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Hölder exponents of arbitrary functions

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Abstract

The functional class of Hölder exponents of continuous function has been completely characterized by P. Andersson, K. Daoudi, S. Jaffard, J. Lévy Véhel and Y. Meyer [1, 2, 6, 9]; these authors have shown that this class exactly corresponds to that of the lower limits of the sequences of nonnegative continuous functions. The problem of determining whether or not the Hölder exponents of discontinuous (and even unbounded) functions can belong to a larger class remained open during the last decade. The main goal of our article is to show that this is not the case: the latter Hölder exponents can also be expressed as lower limits of sequences of continuous functions. Our proof mainly relies on a "wavelet-leader" reformulation of a nice characterization of pointwise Hölder regularity due to P. Anderson.

1. Pointwise regularity

Our purpose is to determine how general the pointwise Hölder exponent of a locally integrable function can be. Let us recall the definition of pointwise regularity mostly used.

Definition 1 *Let* $f \in L^1_{loc}(\mathbb{R}^d)$*, let* $x_0 \in \mathbb{R}^d$ *and* $\alpha \geq 0$*;* $f \in C^\alpha(x_0)$ *if there exist* $R > 0$*,* $C > 0$*, and a polynomial* P *of degree less than* α *such that*

$$\text{(1.1)} \qquad \text{if } |x - x_0| \leq R, \quad \text{then } |f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha.$$

The Hölder exponent of f *at* x_0 *is*

$$\text{(1.2)} \qquad h_f(x_0) = \sup\{\alpha \geq 0 : \quad f \in C^\alpha(x_0)\}.$$

If the set $\{\alpha \geq 0 : \quad f \in C^\alpha(x_0)\}$ *is empty (i.e. if* f *is bounded in no neighbourhood of* x_0*), then the supremum in* (1.2) *is taken on the empty set, and therefore* $h_f(x_0) = -\infty$*.*

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Let $B(x_0, r)$ denote the open ball centered at x_0 and of radius r. The definition of $C^\alpha(x_0)$ can be rewritten as follows: There exist $R > 0$, $C > 0$, and a polynomial P of degree less than α such that

$$\forall r \leq R, \qquad \sup_{x \in B(x_0,r)} |f(x) - P(x - x_0)| \leq C r^\alpha. \tag{1.3}$$

The Hölder exponent takes the value $-\infty$ at x_0 if and only if f is bounded in no neighbourhood of x_0. It follows that

$$A_f^\infty = \{x : \ h_f(x) = -\infty\}$$

is a closed set. The following lemma shows that this closed set can be completely arbitrary.

Lemma 1 *Let A be a closed subset of $\mathbb{R}^d$. There exists $f \in L^1(\mathbb{R}^d)$ such that $\forall x \in A$, $h_f(x) = -\infty$ and $\forall x \notin A$, $h_f(x) = +\infty$.*

Indeed, let x_n be a dense sequence in A, and let

$$\omega(x) = \begin{cases} |x|^{-d/2} & \text{if} \quad |x| \leq 1 \\ 0 & \text{otherwise;} \end{cases}$$

then the function

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n^2}\, \omega(x - x_n)$$

clearly satisfies the requirements of Lemma 1.

In view of the fact that in the L^1 setting, functions are implicitly defined almost everywhere; it may be seem more coherent to replace the space $C^\alpha(x_0)$ by the space $\tilde{C}^\alpha(x_0)$ defined in the following way.

Definition 2 *Let $f : \mathbb{R}^d \to \mathbb{R}$, be a locally integrable function, let $x_0 \in \mathbb{R}^d$ and $\alpha \geq 0$; $f \in \tilde{C}^\alpha(x_0)$ if there exist $R > 0$, $C > 0$, and a polynomial P of degree less than α such that*

$$\forall r < R, \qquad \operatorname*{Ess\,sup}_{x \in B(x_0,r)} |f(x) - P(x - x_0)| \leq C r^\alpha. \tag{1.4}$$

Here Ess sup means that the supremum is taken outside a set of vanishing measure; therefore $C^\alpha(x_0) \subset \tilde{C}^\alpha(x_0)$. Note that Definition 2 differs from the notion of *approximate Hölder regularity* where (1.4) is assumed to hold except on a subset of $B(x_0, r)$ whose measure is a $o(r^d)$, see [4].

