

# The phase of the Daubechies filters

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**Abstract.** We give the first term of the asymptotic development for the phase of the  $N$ -th (minimum-phased) Daubechies filter as  $N$  goes to  $+\infty$ . We obtain this result through the description of the complex zeros of the associated polynomial of degree  $2N + 1$ .

## 0. Introduction.

The *Daubechies filters*  $m_N(\xi)$  are defined in the following way [2]:

i)  $m_N(\xi)$  is a trigonometric polynomial of degree  $2N + 1$

$$(1) \quad m_N(\xi) = \sum_{k=0}^{2N+1} a_{N,k} e^{-ik\xi}$$

with *real-valued* coefficients  $a_{N,k}$ .

ii)  $\sqrt{2} m_N(\xi)$  and  $\sqrt{2} e^{-i\xi} \overline{m}_N(\xi + \pi)$  are *conjugate quadrature filters*

$$(2) \quad |m_N(\xi)|^2 + |m_N(\xi + \pi)|^2 = 1.$$

iii)  $m_N(\xi)$  satisfies at 0 and  $\pi$

$$(3) \quad m_N(0) = 1,$$

$$(4) \quad \frac{\partial^p}{\partial \xi^p} m_N(\pi) = 0, \quad \text{for } p \in \{0, 1, \dots, N\}.$$

The importance of those filters is due to the following facts: the associated *wavelet*  $\psi_N$  defined by

$$\hat{\psi}_N(\xi) = e^{-i\xi/2} \overline{m}_N\left(\frac{\xi}{2} + \pi\right) \prod_{j=2}^{+\infty} m_N\left(\frac{\xi}{2^j}\right),$$

generates an orthonormal basis of  $L^2(\mathbb{R})$   $\{2^{j/2}\psi_N(2^j x - k)\}_{j \in \mathbb{Z}, k \in \mathbb{Z}}$  and satisfies the cancellation properties

$$\int x^p \psi_N(x) dx = 0, \quad \text{for } p \in \{0, 1, \dots, N\},$$

and has a support of minimal length among all orthonormal wavelets satisfying (6).

Conditions (1) to (4) don't define  $m_N$  in an unique way. As a matter of fact, there is exactly  $2^{\lfloor (N+1)/2 \rfloor}$  solutions  $m_N$  (where  $[x]$  is the integer part of  $x$ ). Indeed, conditions (1) to (4) determine only the *modulus* of  $m_N$

$$(7) \quad |m_N(\xi)|^2 = Q_N(\cos \xi),$$

$$(8) \quad Q_N(X) = \left(\frac{1+X}{2}\right)^{N+1} \sum_{k=0}^N \binom{N+k}{k} \binom{1-X}{2}^k.$$

We are going to check easily the following result on the roots of  $Q_N$ .

**Proposition 1.** *The roots of  $Q_N$  are  $X = -1$  with multiplicity  $N + 1$  and  $N$  roots  $X_{N,1}, \dots, X_{N,N}$  with multiplicity 1 such that*

- i) for  $1 \leq k \leq N$ ,  $\text{Re } X_{N,k} > 0$  and  $X_{N,N+1-k} = \overline{X_{N,k}}$ ,
- ii) for  $1 \leq k \leq \lfloor N/2 \rfloor$ ,  $\text{Im } X_{N,k} > 0$ ,
- iii) if  $N$  is odd,  $X_{N,(N+1)/2} > 1$ .

With help of Proposition 1, we may easily describe the solutions  $m_N$  of (1) to (4). Indeed, if  $X_{N,k} = (z_{N,k} + 1/z_{N,k})/2$  with  $|z_{N,k}| > 1$ , then we have

$$(9) \quad m_N(\xi) = \prod_{k=1}^{\lfloor (N+1)/2 \rfloor} S_{N,k}(\xi) \left(\frac{1 + e^{-i\xi}}{2}\right)^{N+1},$$

where, for  $1 \leq k \leq [N/2]$ ,

$$\begin{aligned}
 (10) \quad S_{N,k}(\xi) &= \frac{(e^{-i\xi} - z_{N,k})(e^{-i\xi} - \bar{z}_{N,k})}{|1 - z_{N,k}|^2} \\
 \text{or} \quad S_{N,k}(\xi) &= \frac{(1 - z_{N,k} e^{-i\xi})(1 - \bar{z}_{N,k} e^{-i\xi})}{|1 - z_{N,k}|^2}.
 \end{aligned}$$

