 < t < 1, \ 1 < q < \infty \\
u(0) = u(1) = 0,
\end{cases}$$

where $\alpha_1 > 0$, $\beta_1 > q - 1$ and $0 < \varsigma < \frac{2\alpha_1}{3} \left( \frac{q}{q-1+\alpha_1} \right)^{q-1}$, has two solutions $u_1, u_2 \in C[0, 1] \cap C^1(0, 1)$ with $u_1 > 0$, $u_2 > 0$ on $(0, 1)$ and $\|u_1\|_\infty < 1 < \|u_2\|_\infty$. 

applied to obtain the existence of solutions since the associate functional fails to be Frechet differentiable in $\Omega$. We mainly rely on the Ekeland’s variational principle [6] and careful estimates inspirsed by Lair-Shaker [7] and Tarantello [3].
2. Preliminary lemmas

Let us define
\[
\Lambda = \left\{ u \in W^{1,p}_0(\Omega) : \|u\|^p - \lambda\|u\|^{\beta+1} - \lambda \int_\Omega |u|^{1-\alpha} = 0 \right\}.
\]

It is easy to see that \(\Lambda \setminus \{0\}\) is a Nehari manifold, see [14]. Notice that if \(u\) is a weak of (1), then \(u \in \Lambda\). For the sake of the convenience, we record
\[
A = \frac{p - 1 + \alpha}{\beta + \alpha}, \quad B = \frac{\beta - p + 1}{\beta + \alpha}, \quad D = \frac{p + \alpha - 1}{\beta + 1 - p}, \quad E = \frac{p^*-\beta-1}{p^*(\beta+1)}, \quad F = \frac{\beta + \alpha}{\beta + 1},
\]
(4)

Further, we define \(G : W^{1,p}_0(\Omega) \to R\) by
\[
G(u) = A\|u\|^p - \lambda\|u\|^{\beta+1}.
\]

In succession, let
\[
\Lambda_+ = \{ u \in \Lambda : G(u) > 0 \}, \quad \Lambda_0 = \{ u \in \Lambda : G(u) = 0 \}, \quad \Lambda_- = \{ u \in \Lambda : G(u) < 0 \}.
\]

For the sake of the convenience, we list some inequalities which we will use in the next section. By Sobolev’s embedding Theorem, we have
\[
\|u\|_p \leq C_0\|u\| \quad \forall u \in W^{1,p}_0(\Omega)
\]
(5)

\[
\|u\|_{p^*} \leq \left(\frac{1}{S}\right)^\frac{1}{p} \|u\| \quad \forall u \in W^{1,p}_0(\Omega),
\]
where \(C_0 > 0\) is a constant and \(S > 0\) is the best Sobolev constant. By Hölder inequalities we have
\[
\|u\|_{\beta+1} \leq |\Omega|^E \|u\|_{p^*} \quad \forall u \in W^{1,p}_0(\Omega) \quad (6)
\]
\[
\int_\Omega |u|^{1-\alpha} dx \leq |\Omega|^F \|u\|^{1-\alpha}_{\beta+1} \quad \forall u \in W^{1,p}_0(\Omega). \quad (7)
\]

By (5) and (6), we have
\[
\|u\|_{\beta+1} \leq C_1\|u\| \quad \forall u \in W^{1,p}_0(\Omega)
\]
(8)

where \(C_1 = |\Omega|^E \left(\frac{1}{S}\right)^\frac{1}{p}\). By (7) and (8), we have
\[
\int_\Omega |u|^{1-\alpha} dx \leq C_2\|u\|^{1-\alpha} \quad \forall u \in W^{1,p}_0(\Omega) \quad (9)
\]
where \(C_2 = |\Omega|^{E+\alpha-1}S^{\frac{\alpha-1}{p}}\).
Lemma 2.1. Let
\[ \lambda_1 = \left( \frac{A^\beta B S^{\frac{1}{\beta}}}{D^{E+F}} \right)^{\frac{1}{p}} \]  
where \( A, B, D, E, F, S \) are defined in (4) and (5). Then, for all \( \lambda \in (0, \lambda_1) \), we have the following conclusions:
1. For every \( u \in \Lambda, u \neq 0 \), then \( G(u) \neq 0 \), (i.e., \( \Lambda_0 = \{0\} \));
2. \( \Lambda_- \) is closed in \( W^{1,p}_0(\Omega) \).

Proof. 1. Suppose, by contradiction that there exists some \( u \in \Lambda, u \neq 0 \) such that \( G(u) = 0 \). Then
\[ \|u\|^{\beta+1} = \frac{1}{\lambda} \|u\|^p. \]
So
\[ 0 = \|u\|^p - \lambda \|u\|^{\beta+1} - \lambda \int_{\Omega} \|u\|^{1-\alpha} dx = \|u\|^p - A \|u\|^p - \lambda \int_{\Omega} \|u\|^{1-\alpha} dx. \]

Thus
\[ \int_{\Omega} \|u\|^{1-\alpha} dx = \frac{1 - A}{\lambda} \|u\|^p = \frac{B}{\lambda} \|u\|^p. \]

By (11) and (12) we have
\[ \frac{B}{\lambda} \|u\|^p \left( \frac{A}{\lambda} \right)^D \|u\|^{pD} \left( \|u\|^{(\beta+1)D} - \int_{\Omega} \|u\|^{1-\alpha} dx \right) = 0. \]

On the other hand, by (7) and (8) we have
\[ \frac{B}{\lambda} \|u\|^p \left( \frac{A}{\lambda} \right)^D \|u\|^{pD} \left( \|u\|^{(\beta+1)D} - \int_{\Omega} \|u\|^{1-\alpha} dx \right) \geq \frac{B}{\lambda} \left( \frac{A}{\lambda} \right)^D \frac{S^{\frac{1}{\lambda}}}{\|u\|^{(\beta+1)D}} \|u\|^{(\beta+1)D} - |\Omega|^D \|u\|^{1-\alpha} \]
\[ = \left( \frac{A^\beta B S^{\frac{1}{\beta}}}{\lambda^{E+F} |\Omega|^E} - |\Omega|^F \right) \|u\|^{1-\alpha}. \]