Let us now explain the reason why it is not really necessary to replace $C^\alpha(x_0)$ by $\tilde{C}^\alpha(x_0)$. First we need to prove the following lemma.

Lemma 2 *Let* $f \in L^1_{loc}(\mathbb{R}^d)$ *and let* $\tilde{f}$ *be the function defined, for each* $x_0 \in \mathbb{R}^d$, *as*

$$\tilde{f}(x_0) = \liminf_{r \to 0} \frac{1}{Vol(B(x_0,r))} \int_{B(x_0,r)} f(x)dx. \tag{1.5}$$

Then $f = \tilde{f}$ *almost everywhere, moreover for any* $x_0 \in \mathbb{R}^d$ *and* $\alpha \geq 0$, *one has* $f \in \tilde{C}^\alpha(x_0)$ *if and only if* $\tilde{f} \in C^\alpha(x_0)$.

Note that the way to define $\tilde{f}$ so that Lemma 2 holds is not unique: Any value between the liminf and the limsup in (1.5) would do. Surprisingly, in some cases, values outside of this range can also lead to the same conclusion, as shown by the following example.

Example. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such that f everywhere has a left and a right limit. Define the jump of f at x by

$$J_f(x) = |f(x^+) - f(x^-)| + |f(x) - f(x^-)|.$$

Let I be a closed bounded interval. Clearly, for any $\varepsilon > 0$, the set of points $x \in I$ where $J_f(x) \geq \varepsilon$ is finite; and $J_f(x) = 0$ means that f is continuous at x. It follows that f is continuous except on a countable set, and (1.5) holds everywhere: The corresponding liminf actually is a limit which is equal to $(f(x^+) + f(x^-))/2$. Therefore f coincides a.e. with a càdlàg function; but the standard convention in probability theory for càdlàg functions is different and consists in redefining f so that $f(x) = f(x^+)$; nonetheless, Lemma 2 remains true in this case.

We now prove Lemma 2. It follows from a classical result of Lebesgue (see for example [15, page 138]) that, $f = \tilde{f}$ a.e.; let x_0 be fixed, let r be such that (1.4) holds, and let $x \in B(x_0, r)$. There exists r' small enough such that, if $B = B(x, r')$, then:

- $B \subset B(x_0, r)$,
- $\left| \frac{1}{Vol(B)} \int_B f(y)dy - \tilde{f}(x) \right| \leq r^\alpha$,
- $\forall y \in B, \quad |P(y - x_0) - P(x - x_0)| \leq r^\alpha$.

Using (1.4) and these estimates, it follows that

$$\begin{aligned} &|\tilde{f}(x) - P(x - x_0)| \\ &\quad \leq \left| \tilde{f}(x) - \frac{1}{Vol(B)} \int_B f(y)dy \right| + \frac{1}{Vol(B)} \int_B |f(y) - P(y - x_0)|dy \\ &\qquad + \frac{1}{Vol(B)} \int_B |P(y - x_0) - P(x - x_0)|dy \leq C' r^\alpha. \end{aligned}$$

The following result shows that the setting supplied by $C^\alpha(x_0)$ is more general than the one supplied by $\tilde{C}^\alpha(x_0)$. Therefore, it justifies that we will stick to the $C^\alpha(x_0)$ setting.