If  $N$  is odd,

$$\begin{aligned}
 (11) \quad S_{N,(N+1)/2}(\xi) &= \frac{e^{-i\xi} - z_{N,(N+1)/2}}{1 - z_{N,(N+1)/2}} \\
 \text{or} \quad S_{N,(N+1)/2}(\xi) &= \frac{1 - z_{N,(N+1)/2} e^{-i\xi}}{1 - z_{N,(N+1)/2}}.
 \end{aligned}$$

The case where all the roots of  $M_N(z)$  (the polynomial such that  $m_N(\xi) = M_N(e^{-i\xi})$ ) are outside the unit disk is the *minimum-phased Daubechies filter*

$$(12) \quad m_N(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)^{N+1} \prod_{k=1}^N \frac{e^{-i\xi} - z_{N,k}}{1 - z_{N,k}}.$$

The aim of this paper is to describe the phase of the Daubechies filters as  $N$  goes to  $+\infty$ . Indeed, the modulus of  $m_N$  is described by (7) and (8) and one easily checks that

$$(13) \quad \lim_{N \rightarrow +\infty} |m_N(\xi)| = \begin{cases} 1, & \text{if } |\xi| < \frac{\pi}{2}, \\ \frac{1}{\sqrt{2}}, & \text{if } |\xi| = \frac{\pi}{2}, \\ 0, & \text{if } \frac{\pi}{2} < |\xi| \leq \pi. \end{cases}$$

The phase of  $m_N$ , on the other hand, is much more delicate to study: it depends of course on the choice of the factors  $S_{N,k}$  in (9), but even for the case of minimum-phased filters we are not aware of any previous results on the behaviour of the phase.

We are going to give an approximate value of  $z_{N,k}$  which allows the determination of the phase of  $m_N$ . More precisely, if  $Z_1, \dots, Z_N$  are  $N$  complex numbers such that for  $k \in \{1, \dots, N\}$ ,  $|Z_k| \neq 1$  and if

$$\Pi(Z_1, \dots, Z_N)(\xi) = \prod_{k=1}^N \frac{e^{-i\xi} - Z_k}{1 - Z_k},$$

we define the phase  $\omega(Z_1, \dots, Z_N)(\xi)$  as the  $C^\infty$  real-valued function such that  $\omega(0) = 0$  and

$$\Pi(Z_1, \dots, Z_N)(\xi) = \prod_{k=1}^N \left| \frac{e^{-i\xi} - Z_k}{1 - Z_k} \right| e^{-i\omega(Z_1, \dots, Z_N)(\xi)}.$$

This function is easily computed as

$$(14) \quad \omega(Z_1, \dots, Z_N)(\xi) = \text{Im} \left( \int_0^\xi \sum_{k=1}^N \frac{i e^{-is}}{e^{-is} - Z_k} ds \right).$$

**Theorem 1.** Let  $Q_N(X)$  be given by (8),  $X_{N,1}, \dots, X_{N,N}$  be its roots which are not equal to  $-1$  ordered by:

- for  $1 \leq k \leq [(N+1)/2]$ ,  $\text{Im } X_{N,k} \geq 0$  and  $X_{N,N+1-k} = \overline{X_{N,k}}$ ,
- $|X_{N,1}| < |X_{N,2}| < \dots < |X_{N,[(N+1)/2]}|$

and let  $z_{N,k}$  be defined by  $X_{N,k} = (z_{N,k} + 1/z_{N,k})/2$  and  $|z_{N,k}| > 1$ .

For  $1 \leq k \leq N$ , we approximate  $z_{N,k}$  by  $Z_{N,k}$  where:

i) for  $1 \leq k \leq [(N^{1/5})/\text{Log } N]$ ,  $Z_{N,k} = i - \overline{\gamma_k}/\sqrt{N}$ , where  $\gamma_1, \gamma_2, \dots, \gamma_k, \dots$  are the roots of  $\text{erfc}(z) = 1 - (2/\sqrt{\pi}) \int_0^z e^{-s^2} ds$ , such that  $\text{Im } \gamma_k > 0$  and ordered by  $|\gamma_1| < |\gamma_2| < \dots < |\gamma_k| < \dots$ ,

ii) for  $[(N^{1/5})/\text{Log } N] < k \leq [(N+1)/2]$ ,  $Z_{N,k} = \theta_{N,k} + \sqrt{\theta_{N,k}^2 - 1}$ , where

$$(15.a) \quad \text{Im } \theta_{N,k} > 0,$$

$$(15.b) \quad 1 - \theta_{N,k}^2 = \left( 1 + \frac{1}{N} \text{Log} (2\sqrt{2N\pi \sin \varphi_{N,k}}) \right) e^{-2i\varphi_{N,k}},$$

and

$$(16) \quad \varphi_{N,k} = \frac{8k-1}{8N+6} \pi,$$

iii) for  $[(N+1)/2] < k \leq N$ ,  $Z_{N,k} = \overline{Z_{N,N+1-k}}$ .