If \( 0 < \lambda < \lambda_1 \), then \( \frac{A^\beta B S^{\frac{1}{\beta}}}{\lambda^{E+F} |\Omega|^E} - |\Omega|^F > 0 \). Thus
\[ \frac{B}{\lambda} \|u\|^p \left( \frac{A}{\lambda} \right)^D \|u\|^{pD} \left( \|u\|^{(\beta+1)D} - \int_{\Omega} \|u\|^{1-\alpha} dx \right) > 0, \]
which yields a contraction by (13). So \( \Lambda_0 = \{0\} \).
2. Let \( \{u_n\} \subset \Lambda_- \) be a sequence such that \( u_n \to u_0 \) in \( W^{1,p}_0(\Omega) \). Then \( u_n \to u_0 \) in \( L^{\beta+1}(\Omega) \) and \( u_0 \in \Lambda_- \cup \Lambda_0 \). Now we prove \( u_0 \in \Lambda_- \). Suppose
u_0 \in \Lambda_0$. Since \(\Lambda_0 = \{0\}\), it follows that \(u_0 = 0\). On the other hand, for all \(u \in \Lambda_-\),
\[
\frac{A}{\lambda} \leq \frac{\|u\|^{\beta+1}}{\|u\|^p}.
\]
By (8), we have
\[
\frac{AS}{\lambda|\Omega| E_p} \leq \frac{\|u\|^{\beta+1-p}}{\|u\|^p}.
\]
Thus
\[
\frac{AS}{\lambda|\Omega| E_p} \leq \frac{\|un\|^{\beta+1-p}}{\|un\|^p} \quad \text{for } n \in N.
\]
Let \(n \to \infty\), we have
\[
\frac{AS}{\lambda|\Omega| E_p} \leq \frac{\|u_0\|^{\beta+1-p}}{\|u_0\|^p}.
\]
So \(u_0 \not= 0\). Hence \(u_0 \in \Lambda_-\).

**Lemma 2.2.** Let
\[
\lambda_2 = A \frac{p}{\lambda} \cdot B \frac{1}{E_p} \cdot \frac{S}{|\Omega| E_p + F}
\]
where \(A, B, D, E, F, S\) are defined in (4) and (5). If \(0 < \lambda < \lambda_2\), then for all \(u \in W_0^{1,p}(\Omega)\), \(u \not= 0\), there exists a unique \(t^+ = t^+(u) > 0\) such that \(t^+u \in \Lambda_-\).

**Proof.** For all \(u \in W_0^{1,p}(\Omega)\), \(u \not= 0\), define \(H : [0, \infty) \to (-\infty, \infty)\) by
\[
H(t) = t^{p-1+\alpha}\|u\|^p - \lambda t^{\beta+\alpha}\|u\|^{\beta+1}_{\beta+1}.
\]
Easy computations show that \(H\) achieves its maximum at
\[
t_0 = \left(\frac{A}{\lambda} \frac{\|u\|^p}{\|u\|^{\beta+1}_{\beta+1}}\right)^\frac{1}{\alpha+1-p}.
\]
So
\[
H(t_0) = \left(\frac{A}{\lambda} \right)^D B \cdot \left(\frac{\|u\|^{p(\beta+\alpha)}}{(\|u\|^{\beta+1}_{\beta+1})(\|u\|^{\alpha+1}_{\beta+1})}\right)^\frac{1}{\alpha+1-p}.
\]
If \(\lambda \in (0, \lambda_2)\), then \(\lambda|\Omega| E_p \|u\|^{1-\alpha}_{\beta+1} < H(t_0)\). By (7), \(\lambda \int |u|^{1-\alpha} dx \leq \lambda|\Omega| E_p \|u\|^{1-\alpha}_{\beta+1}\). So \(\lambda \int |u|^{1-\alpha} dx < H(t_0)\).

On the other hand, \(H'(t) < 0\) for \(t \in (t_0, \infty)\) and \(\lim_{t \to +\infty} H(t) = -\infty\). So, there exists a unique \(t^+ \in (t_0, \infty)\) such that \(H(t^+) = \lambda \int |u|^{1-\alpha} dx\), i.e., \(\|t^+u\|^p - \lambda \|t^+u\|^{\beta+1}_{\beta+1} = \lambda \int |u|^{1-\alpha} dx\). So \(t^+u \in \Lambda_-\). By
\[
H'(t^+) = (p - 1 + \alpha)(t^+)^{p-2+\alpha}\|u\|^p - \lambda(\beta + \alpha)(t^+)^{\beta+\alpha-1}\|u\|^{\beta+1}_{\beta+1} < 0,
\]
we have \(G(t^+u) = (A\|t^+u\|^p - \lambda \|t^+u\|^{\beta+1}_{\beta+1}) \leq 0\). So \(t^+u \in \Lambda_-\).

**Remark 2.3.** From Lemma 2.2 it follows that the set \(\Lambda_-\) is nonempty.
Lemma 2.4. Given \( u \in \Lambda_\cdot \), then there exist \( \varepsilon > 0 \) and a continuous function \( f = f(w) > 0, w \in W^{1,p}_0(\Omega), \|w\| < \varepsilon \), satisfying

\[
f(0) = 1, \quad f(w)(u + w) \in \Lambda_- \text{ for all } w \in W^{1,p}_0(\Omega), \|w\| < \varepsilon.
\]

Proof. Define \( F : R \times W^{1,p}_0(\Omega) \to R \) as follows:

\[
F(t, w) = t^{p-1+\alpha}\|u + w\|^p - \lambda t^{\beta+\alpha}\|u + w\|^{\beta+1} - \lambda \int_\Omega |u + w|^{1-\alpha}dx.
\]