Remark 1 *It follows from Lemma* 2 *that any* $f \in L^1_{loc}(\mathbb{R}^d)$ *satisfies the following property: There exists* $\tilde{f} \in L^1_{loc}(\mathbb{R}^d)$ *such that* $f = \tilde{f}$ *almost everywhere, moreover*

$$\forall x_0 \in \mathbb{R}^d, \qquad h_{\tilde{f}}(x_0) = \tilde{h}_f(x_0),$$

where $h_{\tilde{f}}$ *denotes the Hölder exponent of* $\tilde{f}$ *in the sense of Definition* 1 *and* $\tilde{h}_f$ *the Hölder exponent of* $\tilde{f}$ *in the sense of Definition* 2*.*

Let us now give the main motivations of our article and state our main result. Determining the Hölder exponent of a function, or of a signal, can be of interest in its own right, or as an intermediate step in the context of multifractal analysis (in which case the goal is to determine the Hausdorff dimensions of the sets where the Hölder exponent takes a given value). Even if f is a smooth function, its Hölder exponent can be very irregular. Therefore, an important question is to determine how "bad" it can behave. When f satisfies a uniform Hölder regularity condition (i.e. if there exists $\varepsilon > 0$ such that $f \in C^\varepsilon(\mathbb{R}^d)$), this problem was considered by S. Jaffard in [9] and by K. Daoudi, J. Lévy-Véhel and Y. Meyer in [6]. They proved that the Hölder exponent of such a function f is a lower limit of a sequence of non-negative continuous functions. Furthermore, they gave explicit constructions showing that, conversely, any lower limit of a sequence of nonnegative continuous functions is indeed the Hölder exponent of a continuous function. These constructions were of different types: Explicit wavelet expansions, variants of the Weierstrass functions, or variants of the Takagi function. An important improvement was later obtained by P. Andersson in his PhD Thesis: He showed that this condition actually characterizes Hölder exponents of the larger class of *continuous* functions, see [1, 2]. (See also [3] and references therein for recent results concerning Hölder exponents of stochastic processes). Additional results concerning pointwise Hölder regularity were obtained by Y. Meyer in [14] and by P. Andersson in [1]. However the problem of determining whether the Hölder exponent of discontinuous (and even unbounded) functions can belong to a larger class of functions remained open during the last decade. The purpose of the present paper is to prove the following statement, which answers this question.

Theorem 1 *Let* $f \in L^1_{loc}(\mathbb{R}^d)$*. The Hölder exponent of* f *is the lower limit of a sequence of continuous functions.*

Conversely, any function $h(x)$ *which is the lower limit of a sequence of continuous functions and takes values in* $\mathbb{R}^+ \bigcup \{-\infty, +\infty\}$ *is the Hölder exponent of a function* $f \in L^1_{loc}(\mathbb{R}^d)$*.*

2. Proof of Theorem 1

The proof is a consequence of a characterization of pointwise Hölder regularity which is based on the local rate of approximation of f by smooth functions. First note that, since the result is local, we can assume without loss of generality that f actually belongs to $L^1(\mathbb{R}^d)$. Let $\varphi \in \mathcal{S}$ be such that $\hat{\varphi}(\xi)$ vanishes for $|\xi| \geq 2$ and is identically 1 for $|\xi| \leq 1$. The "smooth approximation" of f at the scale j is

$$S_j f = f * \varphi_j \qquad \text{where} \qquad \varphi_j(x) = 2^{dj}\varphi\left(2^j t\right).$$

P. Andersson proved the following result in [2].

Theorem 2 *Let $f \in L^1(\mathbb{R}^d)$, and let $\alpha > 0$; $f \in C^\alpha(x_0)$ if and only if the following condition holds:*

$$\begin{aligned} &\exists C, R > 0 : \textit{if } \ 2^{-j} + |x - x_0| \leq R, \\ &\qquad\qquad \textit{then } \ |f(x) - (S_j f)(x)| \leq C(2^{-j} + |x - x_0|)^\alpha. \end{aligned} \tag{2.1}$$

Remark 2 *If f is bounded in a neighbourhood of x_0, then the sequence $f - (S_j f)$ is uniformly bounded in j in a neighbourhood of x_0, so that the criterion given by Theorem 2 extends to the case $\alpha = 0$.*

We will start by rewriting this criterion in a way which will be more convenient to handle. First, we recall a few classical definitions. Let $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, and $j \in \mathbb{N}$; λ $(= \lambda(j,k))$ denotes the dyadic cube

$$\lambda = \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_d}{2^j}, \frac{k_d+1}{2^j}\right).$$