Then for any choice

$$m_N(\xi) = \left( \frac{1 + e^{-i\xi}}{2} \right)^{N+1} \Pi(z_{N,1}^{\varepsilon_1}, \dots, z_{N,N}^{\varepsilon_N})(\xi)$$

of the Daubechies filter  $m_N$  (where  $\varepsilon_k = \pm 1$  and  $\varepsilon_{N+1-k} = \varepsilon_k$ ), the approximation

$$\tilde{m}_N(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)^{N+1} \Pi(Z_{N,1}^{\varepsilon_1}, \dots, Z_{N,N}^{\varepsilon_N})(\xi)$$

satisfies

$$(17) \quad |\omega(z_{N,1}^{\varepsilon_1}, \dots, z_{N,N}^{\varepsilon_N})(\xi) - \omega(Z_{N,1}^{\varepsilon_1}, \dots, Z_{N,N}^{\varepsilon_N})(\xi)| \leq C_0 \frac{(\text{Log } N)^2}{N^{1/5}},$$

for all  $\xi \in \mathbb{R}$ , where  $C_0$  doesn't depend neither on  $N \geq 2$  nor on  $\xi$  nor on the  $\varepsilon_k$ 's.

Thus, due to Theorem 1, we may give the phase of  $m_N$  with an  $o(1)$  precision! Of course, we need the knowledge of the roots of the complementary error function; these roots are described in [3] and our results give again the same estimates, as we shall see.

We may greatly simplify the approximating  $Z_{N,k}$ 's if we accept to get a greater error. For instance, we may characterize easily the minimum-phased filters with an  $O(\sqrt{N})$  error:

**Theorem 2.** *Let*

$$m_N(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)^{N+1} \Pi(z_{N,1}, \dots, z_{N,N})(\xi)$$

be the  $N$ -th minimum-phased Daubechies filter. Then the phase

$$\omega(z_{N,1}, \dots, z_{N,N})(\xi)$$

satisfies

$$(18) \quad |\omega(z_{N,1}, \dots, z_{N,N})(\xi) - N\omega(\xi)| \leq C_0\sqrt{N}, \quad \text{for all } \xi \in \mathbb{R},$$

where  $C_0$  doesn't depend on  $\xi$  nor on  $N$  and where

$$(19) \quad \omega(\xi) = \frac{1}{2\pi} (\text{Li}_2(-\sin \xi) - \text{Li}_2(\sin \xi)) = \frac{-1}{\pi} \sum_{k=0}^{+\infty} \frac{(\sin \xi)^{2k+1}}{(2k+1)^2}.$$

The  $Li_2$  function is the polylogarithm of order 2

$$(20) \quad Li_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = \int_0^z \frac{1}{u} \operatorname{Log} \frac{1}{1-u} du.$$

The function  $(Li_2(z) - Li_2(-z))/2$  is known under the name of Legendre's  $\chi_2$  function.

Theorem 2 will be proved by approximating  $m_N$  by

$$\tilde{m}_N(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)^{N+1} \pi(\tilde{Z}_{N,1}, \dots, \tilde{Z}_{N,N})(\xi)$$

with

$$\tilde{Z}_{N,k} = \sqrt{e^{-i\theta_{N,k}} + 1} + \sqrt{1 + e^{-i\theta_{N,k}}}, \quad \theta_{N,k} = -\pi + \frac{16k - 2}{8N + 6} \pi,$$

Then  $\omega(\tilde{Z}_{N,1}, \dots, \tilde{Z}_{N,N})/N$  is identified with a Riemann sum for the integral

$$\frac{1}{2\pi} \operatorname{Im} \int_{-\pi}^{\pi} \operatorname{Log} \frac{1}{\sqrt{e^{-i\theta}} + \sqrt{1 + e^{-i\theta}} - e^{-i\xi}} d\theta = \omega(\xi).$$

This approximating  $\tilde{Z}_{N,k}$  is a simplified version of the approximating  $Z_{N,k}$  of Theorem 1, obtained by neglecting the term

$$\frac{1}{N} \operatorname{Log} 2\sqrt{2N\pi \sin \varphi_{N,k}}.$$

We will be also able to give a description of a family of almost linear-phased Daubechies filters:

**Theorem 3.** *Let*

$$m_N(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)^{N+1} \pi(z_{N,1}^{\varepsilon_{N,1}}, \dots, z_{N,N}^{\varepsilon_{N,N}})(\xi)$$

*be the  $N$ -th Daubechies filter with  $N = 4q$  and with the following choice of  $\varepsilon_{N,k}$ : for  $1 \leq p \leq q$ ,  $\varepsilon_{N,4p-3} = \varepsilon_{N,4p} = 1$  and  $\varepsilon_{N,4p-2} = \varepsilon_{4p-1} = -1$  (so that  $\varepsilon_{N,N+1-k} = \varepsilon_{N,k}$ ). Then the phase  $\omega(z_{N,1}^{\varepsilon_{N,1}}, \dots, z_{N,N}^{\varepsilon_{N,N}})(\xi)$  satisfies:*

$$(21) \quad \left| \omega(z_{N,1}^{\varepsilon_{N,1}}, \dots, z_{N,N}^{\varepsilon_{N,N}})(\xi) - \frac{1}{2} N\xi \right| \leq C_0, \quad \text{for all } \xi \in \mathbb{R},$$

where  $C_0$  doesn't depend on  $\xi$  nor on  $N$ .

We are now going to prove Theorem 1 (and obtain theorems 2 and 3 as corollaries). Of course, it amounts to give a precise description of the roots  $X_{N,k}$  of  $Q_N(X)$ . If we neglect the term  $\text{Log } 2 \sqrt{2N\pi \sin \varphi_{N,k}}/N$  in  $Z_{N,k}$ , we obtain as a first approximation that the  $z_{N,k}$  are close to the arc  $\{|z - 1| = \sqrt{2}, \text{Re } z \geq 0\}$  (which can be parameterized as  $\{\sqrt{e^{-i\theta}} + \sqrt{1 + e^{-i\theta}}, -\pi \leq \theta \leq \pi\}$ ), or equivalently that the  $X_{N,k}$  are close to the half-lemniscate  $\{|1 - X_{N,k}^2| = 1, \text{Re } X_{N,k} \geq 0\}$ . This will be obtained by representing  $Q_N(X)$  as a Bernstein polynomial on  $[-1, 1]$  approximating the piecewise analytical function  $\chi_{[0,1]}$

$$(22) \quad Q_N(X) = \sum_{k=N+1}^{2N+1} \binom{2N+1}{k} \left(\frac{1+X}{2}\right)^k \left(\frac{1-X}{2}\right)^{2N+1-k}$$

(a formula pointed by many authors [1], [6], [11]). In that form,  $Q_N(X)$  corresponds to a Herrmann filter [4] and it is precisely the figure in Herrmann's paper representing the  $z_{N,k}$ 's for  $Q_{21}$  which lead us to conjecture the behaviour of the  $z_{N,k}$ 's.

A classical theorem of Kantorovitch [5], [7] on the behaviour of Bernstein polynomials of piecewise analytical functions ensures that  $Q_N(X)$  converges to 0 uniformly on any compact subset of the interior of the half lemniscat  $\{|1 - x^2| < 1, \text{Re } x < 0\}$  and to 1 uniformly on any compact subset of  $\{|1 - x^2| < 1, \text{Re } x > 0\}$ . We will use similar tools to study  $Q_N(X)$  *outside* of the convergence subsets.

Near the critical point  $X = 0$ , the approximation by points on the lemniscat is no longer precise enough, and we will show that for the small roots  $X_{N,k}$ ,  $-\sqrt{N}X_{N,k}$  is to be approximated by a root of the complementary error function. Such an approximation occurs for instance in the study of the (spurious) zeros of the Taylor polynomials of the exponential function [12] and we will use quite similar tools to get our description. The main difference, however, is maybe that we are dealing with a divergent family of polynomials.

NOTATIONS. We will define as usually  $\text{Log } z$  and  $\sqrt{z}$  as the reciprocal functions of

$$z = \text{Log } w \in \{z \in \mathbb{C} : |\text{Im } z| < \pi\} \mapsto w = e^z \in \{w \in \mathbb{C} : w \notin (-\infty, 0]\},$$

$$z = \sqrt{w} \in \{z \in \mathbb{C} : \text{Re } z > 0\} \mapsto w = z^2 \in \{w \in \mathbb{C} : w \notin (-\infty, 0]\}.$$

The paper will be organized in the following way:

1.  $Q_N$  as a Bernstein polynomial and other preliminary results.
2. Small roots of  $Q_N$ : first estimates.
3. Big roots of  $Q_N$ : first estimates.
4. Big roots of  $Q_N$ : further estimates.
5. Small roots of  $Q_N$ : further estimates.
6. The phase of a general Daubechies filter.
7. Minimum-phased Daubechies filters.
8. Almost linear-phased Daubechies filters.

