

On Falconer's Distance Set Conjecture

M. Burak Erdoğan

Abstract

In this paper, using a recent parabolic restriction estimate of Tao, we obtain improved partial results in the direction of Falconer's distance set conjecture in dimensions $d \geq 3$.

1. Introduction

Let E be a compact subset of \mathbb{R}^d . The distance set, $\Delta(E)$, of E is defined as

$$\Delta(E) = \{|x - y| : x, y \in E\}.$$

Erdős' famous distinct distances conjecture [7] states that for any $\varepsilon > 0$ and for any finite set $E \subset \mathbb{R}^d$, $d \geq 2$,

$$\#\Delta(E) \geq C_{d,\varepsilon}(\#E)^{\frac{2}{d}-\varepsilon}.$$

This conjecture is still open in all dimensions $d \geq 2$. For various partial results and references see [17], [1] and [13].

Falconer's conjecture [8] is a variant of Erdős' conjecture:

Conjecture. Let $d \geq 2$. Let E be a compact subset of \mathbb{R}^d . Then,

$$\dim(E) > \frac{d}{2} \implies |\Delta(E)| > 0.$$

Here $|\cdot|$ is the Lebesgue measure and $\dim(\cdot)$ is the Hausdorff dimension.

Like Erdős' conjecture, Falconer's conjecture is open in every dimension. In [8], Falconer gave an example showing that $\frac{d}{2}$ in the conjecture is optimal and proved that $\dim(E) > \frac{d+1}{2}$ implies $|\Delta(E)| > 0$. Bourgain [3] improved this result in every dimension, and in particular proved that in \mathbb{R}^2 ,

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$\dim(E) > \frac{13}{9}$ suffices. Later, Wolff [24] proved that in \mathbb{R}^2 , $\dim(E) > \frac{4}{3}$ suffices. In [6], the author obtained a simplified proof of Wolff's result and noted that it is possible to obtain the following improved partial result in higher dimensions using the method in [6] and a bilinear Fourier restriction estimate by Tao [22]. In this paper, we prove¹

Theorem 1. *Let $d \geq 2$. Let E be a compact subset of \mathbb{R}^d with*

$$\dim(E) > \frac{d(d+2)}{2(d+1)}.$$

Then $|\Delta(E)| > 0$.

There are other positive results in the direction of Falconer's conjecture. For example, Mattila [14] proved that in \mathbb{R}^2 , $\dim(E) > 1$ implies $\dim(\Delta(E)) \geq \frac{1}{2}$. Recently, Bourgain [4] improved this result and proved that there exists $c > 0$ such that in \mathbb{R}^2 , $\dim(E) > 1$ implies $\dim(\Delta(E)) > \frac{1}{2} + c$. Bourgain's result relies on a paper by Katz and Tao [12] which relates the Falconer's conjecture to various other problems in harmonic analysis.

There are lots of variations of Falconer's problem. Notably, Mattila and Sjölin [16] proved that $\Delta(E)$ has interior points if $\dim(E) > \frac{d+1}{2}$. Peres and Schlag [18] considered pinned distance sets,

$$\Delta(x, E) = \{|x - y| : y \in E\},$$

and proved that if $\dim(E) > \frac{d+1}{2}$ then $|\Delta(x, E)| > 0$ for almost every $x \in E$.

One can also consider distance sets with respect to general metrics. Let K be a convex symmetric body in \mathbb{R}^d , $d \geq 2$. Define $\Delta_K(E) = \{d_K(x, y) : x, y \in E\}$, where d_K is the distance induced by K . Iosevich and Laba [10] investigated the relation between the curvature of the boundary of K and the size of the distance sets. Hofmann and Iosevich [9] (also see [2] for a similar result in higher dimensions) proved that in \mathbb{R}^2 if $\dim(E) > 1$ then $|\Delta_K(E)| > 0$ for almost every ellipse K centered at the origin. We note that our main result, Theorem 1, remains valid for Δ_K in the case when the boundary of K is smooth and has non-vanishing Gaussian curvature (see Remark 1 below).

List of notations.

χ_A : characteristic function of the set A .

$B(x, r) := \{y : |x - y| < r\}$.

$d(A, B)$: the distance between the sets A and B .

¹Recently, Theorem 1 has been improved by the author. The current best known exponent in Falconer's conjecture is $d/2 + 1/3$ in every dimension $d \geq 2$.

$A_R(C) := \{x \in \mathbb{R}^d : ||x| - R| \leq C\}$.

C : a constant which may vary from line to line.

$A \lesssim B$: $A \leq CB$.

$A \approx B$: $A \lesssim B$ and $B \lesssim A$.

$A \ll B$: $A \leq \frac{1}{C}B$, for some large constant C .

$|A|$: length of the vector A or the measure of the set A .