If $x_0 \in \mathbb{R}^d$, $\lambda_j(x_0)$ is the only dyadic cube of width 2^{-j} which contains x_0. If $K > 0$, $K\lambda$ denotes the cube of same center as λ, but K times wider i.e.

$$K\lambda = \left[\frac{k_1+\frac{1}{2}}{2^j} - \frac{K}{2^{j+1}}, \frac{k_1+\frac{1}{2}}{2^j} + \frac{K}{2^{j+1}}\right) \times \cdots \times \left[\frac{k_d+\frac{1}{2}}{2^j} - \frac{K}{2^{j+1}}, \frac{k_d+\frac{1}{2}}{2^j} + \frac{K}{2^{j+1}}\right).$$

Definition 3 *Let $A \subset \mathbb{R}^d$;* the oscillation *of f at scale j over A is*

$$\mathcal{O}sc_j(f, A) = \sup_{j' \geq j,\, x \in A} |f(x) - (S_{j'} f)(x)|. \tag{2.2}$$

Note that $S_j f$ is a regularization of f at scale j; therefore $f - S_j f$ measures a "high frequency" part of f, and $\mathcal{O}sc_j(f, A)$ thus measures its oscillations at scales larger than j. Definition 3 is not the standard definition of the oscillation of a function over a set; in particular, it depends on the function φ which is chosen in the definition of the smooth approximation.

Lemma 3 *Let $f \in L^1(\mathbb{R}^d)$, and let $\alpha \geq 0$; let $K \geq 3$ be given; then (2.1) is equivalent to:*

$$(2.3) \qquad \exists C > 0 \quad \text{such that, for all } j \in \mathbb{N}, \quad \mathcal{O}sc_j(f, K\lambda_j(x_0)) \leq C2^{-\alpha j}.$$

Remark 3 *This equivalence is in the same spirit as the "wavelet leader" reformulation of the two-microlocal conditions $C^{\alpha,-\alpha}(x_0)$, see [10].*

We now prove Lemma 3. Assume that (2.1) holds. Let x_0 and j be given. If $j' \geq j$ and $x \in K\lambda_j(x_0)$, then $2^{-j'} + |x - x_0| \leq (K\sqrt{d} + 1) \cdot 2^{-j}$ and (2.1) implies that

$$|f(x) - (S_{j'}f)(x)| \leq C2^{-\alpha j},$$

so that (2.3) holds.

Conversely, let $x_0 \in \mathbb{R}^d$ and assume that (2.3) holds. Let $j \in \mathbb{N}$ and $x \in \mathbb{R}^d$ be fixed. We suppose that

$$(2.4) \qquad 2^{-j} + |x - x_0| \leq 1/4.$$

Observe that (2.4) implies that

$$(2.5) \qquad j \geq 2 \text{ and } x \in K\lambda_0(x_0).$$

First case: We assume that

$$(2.6) \qquad |x - x_0| \leq 3 \cdot 2^{-j}.$$

Then using (2.6) and the fact that $K \geq 3$, it follows that $x \in K\lambda_{j-2}(x_0)$. Therefore (2.2) implies that

$$(2.7) \qquad |f(x) - (S_jf)(x)| \leq \mathcal{O}sc_j(f, K\lambda_{j-2}(x_0)).$$

Next, putting together (2.7) and (2.3) (we replace j by $j-2$) we obtain that

$$|f(x) - (S_jf)(x)| \leq C2^{-\alpha(j-2)},$$

which proves that (2.1) holds.