Since \( u \in \Lambda_- (\subset \Lambda) \), it follows that \( F(1, 0) = 0 \) and

\[
F_t(1, 0) = (p - 1 + \alpha)\|u\|^p - \lambda(\beta + \alpha)\|u\|^{\beta+1} < 0,
\]

then we can apply the implicit function theorem at the point \((1, 0)\) and obtain \( \varepsilon > 0 \) and a continuous function \( f = f(w) > 0, w \in W^{1,p}_0(\Omega), \|w\| < \varepsilon \), satisfying \( f(0) = 1, F(f(w), w) = 0 \) for all \( w \in W^{1,p}_0(\Omega), \|w\| < \varepsilon \). Hence \( f(w)(u + w) \in \Lambda_- \). Let \( \varepsilon \in (0, \varepsilon) \) small enough, we have \( f(w)(u + w) \in \Lambda_- \) for all \( w \in W^{1,p}_0(\Omega), \|w\| < \varepsilon \). \( \square \)

Lemma 2.5. Let

\[
\lambda_3 = \left( \frac{\beta + 1}{1 - \alpha} \right)^B D^B|\Omega|^{FB} \frac{AS}{|\Omega|^{Ep}}.
\]

Then, for all \( \lambda \in (0, \lambda_3) \), the whole set \( \Lambda_\cdot \) lies at the nonnegative level, that is \( I(u) \geq 0 \), for all \( u \in \Lambda_- \).

Proof. We argue by contradiction. Suppose that exists \( u_0 \in \Lambda_\cdot \subset \Lambda \) such that \( I(u_0) < 0 \), i.e.,

\[
\frac{1}{p}\|u_0\|^p - \frac{\lambda}{\beta + 1}\|u_0\|^{\beta+1} - \frac{\lambda}{1 - \alpha} \int_\Omega |u_0|^{1-\alpha}dx < 0.
\]

By \( u_0 \in \Lambda \), we have \( \|u_0\|^p = \lambda\|u_0\|^{\beta+1} + \lambda \int_\Omega |u_0|^{1-\alpha}dx \). By (17), we have

\[
\lambda \left( \frac{1}{p} - \frac{1}{\beta + 1} \right) \|u_0\|^{\beta+1} + \lambda \left( \frac{1}{p} - \frac{1}{1 - \alpha} \right) \int_\Omega |u_0|^{1-\alpha}dx < 0,
\]

and by (7), we have

\[
\|u_0\|^{\beta+\alpha} \leq \frac{D(1 + \beta)}{1 - \alpha}|\Omega|^F.
\]

By (14) (noticing \( u_0 \in \Lambda_- \)), we have

\[
\left( \frac{AS}{\lambda|\Omega|^{Ep}} \right)^{\frac{\beta+\alpha}{p+1-p}} \leq \|u_0\|^{\beta+\alpha}.
\]

If \( 0 < \lambda < \lambda_3 \), we have

\[
\|u_0\|^{\beta+\alpha} \leq \frac{D(1 + \beta)}{1 - \alpha}|\Omega|^F < \left( \frac{AS}{\lambda|\Omega|^{Ep}} \right)^{\frac{\beta+\alpha}{p+1-p}} \leq \|u_0\|^{\beta+\alpha}.
\]

This is a contradiction. So \( I(u_0) \geq 0 \). \( \square \)
3. Proof of Theorem 1.2

In this section, we prove that there exist $\lambda_0 > 0$ such that, for all $\lambda \in (0, \lambda_0)$, there exist at least two positive functions $u_1(\cdot), u_2(\cdot) \in W_0^{1,p}(\Omega)$ such that
\[
\int_{\Omega} |\nabla u_i|^{p-2} \nabla u_i \cdot \nabla \varphi dx = \lambda \int_{\Omega} u_i^\beta \varphi dx + \lambda \int_{\Omega} \frac{\varphi}{u_i^\alpha} dx \quad \text{for all } \varphi \in W_0^{1,p}(\Omega), \; i = 1, 2.
\]
Moreover, $u_1$ is a local minimizer of $I$ in $W_0^{1,p}(\Omega)$ with $I(u_1) < 0$; and $u_2$ is a minimizer of $I$ on $\Lambda_-.$

**Proof of Theorem 1.2.** Using the inequalities (8) and (9), we have
\[
I(u) \geq \frac{1}{p} \|u\|^p - \lambda C_3 \|u\|^\beta + \lambda C_4 \|u\|^{1-a}, \quad \forall u \in W_0^{1,p}(\Omega),
\]
where $C_3, C_4 > 0$ are positive constants. From this we readily find that there exists $\lambda_4 > 0$ such that for all $\lambda \in (0, \lambda_4]$ there are $r, a > 0$ such that
(i) $I(u) \geq a$ for all $\|u\| = r$;
(ii) $I$ is bounded on $B_r = \{u \in W_0^{1,p}(\Omega) : \|u\| \leq r\}$.

Let $\lambda_0 = \min \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ where $\lambda_i(i = 1, 2, 3)$ are the values found in (10), (15), (16), and $\lambda_4$ is defined as above. Next, we fix $\lambda \in (0, \lambda_0)$.