### 1. $Q_N$ as a Bernstein polynomial and other preliminary results.

We begin by proving a first localization result:

**Result 1.** *For  $N \geq 2$  and  $t \neq -1$ , if  $Q_N(t) = 0$  then  $|1 - t| < 1$ .*

PROOF. This will be the only time where we use the Daubechies formula (8) for  $Q_N(X)$ . This formula gives that if  $Q_N(t) = 0$  and  $t \neq -1$ , then

$$(23) \quad \sum_{k=0}^N \frac{1}{2^k} \binom{N+k}{k} (1-t)^k = 0.$$

If we define  $\alpha_k$  as  $\alpha_k = \binom{N+k}{k}/2^k$ ,  $0 \leq k \leq N$ , then we have obviously  $0 < \alpha_0 < \alpha_1 < \dots < \alpha_{N-1} = \alpha_N$ , and we may apply a very classical lemma of Eneström, Takeya and Hurwitz (quoted by G. Pólya and Szegő [10, Exercise III-22]):

**Lemma 1.** *If  $0 < a_0 < a_1 < \dots < a_{N-1} = a_N$  and if  $\sum_{k=0}^N a_k s^k = 0$  then  $|s| < 1$ .*

PROOF OF THE LEMMA. If  $s \geq 0$  then  $\sum_{k=0}^N a_k s^k > 0$ ; if  $s \notin [0, +\infty)$ , then

$$\left| a_0 + \sum_{k=1}^N (a_k - a_{k-1}) s^k \right| < a_0 + \sum_{k=1}^N (a_k - a_{k-1}) |s|^k,$$

thus if  $|s| \geq 1$  (so that  $|s|^k \leq |s|^{N+1}$ ) and  $s \notin [0, +\infty)$ , we get

$$\left| (1-s) \sum_{k=0}^N a_k s^k \right| > |s|^{N+1} \left( a_N - \sum_{k=1}^N (a_k - a_{k-1}) - a_0 \right) = 0.$$



Thus, we have shown that the roots  $t$  of  $Q_N$  such that  $t \neq -1$  are located in the open disk of radius 1 and of center 1, and that the associated values  $1 - t^2$  are located in the interior of a cardioid.

From now until the end, we will use formula (22) instead of formula (8) to represent  $Q_N$ . The main interest in the representation of  $Q_N$  as a Bernstein polynomial is that  $Q_N$  is easily differentiated: (22) gives

$$(24) \quad \frac{d}{dt}Q_N(t) = \frac{(2N + 1)!}{4^N(N!)^2} \frac{1}{2} (1 - t^2)^N .$$

This expression can be easily related to the expression of  $Q_N(\cos \xi)$  given by Y. Meyer ([8])

$$\begin{aligned} Q_N(\cos \xi) &= \int_{-1}^{\cos \xi} \frac{(2N + 1)!}{4^N(N!)^2} \frac{1}{2} (1 - t^2)^N dt \\ &= \int_{\xi}^{\pi} \frac{(2N + 1)!}{4^N(N!)^2} \frac{1}{2} (\sin \theta)^{2N+1} d\theta . \end{aligned}$$

We will use intensively formula (24) in the following. If  $t$  is small, we approximate  $Q_N(t)$  by  $Q_N(0) = 1/2$  and obtain

$$(25) \quad Q_N(t) = \frac{1}{2} \left( 1 + \frac{(2N + 1)!}{4^N(N!)^2} \int_0^t (1 - s^2)^N ds \right) ,$$

while for a bigger  $t$  (with  $\text{Re } t > 0$ ) we approximate  $Q_N(t)$  by  $Q_N(1) = 1$  and obtain

$$(26) \quad Q_N(t) = 1 - \frac{1}{2} \frac{(2N + 1)!}{4^N(N!)^2} \int_t^1 (1 - s^2)^N ds .$$

Stirling's formula  $N! = (N/e)^N \sqrt{2\pi N} (1 + 1/(12N) + O(1/N^2))$  allows one to simplify formulas (25) and (26)

$$(27) \quad \frac{(2N + 1)!}{4^N(N!)^2} = 2 \sqrt{\frac{N}{\pi}} \left( 1 + O\left(\frac{1}{N^2}\right) \right) .$$