Second case: We assume that

$$(2.8) \qquad |x - x_0| > 3 \cdot 2^{-j}.$$

Let

$$(2.9) \qquad l = \sup\{0 \leq m \leq j : \quad x \in K\lambda_m(x_0)\}.$$

In view of (2.5), l is well-defined. Moreover (2.3) implies that

$$\mathcal{O}sc_l(f, K\lambda_l(x_0)) \leq C 2^{-\alpha l}. \tag{2.10}$$

Let us now prove that (2.1) holds when $l \leq j-1$. If this condition is satisfied, then it follows from (2.9) that $x \notin K\lambda_{l+1}(x_0)$, which entails that

$$|x - x_0| > 3.2^{-l-3}. \tag{2.11}$$

Using (2.2), the fact that $j \geq l$, (2.10) and (2.11) we get that

$$|f(x) - (S_j f)(x)| \leq \mathcal{O}sc_l(f, K\lambda_l(x_0)) \leq C 2^{-\alpha l} \leq C|x - x_0|^\alpha,$$

which means that (2.1) is satisfied. Finally, following the same method as before and using (2.8) instead of (2.11) we obtain that (2.1) also holds when $j = l$.

We now prove Theorem 1. We define a family of continuous functions $\mathcal{O}sc_j(x)$, which will essentially extrapolate these local oscillations at scale j. First, we define them at the dyadic points $2^{-j}k$:

$$\text{if } \lambda = \lambda(j,k), \qquad \mathcal{O}sc_j(2^{-j}k) = \inf(2^{j^2}, \mathcal{O}sc_j(f, 5\lambda)) + 2^{-j^2}. \tag{2.12}$$

The purpose of taking an infimum with 2^{j^2} is to deal with bounded quantities even when the oscillation is infinite (i.e. when f is not locally bounded) and adding 2^{-j^2} prevents the quantity defined by (2.12) to vanish; note that both modifications do not alter logarithmic orders of magnitude when $j \to \infty$. The function $\mathcal{O}sc_j(x)$ is then interpolated on $\mathbb{R}^d$ in a continuous way, and so that its value at any point x lies between its values at the extreme points of the cube $\lambda_j(x)$. An interpolation scheme which satisfies these continuity and monotonicity properties will be called a *regular interpolation* in the following. The precise method used is not important; however, in order to fix ideas, one can choose the following standard polynomial interpolation which we state for $j = 0$ and $\lambda = [0,1]^d$. Let $P_0(x) = 1 - x$ and $P_1(x) = x$; then the interpolation of a function f defined at the 2^d extreme points of the unit cube λ is

$$f(x) = \sum_i f(i) \prod_k P_{i_k}(x_k),$$

where the sum is taken on all d-uples $i = (i_1, \ldots, i_k, \ldots i_d) \in \{0,1\}^d$ (note that, if $d = 1$, this interpolation method boils down to the standard linear approximation). The interpolation formula follows for all cubes and all scales, by translation and dilation, and one easily checks that it satisfies the regularity properties mentioned above. In particular, each function $\mathcal{O}sc_j(x)$ is continuous. Since

$$2^{-j^2} \leq \mathcal{O}sc_j(x) \leq 2^{j^2} + 2^{-j^2},$$

each function $\log(\mathcal{O}sc_j(x))/\log(2^{-j})$ is also continuous.

Let x_0 be a fixed point of $\mathbb{R}^d$. If f is not bounded in any neighbourhood of x_0 then, because of (2.12), for each point $k2^{-j}$ which is at a corner of a dyadic cube of width 2^{-j} which contains x, $\mathcal{O}sc_j(2^{-j}k) = 2^{j^2}$, so that $\mathcal{O}sc_j(x) = 2^{j^2}$ at x_0, and

$$\log\left(\frac{\mathcal{O}sc_j(x_0)}{\log(2^{-j})}\right) \longrightarrow -\infty \quad \text{when} \quad j \to +\infty.$$

Else, f is bounded in a neighbourhood of x_0; let $(\omega_{j,l})_{l=1,\dots,2^d}$ be the 2^d extreme points of the cube $\lambda_j(x_0)$. By the regularity of the interpolation method,

$$\inf_l(\mathcal{O}sc_j(\omega_{j,l})) \leq \mathcal{O}sc_j(x_0) \leq \sup_l(\mathcal{O}sc_j(\omega_{j,l})).$$