2. Mattila's approach to distance set problem

In [14], Mattila developed a method to attack the distance set problem. For a very good exposition of this method, see [26]. Mattila's approach was used in [14, 3, 24, 9, 6, 2].

Let μ be a probability measure supported in E . Let ν_μ be the push forward of $\mu \times \mu$ under the distance map $(x, y) \mapsto |x - y|$, i.e.,

$$\nu_\mu(A) = \mu \times \mu(\{(x, y) : |x - y| \in A\}), \text{ for Borel sets } A \subset \mathbb{R}.$$

It is easy to check that ν_μ is a probability measure supported in $\Delta(E)$. Note that if the Fourier transform of ν_μ ,

$$\widehat{\nu}_\mu(\xi) := \int e^{-ix \cdot \xi} d\nu_\mu(x),$$

is an L^2 function, then ν_μ should be absolutely continuous with an L^2 density and hence

$$|\Delta(E)| \geq |\text{Supp}(\nu_\mu)| > 0.$$

Using this idea and the Fourier asymptotics of the surface measure of the unit sphere in \mathbb{R}^d , Mattila proved [14]:

Theorem A. *Let $\alpha \in (0, d)$. Let E be a compact subset of \mathbb{R}^d with $\dim(E) > \alpha$. Assume that there is a probability measure μ supported in E such that*

$$(2.1) \quad \|\widehat{\mu}(R \cdot)\|_{L^2(S^{d-1})} \leq C_\mu R^{\frac{\alpha-d}{2}}, \quad \forall R > 1.$$

Then $|\Delta(E)| > 0$.

Note that Theorem A proves the distance set conjecture for Salem sets [19, 11]. A set $E \subset \mathbb{R}^d$ is called a Salem set if for each $\beta < \dim(E)$, there exists a probability measure μ supported in E such that

$$|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\frac{\beta}{2}}, \quad \forall \xi \in \mathbb{R}^d.$$

To apply Theorem A to arbitrary compact sets, one needs Frostman's lemma (see, e.g., [15]).

Definition 1. A compactly supported probability measure μ is called α -dimensional if it satisfies

$$(2.2) \quad \mu(B(x, r)) \leq C_\mu r^\alpha, \quad \forall r > 0, \forall x \in \mathbb{R}^d.$$

Frostman's Lemma. If E is a compact subset of \mathbb{R}^d with $\dim(E) > \alpha$, then there is an α -dimensional measure μ supported in E .

Frostman's lemma and Mattila's theorem imply:

Lemma 2.1. Fix $\alpha \in (0, d)$. Assume that the inequality (2.1) holds for all α -dimensional measures. Then for any compact $E \subset \mathbb{R}^d$

$$\dim(E) > \alpha \implies |\Delta(E)| > 0.$$

In view of Lemma 2.1, Theorem 1 is a corollary of the following:

Theorem 2. Let $d \geq 2$ and $\alpha \in (0, d)$. Let μ be an α -dimensional measure. Then for each $q > \frac{d+2}{d}$,

$$\|\widehat{\mu}(R \cdot)\|_{L^2(S^{d-1})} \leq C_{q,\mu} R^{-\frac{\alpha}{2q}}, \quad \forall R > 1.$$

Like Theorem 1, Theorem 2 was first proved in [24] for $d = 2$. Under the hypothesis of Theorem 2, it is also known that [14, 20] (also see [21, 6])

$$(2.3) \quad \|\widehat{\mu}(R \cdot)\|_{L^2(S^{d-1})} \lesssim R^{-\max(\frac{\alpha-1}{2}, \min(\frac{\alpha}{2}, \frac{d-1}{4}))}, \quad \forall R > 1.$$

Theorem 2 and (2.3) give optimal bounds for each $\alpha \in (0, 2)$ for $d = 2$ (see, e.g., [20, 24, 6]). Therefore, one can not improve the result in Theorem 1 for $d = 2$ using Mattila's approach. In higher dimensions, (2.3) is optimal for $\alpha \leq \frac{d-1}{2}$ (see [20]); however, there is no reason to believe that Theorem 2 and (2.3) give optimal bounds for $\alpha > \frac{d-1}{2}$.

It is essential that in Theorem 2, we are averaging $\widehat{\mu}(R \cdot)$ on a surface with non-vanishing Gaussian curvature. In general, the Fourier transform $\widehat{\mu}(\xi)$ of an α -dimensional measure μ does not have to converge to zero as $|\xi| \rightarrow \infty$. In fact, for any $d \geq 1$ and for any $\alpha \in (0, d)$, there are Cantor-type measures in \mathbb{R}^d of dimension greater than α whose Fourier transform does not converge to 0 at infinity [19].