Thus  $Q_N(t) = 0$  may be rewritten as

$$(28) \quad 1 + \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{N}t} \left( 1 - \frac{s^2}{N} \right)^N ds = 1 - 2 \frac{\sqrt{N}}{\sqrt{\pi}} \frac{4^N(N!)^2}{(2N + 1)!} = O\left(\frac{1}{N^2}\right)$$

or as

$$(29) \quad \sqrt{N} \int_t^1 (1-s^2)^N ds = 2 \frac{4^N (N!)^2}{(2N+1)!} = \sqrt{\pi} + O\left(\frac{1}{N^2}\right).$$

Formula (28) will be used for the small roots (sections 2 and 5) and formula (29) for the big roots (sections 3 and 4).

We mention a further application of (24) (which will not be used in the following): we may compute explicitly the generating series for  $Q_N(t)$  when  $\operatorname{Re} t < 0$ :

**Proposition 2.** *Assume that  $\operatorname{Re} t < 0$  and  $|(1-t^2)u| < 1$ . Then*

$$(30) \quad \sum_{N=0}^{+\infty} Q_N(t) u^N = \frac{1}{2} \frac{1-t^2}{\sqrt{1-u(1-t^2)}(-t+\sqrt{1-u(1-t^2)})}.$$

PROOF. We differentiate  $\sum_{N=0}^{+\infty} Q_N(t) u^N$  with respect to  $t$ . Then (24) gives

$$\begin{aligned} \frac{\partial}{\partial t} \left( \sum_{N=0}^{+\infty} Q_N(t) u^N \right) &= \sum_{N=0}^{+\infty} \frac{1}{2} \frac{(2N+1)!}{4^N N!} \frac{((1-t^2)u)^N}{N!} \\ &= \frac{1}{2} (1-u(1-t^2))^{-3/2}, \end{aligned}$$

hence

$$\sum_{N=0}^{+\infty} Q_N(t) u^N = \int_{-1}^t \frac{1}{2} \frac{ds}{(1-(1-s^2)u)^{3/2}}.$$

On the other hand, if we differentiate  $t/(1-u(1-t^2))^{1/2}$ , we get

$$\frac{\partial}{\partial t} \left( \frac{t}{(1-u(1-t^2))^{1/2}} \right) = \frac{1-u(1-t^2)-t^2u}{(1-u(1-t^2))^{3/2}} = \frac{1-u}{(1-u(1-t^2))^{3/2}}.$$

Thus we have

$$\begin{aligned} \sum_{N=0}^{+\infty} Q_N(t) u^N &= \frac{1}{2(1-u)} \left( \frac{t}{(1-u(1-t^2))^{1/2}} + 1 \right) \\ &= \frac{1}{2(1-u)} \frac{1-u(1-t^2)-t^2}{(1-u(1-t^2))^{1/2}((1-u(1-t^2))^{1/2}-t)} \\ &= \frac{1}{2} \frac{1-t^2}{(1-u(1-t^2))^{1/2}((1-u(1-t^2))^{1/2}-t)}. \end{aligned}$$

As a corollary, we get:

**Result 2.** *If  $t \in \mathbb{C}$  is such that  $|1 - t^2| > 1$ , then*

$$\limsup_{N \rightarrow +\infty} |Q_N(t)| = +\infty.$$

PROOF. If  $\operatorname{Re} t < 0$ , this is obvious by formula (30); the right-hand term of equality (30) has  $1/|1 - t^2|$  as its radius of convergence in  $u$ , so that

$$\limsup_{N \rightarrow +\infty} |Q_N(t)|^{1/N} = |1 - t^2|.$$

If  $\operatorname{Re} t > 0$ , then  $Q_N(t) = 1 - Q_N(-t)$  so that again

$$\limsup_{N \rightarrow +\infty} |Q_N(t)|^{1/N} = |1 - t^2|.$$

If  $\operatorname{Re} t = 0$  and  $t \neq 0$ , then

$$|Q_N(t)| \sim \frac{1}{2} 2 \sqrt{\frac{N}{\pi}} \int_0^{|t|} (1 + \rho^2)^N d\rho \longrightarrow +\infty, \quad \text{as } N \longrightarrow +\infty.$$

A last (and direct) application of formula (24) is Proposition 1.