But, by (2.12),

$$\sup_l(\mathcal{O}sc_j(\omega_{j,l})) \leq \mathcal{O}sc_j(f, 7\lambda_j(x_0)) + 2^{-j^2},$$

and

$$\inf_l(\mathcal{O}sc_j(\omega_{j,l})) \geq \mathcal{O}sc_j(f, 3\lambda_j(x_0)) + 2^{-j^2},$$

therefore

$$\mathcal{O}sc_j(f, 3\lambda_j(x_0)) + 2^{-j^2} \leq \mathcal{O}sc_j(x_0) \leq \mathcal{O}sc_j(f, 7\lambda_j(x_0)) + 2^{-j^2}. \tag{2.13}$$

It follows from Theorem 2 and Lemma 2 that

$$\forall K \geq 3, \forall x_0 \in \mathbb{R}^d, \quad h_f(x_0) = \liminf_{j\to+\infty} \frac{\log(\mathcal{O}sc_j(f, K\lambda_j(x_0)) + 2^{-j^2})}{\log(2^{-j})}. \tag{2.14}$$

Therefore, (2.13) and (2.14) imply that

$$\forall x_0 \in \mathbb{R}^d, \qquad h_f(x_0) = \liminf_{j\to+\infty} \frac{\log(\mathcal{O}sc_j(x_0))}{\log(2^{-j})},$$

and the Hölder exponent h_f has thus been written as a lower limit of a sequence of continuous functions.

Let us now prove the second part of the theorem. It is a local version of the proof which was obtained in [9] in a particular case. We consider a function h which is the lower limit of a sequence of continuous functions $h_n(x)$, and we assume that h takes values in $\mathbb{R}^+ \bigcup\{-\infty, +\infty\}$. We define

$$A^\infty = \{x : \ h(x) = -\infty\}. \tag{2.15}$$

Let $\psi^i \in \mathcal{S}(\mathbb{R}^d)$ be the generators of an orthonormal wavelet basis, see [12]. This means that the

$$2^{dj/2}\psi^i(2^j x - k), \qquad j \in \mathbb{Z}, \ \ k \in \mathbb{Z}^d, \ \ i = 1, \dots, 2^d - 1$$

form an orthonormal basis of $L^2(\mathbb{R}^d)$. The wavelet coefficients of f are denoted

$$C^i_{j,k} = \langle f | \psi^i_{j,k} \rangle = \int_{\mathbb{R}^d} f(x) 2^{dj} \psi^i(2^j x - k) dx \quad (j \in \mathbb{Z}, \ k \in \mathbb{Z}^n).$$

We note $C^{\log}$ the class of functions such that: For all $j \geq 2, \quad k \in \mathbb{Z}^d$, $i = 1, \dots, 2^d - 1$,

$$|C^i_{j,k}| \leq C 2^{-j/\log j}. \tag{2.16}$$

It is a slightly stronger asumption than uniform continuity, but it implies no uniform Hölder regularity, see [9].

The first step of the proof consists in constructing a function $f_1 \in L^1(\mathbb{R}^d)$ such that

$$A^\infty_{f_1} = A^\infty, \tag{2.17}$$

where A^∞ was defined by (2.15) and $A^\infty_{f_1} = \{x : \ \ h_{f_1}(x) = -\infty\}$. Let $(j_n, k_n) \in \mathbb{N} \times \mathbb{Z}^d$ be such that the adherence values of the sequence $k_n 2^{-j_n}$ are exactly A^∞. Then clearly the function

$$f_1(x) = \sum_{n=0}^{\infty} j_n \psi^1(2^{j_n} x - k_n)$$

belongs to L^1; since the wavelet coefficients are bounded in no neighbourhood of any point of A^∞, it follows that

$$\forall x \in A^\infty, \qquad h_{f_1}(x) = -\infty$$

and one easily checks that

$$\forall x \notin A^\infty, \qquad h_{f_1}(x) = +\infty,$$

so that, in particular, (2.17) holds.

The second step of the proof consists in constructing a continuous function on $\mathbb{R}^d$ (denoted by f_2) whose Hölder exponent equals h on the complement of A^∞. To this end, we will use the following result of [9].