Remark 1. Mattila's approach can be modified for distance sets with respect to general metrics. Let K be a convex symmetric body. Assume that the boundary of K is smooth and has non-vanishing Gaussian curvature. Let K^* be the dual of K . One can modify Mattila's approach and prove that the statement of Lemma 2.1 remains valid if $\Delta(E)$ is replaced with

$\Delta_K(E)$ and S^{d-1} in (2.1) is replaced with ∂K^* (see [9, 2]). We note that Theorem 2 remains valid, too, if we replace S^{d-1} with ∂K^* . The proof of this fact follows the same line below with minor changes in the statements and proofs of Corollary 2 and Lemma 5.2. Therefore, Theorem 1 holds for Δ_K if K has a smooth boundary with non-vanishing Gaussian curvature.

3. Tao's bilinear parabolic extension estimate

In the proof of Theorem 2, we use a bilinear restriction estimate for elliptic surfaces by Tao [22]. First let us recall the definition of elliptic surfaces from [23]:

Definition 2. We say $\phi : B(0, 1) \subset \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ is an (M, ε_0) -elliptic phase if ϕ satisfies

- i) $\|\phi\|_{C^\infty} < M$,
- ii) $\phi(0) = \nabla\phi(0) = 0$, and
- iii) for all $x \in B(0, 1)$, all eigenvalues of the Hessian $\phi_{x_i x_j}(x)$ lie in $[1 - \varepsilon_0, 1 + \varepsilon_0]$.

We say S is an (M, ε_0) -elliptic surface if $S = \{(x, y) \in B(0, 1) \times \mathbb{R} \subset \mathbb{R}^d : y = \phi(x)\}$ for some (M, ε_0) -elliptic phase ϕ .

Note that in this definition the term ‘‘elliptic’’ is used in a slightly non-standard way. In classical PDE, a non-vanishing symbol is considered to be elliptic. In the definition above, the non-vanishing of the curvature is required, too, (see II below). A model example for an elliptic phase is $\phi(x) = \frac{|x|^2}{2}$. We recall the following properties of elliptic phases (see, e.g., [23]):

- I) Let ϕ be an (M, ε_0) -elliptic phase and $B(x_0, \eta) \subset B(0, 1)$. Let

$$\tilde{\phi}(x) := \frac{1}{\eta^2} (\phi(x\eta + x_0) - \phi(x_0) - \eta x \cdot \nabla\phi(x_0)), \quad x \in B(0, 1).$$

Then $\tilde{\phi}$ is a $(C_d M, \varepsilon_0)$ -elliptic phase.

- II) Let S be a smooth compact submanifold of \mathbb{R}^d with strictly positive principal curvatures. Note that for any $\varepsilon_0 > 0$ and for any $s \in S$ there is a neighborhood U_s of s and an affine bijection a_s of \mathbb{R}^d such that $a_s(U_s)$ is an (M, ε_0) -elliptic surface, where M depends only on d , $\|\phi\|_{C^\infty}$ and the principal curvatures at s . Moreover, by using a partition of unity, we can write S as a union of affine images of finitely many (M, ε_0) -elliptic surfaces.

These observations are especially important for the extension of Theorem 2 to ∂K^* (see Remark 1 above).

The following theorem is proved in [22] for $d \geq 3$. The $d = 2$ case is basically the Carleson-Sjölin Theorem [5]. In [6], it was used in the proof of Theorem 2 for $d = 2$.

Theorem B. *Let $d \geq 2$. For any $M > 0$, there exists $\varepsilon_0 > 0$ such that the following statement holds.*

Let S_1, S_2 be compact subsets of an (M, ε_0) -elliptic surface in \mathbb{R}^d with $d(S_1, S_2) > \frac{1}{2}$. Let σ_j be the Lebesgue measure on S_j , $j = 1, 2$. Then for all $q > \frac{d+2}{d}$, we have

$$(3.1) \quad \|\widehat{f_1 d\sigma_1} \widehat{f_2 d\sigma_2}\|_{L^q(\mathbb{R}^d)} \leq C_{M,q,d} \|f_1\|_{L^2(S_1, d\sigma_1)} \|f_2\|_{L^2(S_2, d\sigma_2)},$$

for all $f_j \in L^2(S_j, d\sigma_j)$, $j = 1, 2$.

In [22], this theorem is proved explicitly only for the paraboloid. The version we stated here can be proved similarly, see the last section of [22] where the necessary modifications are described.

We need the following scaled and mollified version of this theorem (see e.g. [23]). In view of II) above, choose N_d large enough so that any subset of S^{d-1} of diameter $\lesssim \frac{1}{N_d}$ is an affine image of an elliptic surface which satisfies the hypothesis of Theorem B. Let $A_R(\varepsilon)$ denote the set $\{x \in \mathbb{R}^d : ||x| - R| \leq \varepsilon\}$.