**Existence of $u_1$.** In view of of [6, Theorem 1.2] the infimum of $I$ on $B_r$ can be achieved at a point $u_1 \in B_r.$ Note that, since $1 - \alpha < 1$, it follows that for every $v > 0$, $I(tv) < 0$ as $t \to 0$ small. So there exists $v_1 \in B_r$ such that $I(v_1) < 0$. Hence $I(u_1) = \inf_{u \in B_r} I(u) \leq I(v_1) < 0.$ This, together with (i), implies that $u_1 \notin \partial B_1$. Hence $u_1$ is a local minimizer of $I$ in the $W_0^{1,p}$ topology. Clearly, $u_1 \not\equiv 0.$ Moreover, since $I(\|u\|) = I(u)$, we may assume that $u_1 \geq 0$ in $\Omega.$ Then, for any $\varphi \in W_0^{1,p}, \varphi \geq 0,$
\[
0 \leq I(u_1 + t\varphi) - I(u_1)
= \frac{1}{p} (\|u_1 + t\varphi\|^p - \|u_1\|^p) + \frac{\lambda}{\beta+1} \left( \|u_1\|^{\beta+1} - \|u_1 + t\varphi\|^{\beta+1} \right)
+ \frac{\lambda}{1-\alpha} \left( \int_{\Omega} |u_1|^{1-\alpha} dx - \int_{\Omega} |u_1 + t\varphi|^{1-\alpha} dx \right)
\leq \frac{1}{p} (\|u_1 + t\varphi\|^p - \|u_1\|^p),
\]
i.e.,
\[
0 \leq \frac{1}{p} \int_{\Omega} (|\nabla(u_1 + t\varphi)|^p - |\nabla u_1|^p) dx \quad (18)
\]
provided $t > 0$ small enough. Dividing (18) by $t > 0$ and passing to the limit as $t \to 0$, we derive
\[
\int_{\Omega} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \varphi dx \geq 0 \quad \text{for } \varphi \in W_0^{1,p}(\Omega), \; \varphi \geq 0,
\]
which means \( u_1 \in W_0^{1,p}(\Omega) \) satisfies in a weak sense that \(-\Delta_p u_1 \geq 0\) in \(\Omega\). Since \(u_1 \geq 0\), \(u_1 \not\equiv 0\), then the strong maximum principle yields
\[
u_1 > 0 \quad \text{in } \Omega.
\]

On the other hand, from (18), we have
\[
\frac{\lambda}{1-\alpha} \left( \int_{\Omega} |u_1 + t\varphi|^{1-\alpha} dx - \int_{\Omega} |u_1|^{1-\alpha} dx \right) \\
\leq \frac{1}{p} (\|u_1 + t\varphi\|^p - \|u_1\|^p) - \frac{\lambda}{\beta + 1} \left( \|u_1 + t\varphi\|^{\beta+1} - \|u_1\|^{\beta+1} \right).
\]
(19)

Dividing (19) by \(t > 0\) and passing to the limit, it follows that
\[
\frac{\lambda}{1-\alpha} \liminf_{t \to 0^+} \frac{\int_{\Omega} |u_1 + t\varphi|^{1-\alpha} dx - \int_{\Omega} |u_1|^{1-\alpha} dx}{t} \\
\leq \int_{\Omega} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \varphi dx - \lambda \int_{\Omega} u_1^\theta \varphi dx.
\]
(20)

Observing
\[
\frac{1}{1-\alpha} \int_{\Omega} \frac{(u_1 + t\varphi)^{1-\alpha} - u_1^{1-\alpha}}{t} dx = \int_{\Omega} (u_1 + \theta t\varphi)^{-\alpha} \varphi dx,
\]
where \(\theta \to 0^+\) as \(t \to 0^+\) and \((u_1 + \theta t\varphi)^{-\alpha} \varphi \to u_1^{-\alpha} \varphi\) a.e. in \(\Omega\) as \(t \to 0^+\). Since \(0 \leq (u_1 + \theta t\varphi)^{-\alpha} \varphi\), for all \(x \in \Omega\). By Fatou’s Lemma, we have
\[
\frac{1}{1-\alpha} \liminf_{t \to 0^+} \int_{\Omega} \frac{(u_1 + t\varphi)^{1-\alpha} - u_1^{1-\alpha}}{t} dx \geq \int_{\Omega} u_1^{-\alpha} \varphi dx.
\]
(21)

Combining (20) and (21), we have, for all \(\varphi \in W_0^{1,p}(\Omega), \varphi \geq 0,\)
\[
0 \leq \int_{\Omega} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \varphi dx - \lambda \int_{\Omega} u_1^\theta \varphi dx - \lambda \int_{\Omega} u_1^{-\alpha} \varphi dx.
\]
(22)

On the other hand, there exists \(\eta_1 \in (0, 1)\) such that \(u_1 + tu_1 \in B_{r'}\) for \(|t| \leq \eta_1\). We define \(h_1 : [-\eta_1, \eta_1] \to R\) by \(h_1(t) := I((1+t)u_1)\). We have that \(h_1(t)\) achieves its minimum at \(t = 0\). Therefore,
\[
\frac{dh_1}{dt} \bigg|_{t=0} = \int_{\Omega} \left[ |\nabla u_1|^p - \lambda u_1^{\beta+1} - \lambda u_1^{-\alpha} \right] dx = 0.
\]
(23)

Therefore, \(u_1 \in \Lambda\).

We next prove that \(u_1\) is a positive weakly solution. Suppose \(\varphi \in W_0^{1,p}(\Omega)\) and \(\varepsilon > 0\). Let \(\Psi = (u_1 + \varepsilon \varphi)^+,\) where \((u_1 + \varepsilon \varphi)^+ = \max \{u_1 + \varepsilon \varphi, 0\}\). Then
where \( \Psi \in W_0^{1,p}(\Omega) \) and \( \Psi \geq 0 \). Inserting \( \Psi \) into (22) and using (23) again, we infer that

\[
0 \leq \int_{\Omega} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \Psi \, dx - \lambda \int_{\Omega} u_1^p \Psi \, dx - \lambda \int_{\Omega} u_1^{-\alpha} \Psi \, dx
\]

\[
= \int_{\Omega \setminus \Omega_e} \left[ |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla (u_1 + \varepsilon \phi) - \lambda u_1^\beta (u_1 + \varepsilon \phi) - \lambda u_1^{-\alpha} (u_1 + \varepsilon \phi) \right] \, dx
\]

\[
+ \varepsilon \int_{\Omega} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \phi \, dx - \lambda \varepsilon \int_{\Omega} u_1^\beta \phi \, dx - \lambda \varepsilon \int_{\Omega} u_1^{-\alpha} \phi \, dx
\]

\[
- \int_{\Omega_e} \left[ |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla (u_1 + \varepsilon \phi) - \lambda u_1^\beta (u_1 + \varepsilon \phi) - \lambda u_1^{-\alpha} (u_1 + \varepsilon \phi) \right] \, dx
\]

\[
\leq \varepsilon \int_{\Omega} \left[ |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \phi - \lambda u_1^\beta \phi - \lambda u_1^{-\alpha} \phi \right] \, dx - \varepsilon \int_{\Omega_e} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \phi \, dx,
\]

where \( \Omega_e = \{ x \in \Omega : u_1(x) + \varepsilon \phi(x) < 0 \} \). Since the measure of \( \Omega_e \) tends to zero as \( \varepsilon \to 0 \), it follows that \( \int_{\Omega_e} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \phi \, dx \to 0 \) as \( \varepsilon \to 0 \). Dividing by \( \varepsilon \) and letting \( \varepsilon \to 0 \) therefore shows

\[
\int_{\Omega} \left[ |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \phi - \lambda u_1^\beta \phi - \lambda u_1^{-\alpha} \phi \right] \, dx \geq 0.
\]