**Result 3.**

- i) *If  $t$  is a root of  $Q_N(t)$  and  $t \neq -1$ , then  $t$  has multiplicity 1.*
- ii) *If  $N$  is even,  $t = -1$  is the unique real root of  $Q_N$ .*
- iii) *If  $N$  is odd,  $Q_N$  has only one other real root  $x_{N,(N+1)/2} \neq -1$ , and  $x_{N,(N+1)/2} > 1$ .*

PROOF. By (24), we know that the only roots of  $dQ_N/dt$  are 1 and  $-1$ , so i) is obvious. Moreover, if  $N$  is even,  $dQ_N/dt$  is non-negative on  $\mathbb{R}$  and thus  $Q_N$  is increasing:  $-1$  is the unique real root of  $Q_N$ . If  $N$  is odd, then  $Q_N$  decreases on  $(-\infty, -1]$ , vanishes at  $-1$ , increases between  $-1$  and 1, and decreases again from the value 1 at  $t = 1$  to the value  $-\infty$  at  $t = +\infty$ :  $Q_N$  has another real root  $x_{N,(N+1)/2} > 1$ .

Results 1 and 3 imply obviously Proposition 1.

**2. Small roots of  $Q_N$ : first estimates.**

In this section, we are going to prove the following result:

**Result 4.** *Let  $\varepsilon_0 \in (0, 1/2)$  and  $K = \lceil \varepsilon_0 \text{Log } N / (2\pi) \rceil$ . Then, if  $N$  is big enough, the number of roots  $t$  of  $Q_N(t)$  such that  $\text{Im } t \geq 0$  and  $|t| \leq \sqrt{2K\pi/N}$  is exactly  $K$ . Moreover, if we list those roots as  $x_{N,1}, \dots, x_{N,K}$  with  $|x_{N,k}| < |x_{N,k+1}|$  and fix  $\varepsilon_1 \in (\varepsilon_0, 1/2)$ , we have*

$$(31) \quad \left| x_{N,k} + \frac{1}{\sqrt{N}} \bar{\gamma}_k \right| \leq C(\varepsilon_0, \varepsilon_1) \frac{1}{\sqrt{N} N^{1-2\varepsilon_1}},$$

where  $\gamma_1, \dots, \gamma_K$  are the  $K$  first roots  $\gamma$  of  $\text{erfc}(\gamma) = 0$  with  $\text{Im } \gamma \geq 0$ .

PROOF. Assume that  $|t| \leq \sqrt{\alpha_1 \text{Log } N / N}$  for some fixed  $\alpha_1 > 0$ . Then, using formulas (25) and (27), we write

$$Q_N(t) = \left( \frac{1}{2} + \eta_N \right) \left( 1 + \eta'_N + \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{N}t} \left( 1 - \frac{s^2}{N} \right)^N ds \right),$$

where  $\eta_N, \eta'_N$  are two constants (depending only on  $N$ ) which are  $O(1/N^2)$ . Now, if  $|u| \leq \sqrt{\alpha_1 \text{Log } N}$ , we have

$$\frac{|u^4|}{N} \leq \alpha_1^2 \frac{(\text{Log } N)^2}{N} = o(1),$$

hence one may find  $C_0 \geq 0$  so that for  $N$  big enough ( $N \geq N_0$  where  $N_0$  depends only on  $\alpha_1$ )

$$\left| \left( 1 - \frac{u^2}{N} \right)^N - e^{-u^2} \right| \leq C_0 \left| e^{-u^2} \frac{u^4}{N} \right| \leq C_0 \alpha_1^2 \frac{(\text{Log } N)^2}{N^{1-\alpha_1}}.$$

Hence we get for fixed  $\alpha_1 > 0$  and for  $N \geq N_0(\alpha_1)$

$$(32) \quad \left| \left( \frac{1}{2} + \eta_N \right)^{-1} Q_N(t) - \text{erfc}(-\sqrt{N}t) \right| \leq C_1 \frac{(\text{Log } N)^{5/2}}{N^{1-\alpha_1}},$$

for  $|t| \leq \sqrt{\alpha_1 \text{Log } N / N}$ , where  $C_1$  depends only on  $\alpha_1$ .