Corollary 1. *Fix a spherical cap U in $A_1(\varepsilon)$, ($\varepsilon \ll 1/N_d$), of diameter $\lesssim 1/N_d$. If I_1 and I_2 are subsets of U of diameter η with $d(I_1, I_2) \approx \eta$, then for $q > \frac{d+2}{d}$, we have*

$$\|\widehat{f_1} \widehat{f_2}\|_{L^q(\mathbb{R}^d)} \leq C_{q,d} \varepsilon \eta^{d-1-\frac{d+1}{q}} \|f_1\|_2 \|f_2\|_2,$$

for all $f_j \in L^2(I_j)$, $j = 1, 2$.

Proof. First note that the inequality (3.1) is invariant under translations of one or both of the surfaces S_1, S_2 . Therefore, under the hypothesis of Theorem B, we have

$$(3.2) \quad \|\widehat{f_1} \widehat{f_2}\|_{L^q(\mathbb{R}^d)} \lesssim \varepsilon \|f_1\|_2 \|f_2\|_2,$$

for all $f_j \in L^2(S_j^\varepsilon)$, $j = 1, 2$, where S_j^ε is the ε -neighborhood of S_j . This follows easily from the definition of Lebesgue measure.

Let e be the unit vector in the direction of the center of mass of $I_1 \cup I_2$. Let $\{e_1 = e, e_2, \dots, e_d\}$ be an orthogonal basis for \mathbb{R}^d . Let $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the linear map which satisfies

$$T(e_1) = \frac{1}{\eta^2} e_1, \quad T(e_j) = \frac{1}{\eta} e_j, \quad j = 2, 3, \dots, d,$$

In view of I) and II) above, $C_j = TI_j$ is contained in $\approx \frac{\varepsilon}{\eta^2}$ -neighborhood of an affine image of a surface S_j , $j = 1, 2$, where the surfaces S_1, S_2 satisfy the hypothesis of Theorem B (with M independent of η, I_1, I_2).

Let $g_j(x) = f_j(T^{-1}x)$, $j = 1, 2$. Since g_j is supported in C_j , using (3.2) we obtain

$$(3.3) \quad \|\widehat{g}_1 \widehat{g}_2\|_q \lesssim \frac{\varepsilon}{\eta^2} \|g_1\|_2 \|g_2\|_2.$$

The following elementary identities and (3.3) yield the claim of the corollary:

$$\begin{aligned} \widehat{f}_j(\xi) &= \frac{1}{\det(T)} \widehat{g}_j(T^{-1}(\xi)) = \eta^{d+1} \widehat{g}_j(T^{-1}(\xi)), \quad j = 1, 2, \\ \|\widehat{f}_1 \widehat{f}_2\|_q &= \eta^{(d+1)(2-\frac{1}{q})} \|\widehat{g}_1 \widehat{g}_2\|_q, \\ \|f_j\|_2 &= \eta^{\frac{d+1}{2}} \|g_j\|_2, \quad j = 1, 2. \end{aligned}$$

■

The following Corollary is obtained from Corollary 1 using a dilation:

Corollary 2. *If I_1 and I_2 are subsets of $A_R(\varepsilon)$, ($\varepsilon \ll R/N_d$), of diameter $\eta \lesssim R/N_d$ with $d(I_1, I_2) \approx \eta$, then for $q > \frac{d+2}{d}$, we have*

$$(3.4) \quad \|\widehat{f}_1 \widehat{f}_2\|_{L^q(\mathbb{R}^d)} \leq C_{q,d} \varepsilon R^{\frac{1}{q}} \eta^{d-1-\frac{d+1}{q}} \|f_1\|_2 \|f_2\|_2,$$

for all $f_j \in L^2(I_j)$, $j = 1, 2$.

4. Uncertainty principle

Let φ be a Schwartz function satisfying

$$\varphi(\xi) = 1, \text{ for } |\xi| < 2 \text{ and } \varphi(\xi) = 0, \text{ for } |\xi| > 4.$$

Let D be a ball of radius s in \mathbb{R}^d . Fix an affine bijection a_D of \mathbb{R}^d which maps D to $B(0, 1)$. Let $\varphi_D = \varphi \circ a_D$. Since φ is a Schwartz function, for each $M \in \mathbb{N}$, we have

$$(4.1) \quad |\varphi_D^\vee(x)| = s^d |\varphi^\vee(sx)| \leq C_{M,d} s^d \sum_{j=1}^{\infty} 2^{-Mj} \chi_{B(0, 2^j s^{-1})}(x), \quad \forall x \in \mathbb{R}^d.$$

The following well-known corollary of the uncertainty principle (see, e.g., [26, Chapter 5]) is another important ingredient of the proof of Theorem 2. We give a proof for the sake of completeness.