Noting that \( \phi \) is arbitrary, this holds equally for \(-\phi\). So

\[
\int_{\Omega} \left[ |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \phi - \lambda u_1^\beta \phi - \lambda u_1^{-\alpha} \phi \right] \, dx = 0, \quad \text{for all } \phi \in W_0^{1,p}(\Omega).
\]

Hence, \( u_1 \) is a positive weak solution of (1) and \( I(u_1) < 0 \). Next, we prove that (1) has another positive weakly solution \( u_2 \) such that \( I(u_2) > 0 \). We first show that \( I \) is coercive on \( \Lambda \). Indeed, for \( u \in \Lambda \), we have

\[
\|u\|^p - \lambda \|u\|^{\beta+1} - \lambda \int_{\Omega} |u|^{-\alpha} \, dx = 0. \tag{24}
\]

By (24) and (9), we have

\[
I(u) = \frac{1}{p} \|u\|^p - \frac{\lambda}{\beta+1} \|u\|^{\beta+1} - \frac{\lambda}{1-\alpha} \int_{\Omega} |u|^{-\alpha} \, dx
\]

\[
\geq \left( p - \frac{1}{\beta+1} \right) \|u\|^p - \lambda C_2 \left( \frac{1}{1-\alpha} - \frac{1}{\beta+1} \right) \|u\|^{-\alpha}.
\]

So, \( I \) is coercive on \( \Lambda \). Since \( \Lambda_- \) is a closed set in \( W_0^{1,p}(\Omega) \), we apply Ekeland’s variational Principle to the minimization problem \( \inf_{\Lambda_-} I \). It gives a minimizing sequence \( \{w_n\} \subset \Lambda_- \) with the following properties:
(i) \( I(w_n) < \inf_{\Lambda_-} I + \frac{1}{n} \)
(ii) \( I(w) \geq I(w_n) - \frac{1}{n} \| w - w_n \| , \forall w \in \Lambda_- \).

Since \( I(\| u \|) = I(u) \), we may assume that \( w_n \geq 0 \) in \( \Omega \). By coerciveness, \( \{ w_n \} \) is bounded in \( W^{1,p}_0(\Omega) \), i.e.,
\[
\| w_n \| \leq C_5, \quad n = 1, 2, \ldots ,
\]
where \( C_5 > 0 \) is some constant independent on \( n \). So there exists a subsequence (without loss of generality, suppose it is itself) and a function \( u_2 \geq 0 \) such that
\[
w_n \rightarrow u_2 \quad \text{a.e. } x \in \Omega
\]
\[
w_n \quad \text{strongly} \rightarrow u_2 \quad \text{in } L^{\beta+1}
\]
\[
w_n \quad \text{weakly} \rightarrow u_2 \quad \text{in } W^{1,p}_0.
\]

On the other hand, by (14)
\[
\frac{AS}{\lambda |\Omega|^{p - 1}} \leq \| w_n \|^{\beta+1-p}_{\beta+1},
\]
so \( u_2 \neq 0 \). In addition, for the minimizing sequence \( \{ w_n \} \) there exists a suitable constant \( C_6 > 0 \) such that
\[
A \| w_n \|^{p} - \lambda \| w_n \|^{\beta+1}_{\beta+1} \leq -C_6 \quad n = 1, 2, \ldots .
\]
Suppose, by contradiction, that for a subsequence, which is still denoted by \( \{ w_n \} \), we have
\[
A \| w_n \|^{p} - \lambda \| w_n \|^{\beta+1}_{\beta+1} = o(1).
\]
Using \( \{ w_n \} \subset \Lambda_- \) and (26), we have
\[
I(w_n) = \frac{1}{p} \| w_n \|^{p} - \frac{\lambda}{\beta + 1} \| w_n \|^{\beta+1}_{\beta+1} - \frac{1}{1 - \alpha} \| w_n \|^{p} + \frac{\lambda}{1 - \alpha} \| w_n \|^{\beta+1}_{\beta+1}
\]
\[
= -\frac{\beta + \alpha}{p(1 - \alpha)} G(w_n) - \frac{\lambda(\beta + 1 - p)}{\beta + 1} \| w_n \|^{\beta+1}_{\beta+1}
\]
\[
\leq -\frac{\beta + \alpha}{p(1 - \alpha)} G(w_n) - C_7 \quad \text{for } n = 1, 2, \ldots ,
\]
where \( C_7 > 0 \) is some constant independent of \( n \). Passing to the limit as \( n \rightarrow \infty \), we get \( \lim_{n \rightarrow \infty} I(w_n) \leq -C_7 \). This, together with \( I(w_n) \geq \inf_{\Lambda_-} I(u) \) implies \( \inf_{u \in \Lambda_-} I(u) \leq -C_7 < 0 \), which is clearly impossible because from Lemma 2.5. It follows that \( \inf_{u \in \Lambda_-} I(u) \geq 0 \).
For all \( \varphi \in W_0^{1,p}(\Omega) \), \( \varphi \geq 0 \), applying Lemma 2.4, with \( u = w_n, \) \( w = t\varphi, \) \( t > 0 \) small, we find \( f_n(t) = f_n(t\varphi) \) such that \( f_n(0) = 1 \) and \( f_n(t)(w_n+t\varphi) \in \Lambda_- \). Note that, since