Proposition 1 *Let* $f \in L^1(\mathbb{R}^d)$*. Suppose that* $f \in C^\alpha(x_0)$*; if* $|k\,2^{-j} - x_0| \leq \frac{1}{2}$ *then*

$$|C^i_{j,k}| \leq C 2^{-\alpha j}(1 + |2^j x_0 - k|)^\alpha. \tag{2.18}$$

Conversely, if (2.18) *holds for all* j, k *such that* $|k2^{-j} - x_0| \leq 2^{-j/(\log j)^2}$*, and if* (2.16) *holds for* $k2^{-j}$ *in a neighbourhhod of* x_0*, then there exists a polynomial* P *of degree less than* α *such that*

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha (\log|x - x_0|)^2. \tag{2.19}$$

Let F_n be the complement of $A^\infty + B(0, 1/n)$; F_n is a closed set; since the result is local, we can assume that the F_n are uniformly bounded, so that the h_n are uniformly continuous on F_{2n}. Since their lower limit is nonnegative on each F_{2n}, there exist nonnegative C^1 functions g_n such that $\sup_{x \in F_n} |g_n(x) - h_n(x)| \to 0$; let $B(n) = n + \sup_{x \in F_n} |\nabla g_n(x)|$; we define the wavelet coefficients of f_2 as follows. If j is one of the numbers $[B(n)]$, and if $k2^{-j} \in F_n$, then

$$C^i_{j,k} = \inf(2^{-j/\log j},\ 2^{-jg_n(k2^{-j})})$$

else $C^i_{j,k} = 0$.

Finally let us give the last step of the proof. Set $f = f_1 + f_2$. Since the wavelet coefficients of f_2 are bounded and those of f_1 are unbounded, it follows from the definition of f_1 that for all $x \in A^\infty$ one has $h_f(x) = -\infty$. Let $x \notin A^\infty$; since the wavelet coefficients of f_1 vanish for $k2^{-j}$ in a neighbourhood of x, we only have to consider those of f_2. The direct part of Proposition 1 implies that

$$\begin{aligned} h_f(x) &\leq \liminf_{[k2^{-j} - 2^{-j}, k2^{-j} + 2^{-j}] \ni x} g_n\left(\frac{k}{2^j}\right) \\ &\leq \liminf g_n(x) + 2^{-j}B(n) = \liminf h_n(x), \end{aligned}$$

so that $h_f(x) \leq h(x)$. In order to prove the converse estimate remark that f satisfies (2.16) locally, so that we can use the second part of Proposition 1. We have

$$C^i_{j,k} = \inf(2^{-j/\log j},\ 2^{-jg_n(k2^{-j})}) \leq \inf(2^{-j/\log j},\ 2^{-jg_n(x)} 2^{j|x - k2^{-j}|B(n)}).$$

Since $|k2^{-j} - x| \leq 2^{-j/(\log j)^2}$, it follows that $2^{j|x-k2^{-j}|B(n)} \leq 2$ for n (hence j) large enough and Proposition 1 implies that