Lemma 4.1. *Let μ be an α -dimensional measure in \mathbb{R}^d . Let D be a ball of radius s in \mathbb{R}^d . Then the function $\mu_D := |\varphi_D^\vee| * \mu$ satisfies*

- i) $\|\mu_D\|_\infty \lesssim s^{d-\alpha}$,
- ii) $\|\mu_D\|_1 \lesssim 1$,
- iii) $\mu_D(\mathcal{B}) := \int_{\mathcal{B}} \mu_D(y) dy \lesssim r^\alpha$, for any ball \mathcal{B} of radius $r \geq 100s^{-1}$.

Proof. i) Fix $M > 100d$. Using (4.1) and (2.2), we obtain

$$\begin{aligned} 0 \leq \mu_D(x) &\lesssim s^d \sum_{j=1}^\infty 2^{-Mj} \int \chi_{B(0,2^j s^{-1})}(x-y) d\mu(y) \\ &\lesssim s^d \sum_{j=1}^\infty 2^{-Mj} (2^j s^{-1})^\alpha \lesssim s^{d-\alpha}. \end{aligned}$$

ii) follows from Young’s inequality and the observation $\|\varphi_D^\vee\|_1 \lesssim 1$.

iii) Using (4.1), we get

$$\mu_D(\mathcal{B}) \lesssim s^d \sum_{j=1}^\infty 2^{-Mj} \int \int \chi_{\mathcal{B}}(y) \chi_{B(0,2^j s^{-1})}(y-u) d\mu(u) dy$$

Note that $y \in \mathcal{B}$ and $y - u \in B(0, 2^j s^{-1})$ imply $u \in \mathcal{B} + B(0, 2^j s^{-1})$. Using this, Fubini’s theorem and then (2.2), we obtain

$$\begin{aligned} \mu_D(\mathcal{B}) &\lesssim s^d \sum_{j=1}^\infty 2^{-Mj} \int \int \chi_{\mathcal{B}+B(0,2^j s^{-1})}(u) \chi_{B(0,2^j s^{-1})}(y-u) dy d\mu(u) \\ &\lesssim s^d \sum_{j=1}^\infty 2^{-Mj} (r + 2^j s^{-1})^\alpha (2^j s^{-1})^d \\ &\lesssim \sum_{j=1}^\infty 2^{-\frac{Mj}{2}} (r + 2^j s^{-1})^\alpha \lesssim r^\alpha. \end{aligned}$$

■

5. Proof of Theorem 2

The proof is similar to the proof given in [6]. As in [24, 6], we work with the dual formulation:

Lemma 5.1. *Theorem 2 follows from the following statement: For all $q > \frac{d+2}{d}$, for all α -dimensional measures μ , for all $R > 1$ and for all f supported in $A_R(1)$, we have*

$$(5.1) \quad \left| \int f^\vee(u) d\mu(u) \right| \leq C_{q,\mu} R^{\frac{d-1}{2} - \frac{\alpha}{2q}} \|f\|_2,$$

where f^\vee is the inverse Fourier transform of f .

Proof. ([24]) Fix $q_0 > \frac{d+2}{d}$. Note that by duality, Fubini's theorem and the statement of the lemma, we have

$$\begin{aligned} \|\widehat{\mu}\|_{L^2(A_R(1))} &= \sup_{\|f\|_{L^2(A_R(1))}=1} \left| \int_{A_R(1)} f(u)\widehat{\mu}(u)du \right| = \sup_{\|f\|_{L^2(A_R(1))}=1} \left| \int \widehat{f}(u)d\mu(u) \right| \\ &\leq C_{q,\mu} R^{\frac{d-1}{2}-\frac{\alpha}{2q}}, \quad \forall R > 1. \end{aligned}$$

This easily implies that for any $0 < \varepsilon \ll 1$,

$$(5.2) \quad \|\widehat{\mu}\|_{L^2(A_R(R^\varepsilon))} \leq C_{q,\mu} R^{\frac{d-1}{2}-\frac{\alpha}{2q}+C\varepsilon}, \quad \forall R > 1.$$

Take a Schwartz function ϕ equal to 1 in the support of μ . Note that $\widehat{\mu} = \widehat{\mu} * \widehat{\phi}$. Let $d\sigma_R$ be the surface measure on RS^{d-1} . We have

$$\begin{aligned} \|\widehat{\mu}(R\cdot)\|_{L^2(S^{d-1})}^2 &= C_d R^{-(d-1)} \|\widehat{\mu}\|_{L^2(RS^{d-1})}^2 = C_d R^{-(d-1)} \|\widehat{\mu} * \widehat{\phi}\|_{L^2(RS^{d-1})}^2 \\ &\leq C_d R^{-(d-1)} \|\widehat{\phi}\|_1 \|\widehat{\mu}^2 * \widehat{\phi}\|_{L^1(RS^{d-1})} \\ &\lesssim R^{-(d-1)} \int |\widehat{\mu}|^2(u)(|\widehat{\phi}| * d\sigma_R)(u)du \\ (5.3) \quad &\lesssim R^{-(d-1)} \int |\widehat{\mu}|^2(u)(1 + |R - |u||)^{-M} du. \end{aligned}$$