\[
0 = f_n^p(t)\|w_n+t\varphi\|^p - \lambda f_n^{\beta+1}(t)\|w_n+t\varphi\|^{\beta+1} - f_n^{1-\alpha}(t) \int_\Omega (w_n+t\varphi)^{1-\alpha} dx
\]

and

\[
0 = \|w_n\|^p - \lambda \|w_n\|^{\beta+1} - \lambda \int_\Omega w_n^{1-\alpha} dx, \quad \text{so}
\]

\[
0 = f_n^p(t)\|w_n+t\varphi\|^p - \lambda f_n^{\beta+1}(t)\|w_n+t\varphi\|^{\beta+1} - \lambda f_n^{1-\alpha}(t) \int_\Omega (w_n+t\varphi)^{1-\alpha} dx
\]

\[
- \|w_n\|^p + \lambda \|w_n\|^{\beta+1} + \lambda \int_\Omega w_n^{1-\alpha} dx
\]

\[
= (f_n^p(t) - 1)\|w_n+t\varphi\|^p + (\|w_n+t\varphi\|^p - \|w_n\|^p)
\]

\[
- \lambda(f_n^{\beta+1} - 1)\|w_n+t\varphi\|^{\beta+1} - \lambda\left(\|w_n+t\varphi\|^{\beta+1} - \|w_n\|^{\beta+1}\right)
\]

\[
- \lambda(f_n^{1-\alpha} - 1) \int_\Omega (w_n+t\varphi)^{1-\alpha} dx - \lambda \int_\Omega [(w_n+t\varphi)^{1-\alpha} - w_n^{1-\alpha}] dx
\]

\[
\leq (f_n^p(t) - 1)\|w_n+t\varphi\|^p + (\|w_n+t\varphi\|^p - \|w_n\|^p)
\]

\[
- \lambda(f_n^{\beta+1} - 1)\|w_n+t\varphi\|^{\beta+1} - \lambda\left(\|w_n+t\varphi\|^{\beta+1} - \|w_n\|^{\beta+1}\right)
\]

\[
- \lambda(f_n^{1-\alpha} - 1) \int_\Omega (w_n+t\varphi)^{1-\alpha} dx.
\]

Dividing by \( t > 0 \) and letting \( t \to 0 \), we infer that

\[
0 \leq pf_n^\prime(0)\|w_n\|^p + p \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \varphi dx
\]

\[
- \lambda f_n^\prime(0)(\beta+1)\|w_n\|^{\beta+1} - \lambda \int_\Omega w_n^\beta \varphi dx - \lambda(1-\alpha) \int_\Omega w_n^{1-\alpha} dx
\]

\[
= f_n^\prime(0)\left[p\|w_n\|^p - \lambda(\beta+1)\|w_n\|^{\beta+1} - \lambda(1-\alpha)\|w_n\|^{1-\alpha}\right]
\]

\[
+ p \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \varphi dx - \lambda(\beta+1) \int_\Omega w_n^\beta \varphi dx
\]

\[
= f_n^\prime(0)\left[(p+\alpha-1)\|w_n\|^p - \lambda(\beta+\alpha)\|w_n\|^{\beta+1}\right]
\]

\[
+ p \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \varphi dx - \lambda(\beta+1) \int_\Omega w_n^\beta \varphi dx,
\]

i.e.,

\[
0 \leq \left[(p+\alpha-1)\|w_n\|^p - \lambda(\beta+\alpha)\|w_n\|^{\beta+1}\right]
\]

\[
+ p \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \varphi dx - \lambda(\beta+1) \int_\Omega w_n^\beta \varphi dx, \quad (28)
\]
where \( f'_n(0) = \lim_{t \to 0^+} \frac{f_n(t) - f_n(0)}{t} \). For the sake of simplicity, we assume henceforth that the right derivate of \( f_n \) at \( t = 0 \) exists. Indeed, if it doesn’t exist, we let \( t_k \to 0 \) (instead of \( t \to 0 \), \( t_k > 0 \) is chosen in such a way that \( f_n \) satisfies \( q_n := \lim_{k \to \infty} \frac{f_n(t_k) - f_n(0)}{t_k} \), then replace \( f'_n(0) \) by \( q_n \). We next prove that \( f'_n(0) \neq \pm \infty \).

By (8) and (25)

\[

\left| p \int_\Omega |\nabla w_n|^p - 2 \nabla w_n \cdot \nabla \varphi dx - \lambda (\beta + 1) \int_\Omega w_n^\beta \varphi dx \right|

\leq p\|w_n\|^{p-1}\|\varphi\|^p + \lambda (\beta + 1)\|w_n\|_\beta \|\varphi\|_\beta + \leq C_8,

\]

where \( C_8 > 0 \) is a positive constant. For (27), (28) and (29), we know immediately that \( f'_n(0) \neq +\infty \). Now we prove that \( f'_n(0) \neq -\infty \). By contradiction, we assume that \( f'_n(0) = -\infty \), and so for \( t > 0 \) small there holds \( f_n(t) < 1 \). Then

\[

\|f_n(t)(w_n + t\varphi) - w_n\| = \left( \int_\Omega |f_n(t)(\nabla w_n + t\nabla \varphi) - \nabla w_n|^p dx \right)^{\frac{1}{p}}

= \left( \int_\Omega |(f_n(t) - 1)\nabla w_n + t f_n(t)\nabla \varphi|^p dx \right)^{\frac{1}{p}}

\leq [1 - f_n(t)]\|w_n\| + t f_n(t)\|\varphi\|

\]
powered \( t > 0 \) small. Thus, from (ii) we have \( \frac{1}{n}\|w - w_n\| \geq I(w_n) - I(w) \). So

\[

[1 - f_n(t)]\frac{\|w_n\|}{n} + t f_n(t)\frac{\|\varphi\|}{n}

\geq \frac{1}{n}\|f_n(t)(w_n + t\varphi) - w_n\|

\geq I(w_n) - I(f_n(t)(w_n + t\varphi))