$$h_f(x) = \liminf g_n(x) = \liminf h_n(x) = h(x).$$

3. Concluding remarks

The Hölder exponents of stochastic processes are themselves stochastic processes. In order to be able to deal with them, a preliminary requirement is to determine their measurability properties. Let $X = \{X(t)\}_{t\in\mathbb{R}^d}$ be a real valued jointly measurable real valued stochastic process i.e. $(t,\omega) \mapsto X(t,\omega)$ is a measurable function from $(\mathbb{R}^d \times \Omega, \mathcal{L}_d \otimes \mathcal{A})$ into $(\mathbb{R}, \mathcal{B})$ where $\mathcal{L}_d$ is the Lebesgue σ-field on $\mathbb{R}^d$, $\mathcal{B}$ the Borel σ-field on $\mathbb{R}$ and $(\Omega, \mathcal{A})$ the underlying probability space. For each $\omega \in \Omega$ and $t_0 \in \mathbb{R}^d$, $h_X(t_0,\omega)$ denotes the Hölder exponent of the function $t \mapsto X(t,\omega)$ at t_0. We will show that $\{h_X(t_0)\}_{t_0\in\mathbb{R}^d}$ is a jointly measurable stochastic process when the process X is *separable* and its trajectories almost surely belong $L^1_{loc}(\mathbb{R}^d)$. The separability is a weak condition (see [7]). It means that there exists a countable dense subset D of $\mathbb{R}^d$ and a negligible event $\mathcal{N}$ satisfying the following property: For any closed subset F of $\mathbb{R}$ and any open subset U of $\mathbb{R}^d$ the two sets $\{\omega \in \Omega : X(t,\omega) \in F,\, t \in U\}$ and $\{\omega \in \Omega : X(t,\omega) \in F,\, t \in U \cap D\}$ differ by a subset of $\mathcal{N}$. This property and (2.2) imply that for any dyadic point $2^{-j}k$ the quantity $\mathcal{O}sc_j(2^{-j}k) = \inf(2^{j^2}, \mathcal{O}sc_j(X, 5\lambda)) + 2^{-j^2}$ is a random variable i.e. a measurable function from $(\Omega, \mathcal{A})$ to $(\mathbb{R}, \mathcal{B})$. Next by using the fact that for each $\omega \in \Omega$, $t_0 \mapsto \mathcal{O}sc_j(t_0,\omega)$ is a continuous function on $\mathbb{R}^d$, it follows that $(t_0,\omega) \mapsto \mathcal{O}sc_j(t_0,\omega)$ is a measurable function from $(\mathbb{R}^d \times \Omega, \mathcal{L}_d \otimes \mathcal{A})$ into $(\mathbb{R}, \mathcal{B})$. Finally, the equality

$$h_X(x_0,\omega) = \liminf_{j\to+\infty} \frac{\log(\mathcal{O}sc_j(x_0,\omega))}{\log(2^{-j})}$$

implies that $(t_0,\omega) \mapsto h_X(x_0,\omega)$ is a measurable function from $(\mathbb{R}^d \times \Omega, \mathcal{L}_d \otimes \mathcal{A})$ into $(\mathbb{R}^+ \cup \{-\infty,+\infty\}, \tilde{\mathcal{B}})$, where $\tilde{\mathcal{B}}$ denotes the Borel σ-field of $\mathbb{R}^+ \cup \{-\infty,+\infty\}$.

Our second remark concerns multifractal analysis. Assume that f is a locally integrable function. The spectrum of singularities of f is the function d_f defined for any $H \in \mathbb{R}^+ \cup \{-\infty,+\infty\}$ as

$$d_f(H) = dim(\{x \in \mathbb{R}^d : h_f(x) = H\}),$$

where *dim* denotes the Hausdorff dimension. An important open problem in multifractal analysis is to determine how general the spectrum of singularities of an arbitrary locally integrable function can be, see [8] for partial results on this problem; the same question arises for arbitrary locally integrable *Hölder homogeneous function* (f is Hölder homogeneous if, for any nonempty open set O, $dim(\{x \in O : h_f(x) = H\})$ is independent of O) where the problem is completely open. Theorem 1 shows that the spectrum of singularities of arbitrary locally bounded functions will be the same as for continuous functions. Therefore Theorem 1 implies that the general problems we mentioned are not altered if one makes this additional regularity assumption, which, for instance, allows to use wavelet expansions in such constructions.

Our third remark concerns extensions of the result of this paper to other local exponents. Another local definition of regularity has been considered: $f \in C^\alpha_{loc}(x_0)$ if there exists $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\varphi(x_0) = 1$ and $f\varphi \in C^\alpha(\mathbb{R}^d)$. The local Hölder exponent of f is

$$h_f^{loc}(x_0) = \sup\{\alpha :\ f \in C^\alpha_{loc}(x_0)\}.$$

J. Lévy-Véhel and S. Seuret proved that the local Hölder exponents of continuous functions are exactly the nonnegative lower semi-continuous functions (see [13]). Note that a p-exponent (fitted to the L^p setting) has been proposed by Calderón and Zygmund, see [5] and an abstract general extension has also been proposed in [11]. In all these cases, a natural question is to determine what are the most general exponents.

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