The second line follows from Cauchy-Schwarz inequality (as in (5.7) below); the third line from Fubini's theorem and the last line from the Schwartz decay of ϕ . Here M is a large constant and the implicit constants in the inequalities depend on d, μ, ϕ , and M . Choose $q \in ((d+2)/2, q_0)$. Using (5.2) for small $\varepsilon = \varepsilon(d, \alpha, q, q_0)$ and (5.3) for large $M = M(\varepsilon, d, q, q_0, \alpha)$, we obtain

$$\begin{aligned} \|\widehat{\mu}(R\cdot)\|_{L^2(S^{d-1})}^2 &\lesssim R^{-(d-1)} \left[\|\widehat{\mu}\|_{L^2(A_R(R^\varepsilon))}^2 + \int_{A_R(R^\varepsilon)^c} (1 + |R - |u||)^{-M} du \right] \\ &\lesssim R^{-\frac{\alpha}{q}+2C\varepsilon} + R^{-M\varepsilon/2} \lesssim R^{-\frac{\alpha}{q_0}}. \end{aligned}$$

This yields Theorem 2 and hence finishes the proof of the lemma. ■

Let f be as in Lemma 5.1 with L^2 norm 1. Below, we prove that

$$(5.4) \quad \|f^\vee\|_{L^2(d\mu)} \lesssim R^{\frac{d-1}{2}-\frac{\alpha}{2q}}.$$

(5.1) can be obtained from (5.4) using Cauchy-Schwarz inequality. As in [6], we use the bilinear approach. It suffices to prove (5.4) for functions f supported in a subset of $A_R(1)$ of diameter $\ll R$. Consider a dyadic decomposition of $A_R(1)$ into spherical caps, I , with dimensions $2 \times 2^n \times \dots \times 2^n$ for

$$R^{\frac{1}{2}} \ll 2^n \ll R.$$

We say I has sidelength 2^n and write $\ell(I) = 2^n$. The unique cap of sidelength 2^{n+1} which contains I is called the parent of I . Let I and J be caps with the same sidelength. We say I and J are related, $I \sim J$, if they are not adjacent but their parents are.

Let $f_I := f\chi_I$. As in [6], we have

$$(5.5) \quad \|f^\vee\|_{L^2(d\mu)}^2 \leq \sum_{R^{\frac{1}{2}} \ll 2^n \ll R} \sum_{\ell(I)=2^n, I \sim J} \|f_I^\vee f_J^\vee\|_{L^1(d\mu)} + \sum_{I \in I_E} \|f_I^\vee\|_{L^2(d\mu)}^2 =: S_1 + S_2.$$

Here I_E is a set of dyadic caps with sidelengths $\approx R^{\frac{1}{2}}$ satisfying the finite overlapping property:

$$(5.6) \quad \left\| \sum_{I \in I_E} \chi_I \right\|_\infty \lesssim 1.$$

First, we obtain a bound for S_2 . Since each $I \in I_E$ is contained in a ball D of radius $CR^{\frac{1}{2}}$, we have $f_I^\vee = f_I^\vee * \varphi_D^\vee$ (φ_D is defined in the beginning of Section 4). Using this and Cauchy-Schwarz inequality, we have

$$(5.7) \quad |f_I^\vee| \leq (|f_I^\vee|^2 * |\varphi_D^\vee|)^{\frac{1}{2}} \|\varphi_D^\vee\|_1^{\frac{1}{2}} \lesssim (|f_I^\vee|^2 * |\varphi_D^\vee|)^{\frac{1}{2}}.$$

Using this, Fubini’s theorem and Lemma 4.1, we obtain

$$(5.8) \quad \|f_I^\vee\|_{L^2(d\mu)}^2 \leq \int |f_I^\vee(x)|^2 (\mu * |\varphi_D^\vee|)(x) dx \lesssim \|f_I^\vee\|_2^2 R^{\frac{d-\alpha}{2}} = \|f_I \vee\|_2^2 R^{\frac{d-\alpha}{2}}.$$

Using (5.8) and (5.6), we obtain

$$S_2 = \sum_{I \in I_E} \|f_I^\vee\|_{L^2(d\mu)}^2 \lesssim R^{\frac{d-\alpha}{2}} \sum_{I \in I_E} \|f_I\|_2^2 \lesssim R^{\frac{d-\alpha}{2}} \|f\|_2^2 = R^{\frac{d-\alpha}{2}}.$$

This term is harmless since $\frac{d-\alpha}{2} < d - 1 - \frac{\alpha}{q}$, for $\alpha \in (0, d)$ and $q > \frac{d+2}{d}$.