= \frac{1}{p}\|w_n\| - \frac{\lambda}{\beta + 1}\|w_n\|_\beta + \frac{\lambda}{1 - \alpha}\int_\Omega |w_n|^{1-\alpha} dx - \frac{1}{p}\|f_n(t)(w_n + t\varphi)\|^p

+ \frac{\lambda}{\beta + 1}\|f_n(t)(w_n + t\varphi)\|_\beta + \frac{\lambda}{1 - \alpha}\int_\Omega |f_n(t)(w_n + t\varphi)|^{1-\alpha} dx

\]

Using

\[

-\frac{\lambda}{1 - \alpha}\int_\Omega |w_n|^{1-\alpha} dx = -\frac{1}{1 - \alpha}\|w_n\|^p + \frac{\lambda}{1 - \alpha}\|w_n\|_\beta

\]

and

\[

\frac{\lambda}{1 - \alpha}\int_\Omega |f_n(t)(w_n + t\varphi)|^{1-\alpha} dx = \frac{1}{1 - \alpha} f_n^p(t)\|w_n + t\varphi\|^p

- \frac{\lambda}{1 - \alpha} f_n^{\beta + 1}(t)\|w_n + t\varphi\|_{\beta + 1}^eta,

\]
we have
\[
[1-f_n(t)] \frac{\|w_n\|}{n} + tf_n(t) \|\varphi\|_n \\
\geq \left( \frac{1}{p} - \frac{1}{1-\alpha} \right) \|w_n\|^p - \left( \frac{\lambda}{\beta + 1} - \frac{\lambda}{1-\alpha} \right) \|w_n\|^\beta \|w_n\|^{\beta+1} \\
+ \lambda \left( \frac{1}{\beta + 1} - \frac{1}{1-\alpha} \right) f_n^p(t) \|w_n + t\varphi\|^p \\
= \frac{p + \alpha - 1}{p(1-\alpha)} (\|w_n + t\varphi\|^p - \|w_n\|^p) + \frac{p + \alpha - 1}{p(1-\alpha)} (f_n^p(t) - 1) \|w_n + t\varphi\|^p \\
- \lambda \frac{\beta + \alpha}{(\beta + 1)(1-\alpha)} f_n^{\beta+1}(t) (\|w_n + t\varphi\|^{\beta+1} - \|w_n\|^{\beta+1}) \\
- \lambda \frac{\beta + \alpha}{(\beta + 1)(1-\alpha)} [f_n^{\beta+1}(t) - 1] \|w_n\|^{\beta+1}.
\]
Dividing by \( t > 0 \) and passing to the limit as \( t \to 0 \), we have
\[
-f_n'(0) \frac{\|w_n\|}{n} + \frac{\|\varphi\|}{n} \\
\geq \frac{p + \alpha - 1}{p(1-\alpha)} \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \varphi \, dx + \frac{p + \alpha - 1}{1-\alpha} f_n'(0) \|w_n\|^p \\
- \frac{\beta + \alpha}{1-\alpha} \int_\Omega |w_n|^\beta \varphi \, dx - \lambda \frac{\beta + \alpha}{1-\alpha} f_n'(0) \|w_n\|^{\beta+1} \\
= \frac{1}{1-\alpha} \left( (p + \alpha - 1) \|w_n\|^p - \lambda (\beta + \alpha) \|w_n\|^{\beta+1} \right) f_n'(0) \\
+ \frac{1}{1-\alpha} \left( (p + \alpha - 1) \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \varphi \, dx - \lambda (\beta + \alpha) \int_\Omega |w_n|^\beta \varphi \, dx \right),
\]
i.e.,
\[
\|\varphi\|_n \geq \frac{1}{1-\alpha} \left( (p + \alpha - 1) \|w_n\|^p - \lambda (\beta + \alpha) \|w_n\|^{\beta+1} + \frac{1-\alpha}{n} \|w_n\| \right) f_n'(0) \\
+ \frac{1}{1-\alpha} \left( (p + \alpha - 1) \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \varphi \, dx - \lambda (\beta + \alpha) \int_\Omega |w_n|^\beta \varphi \, dx \right).
\]
By (25) and (27), there exist \( N_0 > 0 \) and \( C_0 > 0 \) (independent of \( n \)) such that, for \( n \geq N_0 \),
\[
\frac{1}{1-\alpha} \left( (p + \alpha - 1) \|w_n\|^p - \lambda (\beta + \alpha) \|w_n\|^{\beta+1} + \frac{1-\alpha}{n} \|w_n\| \right) \leq -C_0.
\]
On the other hand, by (8) and (25), we have, for \( n \geq N_0 \),
\[
\left| \frac{1}{1 - \alpha} \left( (p + \alpha - 1) \int_{\Omega} |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \varphi dx - \lambda(\beta + \alpha) \int_{\Omega} |w_n|^\beta \varphi dx \right) \right| \leq C_{10},
\]
where \( C_{10} > 0 \) (independent of \( n \)) is a suitable constant. By (30), it is impossible that \( f'_n(0) = -\infty \). Furthermore, (28) and (30) imply that \( |f'_n(0)| \leq C_{11} \) for \( n = 1, 2, \ldots \), where \( C_{11} > 0 \) is a suitable constant.