In the remaining part of the paper we prove that for $q > \frac{d+2}{d}$,

$$S_1 \lesssim R^{d-1-\frac{\alpha}{q}}.$$

Fix n and $I \sim J$ with $|I| = |J| = 2^n$. First, we prove that

$$(5.9) \quad \|f_I^\vee f_J^\vee\|_{L^1(d\mu)} \leq C_{\alpha,q,d} R^{d-1-\frac{\alpha}{q}} \|f_I\|_2 \|f_J\|_2.$$

Note that $I + J$ is contained in a ball of radius $C2^n$. Hence, $f_I * f_J$ is supported in a ball D of radius $C2^n$. Using this as in (5.8), we obtain

$$(5.10) \quad \|f_I^\vee f_J^\vee\|_{L^1(d\mu)} \leq \int |f_I^\vee(x) f_J^\vee(x)| \mu_D(x) dx,$$

where $\mu_D = \mu * |\varphi_D^\vee|$.

Let e be the unit vector which is in the direction of the center of mass of $I \cup J$. Consider a tiling of \mathbb{R}^d with rectangles P of dimensions $100 \times 100 \frac{2^n}{R} \times \dots \times 100 \frac{2^n}{R}$, the long axis being in the direction e . For each P , let a_P be an affine bijection from \mathbb{R}^d to \mathbb{R}^d which maps P to the unit cube. Let ϕ be a Schwartz function satisfying

$$(5.11) \quad \phi(x) \geq \chi_{B(0,1)}(x), \quad x \in \mathbb{R}^d, \quad \text{and} \quad \text{supp}(\widehat{\phi}) \subset B(0,1).$$

Let $\phi_P := \phi \circ a_P$ and $f_{I,P} := \widehat{f_I^\vee \phi_P}$. Using (5.11) and the fact that the rectangles P tile \mathbb{R}^d , we obtain

$$(5.10) \lesssim \sum_P \int |f_{I,P}^\vee(x) f_{J,P}^\vee(x)| \mu_D(x) \phi_P(x) dx$$

$$(5.12) \lesssim \sum_P \|f_{I,P}^\vee f_{J,P}^\vee\|_q \|\mu_D \phi_P\|_{q'},$$

where $q > \frac{d+2}{d}$ and $q' = \frac{q}{q-1}$.

To estimate $\|f_{I,P}^\vee f_{J,P}^\vee\|_q$, we use the Corollary 2 of Tao's theorem. Let I_P be the support of $f_{I,P}$. Note that I_P is contained in $I + \text{supp}(\widehat{\phi_P}) \subset I + P_{dual}$, where P_{dual} is the dual of P centered at the origin. We have

Lemma 5.2. *$I + P_{dual}$ is contained in a spherical cap of dimensions $10 \times \frac{11}{10} 2^n \times \dots \times \frac{11}{10} 2^n$ in $A_R(10)$ which contains I .*

Proof. Note that P_{dual} is a rectangle of dimensions $100^{-1} \times 100^{-1} R 2^{-n} \times \dots \times 100^{-1} R 2^{-n}$, the short axis being in the direction e . For each $p \in P_{dual}$ and $x \in I$, the angle between $p - e \langle p, e \rangle$ and the hyperplane H_x with normal x is $\leq 10 \frac{2^n}{R}$. Therefore P_{dual} is contained in $\frac{1}{10}$ -neighborhood of $H_x \cap B(0, 100^{-1} R 2^{-n})$. Note that if $|x| \approx R$, and $r \ll R^{\frac{1}{2}}$, then $x + (H_x \cap B(0, r))$ is contained in a spherical cap containing x of dimensions $\approx 1 \times r \times \dots \times r$ in $A_{|x|}(1)$. This finishes the proof since

$$100^{-1} R 2^{-n} \leq 100^{-1} R^{\frac{1}{2}} \ll 2^n. \quad \blacksquare$$

Using Lemma 5.2 for I and J , we see that I_P and J_P have diameter $\lesssim 2^n$; they are contained in $A_R(10)$ and $d(I_P, J_P) \gtrsim 2^n$. Therefore, Corollary 2 implies that

$$(5.13) \quad \|f_{I,P}^\vee f_{J,P}^\vee\|_q \lesssim R^{\frac{1}{q}} 2^{n(d-1-\frac{d+1}{q})} \|f_{I,P}\|_2 \|f_{J,P}\|_2.$$

We bound $\|\mu_D \phi_P\|_{q'}$ by interpolating between L^1 and L^∞ . Using the Schwarz decay of ϕ_P , we have

$$\|\mu_D \phi_P\|_1 \leq \sum_{j=1}^{\infty} 2^{-Mj} \int \mu_D(x) \chi_{2^j P}(x) dx.$$

Note that $2^j P$ can be covered by $\approx \frac{R}{2^n}$ balls of radius $\approx \frac{2^j 2^n}{R}$. Therefore, using Lemma 4.1, we get

$$(5.14) \quad \|\mu_D \phi_P\|_1 \lesssim \sum_{j=1}^{\infty} 2^{-\frac{Mj}{2}} 2^{n\alpha-n} R^{1-\alpha} \lesssim 2^{n\alpha-n} R^{1-\alpha}.$$

Using Lemma 4.1 once again, we obtain

$$(5.15) \quad \|\mu_D \phi_P\|_{\infty} \lesssim \|\mu_D\|_{\infty} \lesssim 2^{nd-n\alpha}.$$

Using (5.15) and (5.14), we obtain

$$(5.16) \quad \|\mu_D \phi_P\|_{q'} \leq \|\mu_D \phi_P\|_{\infty}^{1/q} \|\mu_D \phi_P\|_1^{1/q'} \lesssim 2^{n\frac{d-\alpha}{q}} (2^{n\alpha-n} R^{1-\alpha})^{1/q'}.$$

Using (5.12), (5.13), (5.16) and then Cauchy-Schwarz inequality, we get

$$\begin{aligned} \|f_I^{\vee} f_J^{\vee}\|_{L^1(d\mu)} &\lesssim R^{1-\frac{\alpha}{q'}} 2^{n(\alpha(1-\frac{2}{q})+d-2)} \sum_P \|f_{I,P}\|_2 \|f_{J,P}\|_2 \\ &\lesssim R^{1-\frac{\alpha}{q'}} 2^{n(\alpha(1-\frac{2}{q})+d-2)} \left[\sum_P \|f_{I,P}\|_2^2 \right]^{\frac{1}{2}} \left[\sum_P \|f_{J,P}\|_2^2 \right]^{\frac{1}{2}} \end{aligned}$$

Using the Schwartz decay of ϕ , the fact that the rectangles P tile \mathbb{R}^d and Plancherel formula, we get

$$(5.17) \quad \|f_I^{\vee} f_J^{\vee}\|_{L^1(d\mu)} \lesssim R^{1-\frac{\alpha}{q'}} 2^{n(\alpha(1-\frac{2}{q})+d-2)} \|f_I\|_2 \|f_J\|_2.$$

The exponent of 2^n in (5.17) is non-negative and $2^n \lesssim R$. Therefore

$$(5.18) \quad \|f_I^{\vee} f_J^{\vee}\|_{L^1(d\mu)} \lesssim R^{1-\frac{\alpha}{q'}} R^{\alpha(1-\frac{2}{q})+d-2} \|f_I\|_2 \|f_J\|_2 \lesssim R^{d-1-\frac{\alpha}{q}} \|f_I\|_2 \|f_J\|_2.$$

Finally, using (5.18) and L^2 -orthogonality, as in [23] and [25], we bound S_1 . Note that for each dyadic cap I , there are finitely many (depending on d) dyadic caps J related to I . Therefore, for each I ,

$$\sum_{J \sim I} \|f_J\|_2 \lesssim \|f_{I'}\|_2,$$

for a cap I' of sidelength $C2^n$ which contains I . Also note that for each n , the caps $\{I' : \ell(I) = 2^n\}$ are finitely overlapping. Thus,

$$\sum_{\ell(I)=2^n} \|f_I\|_2^2 \approx \sum_{\ell(I)=2^n} \|f_{I'}\|_2^2 \approx \|f\|_2^2.$$

Therefore,

$$\sum_{\ell(I)=2^n, I \sim J} \|f_I\|_2 \|f_J\|_2 \leq \left[\sum_{\ell(I)=2^n} \|f_I\|_2^2 \right]^{1/2} \left[\sum_{\ell(I)=2^n} \left(\sum_{J \sim I} \|f_J\|_2 \right)^2 \right]^{1/2} \lesssim \|f\|_2^2.$$

Using this, (5.18) and the fact that there are $\lesssim \log(R)$ values of n in the sum for S_1 in (5.5), we obtain (for each $q > \frac{d+2}{d}$)

$$S_1 \lesssim R^{d-1-\frac{\alpha}{q}}.$$

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M. Burak Erdoğan
Department of Mathematics
University of California
Berkeley, CA 94720-3840

Current address: Department of Mathematics
University of Illinois
Urbana, IL 61801
berdogan@math.uiuc.edu

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