Now we prove that \( u_2 \in \Lambda_- \) is a positive weakly solution of (1). From condition (ii) we infer \( \frac{1}{n} \| w - w_n \| \geq I(w_n) - I(w) \), i.e.,
\[
\frac{1}{n} \| f_n(t) - 1 \| w_n \| + tf_n(t) \| \varphi \| \geq \frac{1}{n} \| f_n(t)(w_n + t\varphi) - w_n \|
\]
\[
\geq I(w_n) - I(f_n(t)(w_n + t\varphi))
\]
\[
= \frac{1}{p} \| w_n \|^p - \frac{\lambda}{\beta + 1} \| w_n \|^\beta + 1 \| w_n \|^\beta + 1 - \frac{\lambda}{1 - \alpha} \int_{\Omega} |w_n|^{1-\alpha} dx
\]
\[
- \frac{1}{p} \| f_n(t)(w_n + t\varphi) \|^p + \frac{\lambda}{\beta + 1} \| f_n(t)(w_n + t\varphi) \|^\beta + 1
\]
\[
+ \frac{\lambda}{1 - \alpha} \int_{\Omega} |f_n(t)(w_n + t\varphi)|^{1-\alpha} dx
\]
\[
= - \frac{f_n^p(t)}{p} - \frac{1}{p} \| w_n \|^p + \frac{\lambda f_n^{\beta + 1}(t) - 1}{\beta + 1} \| w_n \|^\beta + 1 + \frac{\lambda f_n^{1-\alpha}(t) - 1}{1 - \alpha} \int_{\Omega} |w_n|^{1-\alpha} dx
\]
\[
- \frac{f_n^p(t)}{p} \left( \| w_n + t\varphi \|^p - \| w_n \|^p \right) + \frac{\lambda}{\beta + 1} f_n^{\beta + 1}(t) \left( \| w_n + t\varphi \|^\beta + 1 - \| w_n \|^\beta + 1 \right)
\]
\[
+ \frac{\lambda}{1 - \alpha} f_n^{1-\alpha}(t) \int_{\Omega} ((w_n + t\varphi)^{1-\alpha} - w_n^{1-\alpha}) dx.
\]
Dividing by \( t > 0 \) and passing to the limit as \( t \to 0 \), this yields
\[
\frac{1}{n} \| f'_n(0) \| w_n \| + \| \varphi \|
\]
\[
\geq - f'_n(0) \| w_n \|^p + \lambda f'_n(0) \| w_n \|^\beta + 1 + \lambda f'_n(0) \int_{\Omega} |w_n|^{1-\alpha} dx
\]
\[
- \int_{\Omega} |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \varphi dx + \lambda \int_{\Omega} w_n^{\beta} \varphi dx
\]
\[
+ \liminf_{t \to 0^+} \frac{\lambda}{1 - \alpha} \int_{\Omega} \frac{(w_n + t\varphi)^{1-\alpha} - w_n^{1-\alpha}}{t} dx
\]
\[
= - \int_{\Omega} |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \varphi dx + \lambda \int_{\Omega} w_n^{\beta} \varphi dx
\]
\[
+ \liminf_{t \to 0^+} \frac{\lambda}{1 - \alpha} \int_{\Omega} \frac{(w_n + t\varphi)^{1-\alpha} - w_n^{1-\alpha}}{t} dx.
\]
Since \((w_n(x) + t\varphi(x))^{1-\alpha} - w_n^{1-\alpha}(x) \geq 0\), for all \(x \in \Omega, t > 0\), then by Fatou’s Lemma, we have
\[
\lambda \int_{\Omega} w_n^{1-\alpha} \varphi \, dx \leq \liminf_{t \to 0^+} \frac{\lambda}{1-\alpha} \int_{\Omega} \frac{(w_n + t\varphi)^{1-\alpha} - w_n^{1-\alpha}}{t} \, dx.
\]
So
\[
\lambda \int_{\Omega} w_n^{1-\alpha} \varphi \, dx \leq \frac{1}{n} \left[ |f_n'(0)| \|w_n\| + \|\varphi\| \right] + \int_{\Omega} |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \varphi \, dx - \lambda \int_{\Omega} w_n^{\beta} \varphi \, dx 
\leq C_{11} C_5 + \frac{\|\varphi\|}{n} + \int_{\Omega} |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla \varphi \, dx - \lambda \int_{\Omega} w_n^{\beta} \varphi \, dx.
\]
Let \(n \to \infty\), we have
\[
\liminf_{n \to \infty} \lambda \int_{\Omega} w_n^{1-\alpha} \varphi \, dx \leq \int_{\Omega} |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla \varphi \, dx - \lambda \int_{\Omega} u_2^{\beta} \varphi \, dx;
\]
then using once more Fatou’s Lemma, we infer that, for all \(\varphi \in W_0^{1,p}(\Omega), \varphi \geq 0\),
\[
\int_{\Omega} |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla \varphi \, dx - \lambda \int_{\Omega} u_2^{\beta} \varphi \, dx - \lambda \int_{\Omega} w_n^{1-\alpha} \varphi \, dx \geq 0,
\]
which means that \(u_2\) satisfies \(-\Delta_p u_2 \geq 0\) in \(\Omega\). Since \(u_2 \geq 0\) and \(u_2 \not\equiv 0\) in \(\Omega\), then the strong maximum principle yields \(u_2 > 0\) in \(\Omega\). In particular, using (31) with \(\varphi = u_2\), we infer that
\[
\|u_2\|^p - \lambda \|u_2\|_{\beta+1} - \lambda \int_{\Omega} u_2^{1-\alpha} \, dx \geq 0.
\]
On the other hand, by weakly lower semi-continuity of the norm
\[
\|u_2\|^p \leq \lambda \|u_2\|_{\beta+1} + \lambda \int_{\Omega} u_2^{1-\alpha} \, dx.
\]
So
\[
\|u_2\|^p = \lim_{n \to \infty} \|w_n\|^p = \lambda \|u_2\|_{\beta+1} + \lambda \int_{\Omega} u_2^{1-\alpha} \, dx.
\]
Consequently
\[
\lim_{n \to \infty} w_n \text{ strongly } u_2 \text{ in } W_0^{1,p}(\Omega)
\]
and \(I(u_2) = \inf_{\Lambda_-} I\). Also from Lemma 2.1, it follows that necessarily \(u_2 \in \Lambda_-\). Then, following the same arguments as in proving the existence of \(u_1\) and using (31)–(32), we obtain \(u_2 \in \Lambda_-\) is a positive weakly solution of (1). This completes the proof of Theorem 1.2.
References


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