Now, assume that  $\theta$  is such that  $Q_N(\theta) = 0$  or  $\text{erfc}(-\sqrt{N}\theta) = 0$  and that  $|\theta| \leq \sqrt{\alpha_1 \text{Log } N / N}$ ; in every case we have

$$|\text{erfc}(-\sqrt{N}\theta)| \leq C_1 \frac{(\text{Log } N)^{5/2}}{N^{1-\alpha_1}}.$$

We are going to show that for  $\delta_0$  small enough,  $\operatorname{erfc}(-\sqrt{N}\theta + z)$  is not too small on  $|z| = \delta_0$ . Indeed we have

$$\begin{aligned} |\operatorname{erfc}(-\sqrt{N}\theta + z) - \operatorname{erfc}(-\sqrt{N}\theta)| &= \frac{2}{\sqrt{\pi}} \left| \int_0^z e^{-N\theta^2} e^{2\sqrt{N}\theta s} e^{-s^2} ds \right| \\ &\geq \frac{1}{2} \frac{2}{\sqrt{\pi}} |e^{-N\theta^2}| |z| \geq \frac{1}{\sqrt{\pi}} N^{-\alpha_1} |z|, \end{aligned}$$

provided that

$$|z| \leq \min \left\{ 2 \sqrt{\alpha_1 \operatorname{Log} N}, \frac{1}{8 C_2 \sqrt{\alpha_1 \operatorname{Log} N}} \right\},$$

where  $C_2 = \max_{|w| \leq 1} |(e^w - 1)/w|$ .

Thus, if  $|\theta| \leq \sqrt{\alpha_2 \operatorname{Log} N/N}$ , where  $\alpha_2 < \alpha_1 < 1/2$ , and if  $N$  is big enough so that

$$\sqrt{\alpha_2 \frac{\operatorname{Log} N}{N}} + \frac{1}{8 C_2 \sqrt{\alpha_1 N \operatorname{Log} N}} < \sqrt{\alpha_1 \frac{\operatorname{Log} N}{N}}$$

and

$$C_1 \sqrt{\pi} \frac{(\operatorname{Log} N)^{5/2}}{N^{1-2\alpha_1}} < \frac{1}{8 C_2 \sqrt{\alpha_1 \operatorname{Log} N}} < 2 \sqrt{\alpha_1 \operatorname{Log} N},$$

we obtain that  $Q_N(t)$  and  $\operatorname{erfc}(-\sqrt{N}t)$  have the same number of zeros inside the open disk  $D(\theta, C_1 \sqrt{\pi} (\operatorname{Log} N)^{5/2}/N^{3/2-2\alpha_1})$  (by Rouché's theorem).

In order to conclude, we need some information on the zeros of  $\operatorname{erfc}(z)$ . A theorem by Fettis, Cuslin and Cramer ([3]) gives a development of  $\gamma_k$

$$\begin{aligned} (33) \quad \gamma_k &= e^{3i\pi/4} \left( \sqrt{\left(2k - \frac{1}{4}\right)\pi} \right. \\ &\quad \left. - \frac{i}{2 \sqrt{\left(2k - \frac{1}{4}\right)\pi}} \operatorname{Log} \left( 2 \sqrt{\pi} \sqrt{\left(2k - \frac{1}{4}\right)\pi} \right) \right. \\ &\quad \left. + O\left(\frac{(\operatorname{Log} k)^2}{k \sqrt{k}}\right) \right). \end{aligned}$$

Thus if  $M_0$  is a fixed number in  $(-\pi/4, 3\pi/4)$ , the number of roots  $\gamma$  of  $\operatorname{erfc}(\gamma) = 0$  such that  $\operatorname{Im} \gamma \geq 0$  and  $|\gamma| \leq \sqrt{2k\pi + M_0}$  is exactly  $k$  when  $k$  is large enough.

Now we may prove Result 4. Let  $\varepsilon_0 < 1/2$  and  $K = \lceil \varepsilon_0 \text{Log } N / (2\pi) \rceil$ . For each root  $t$  of  $Q_N(s)$  such that  $|\text{Im } t| \geq 0$  and  $|t| \leq \sqrt{2K\pi/N} \leq \sqrt{\varepsilon_0 \text{Log } N / N}$  there is a root  $\theta$  of  $\text{erfc}(-\sqrt{N}s)$  such that

$$|\theta - t| \leq C_1 \sqrt{\pi} \frac{(\text{Log } N)^{5/2}}{N^{3/2-2\varepsilon_1}},$$

(where  $\varepsilon_0 < \varepsilon_1 < 1/2$  and  $N \geq N_1(\varepsilon_1)$ ). Then we have

$$\begin{aligned} |\sqrt{N}\theta| &\leq \sqrt{2K\pi} + C_1 \sqrt{\pi} \frac{(\text{Log } N)^{5/2}}{N^{1-2\varepsilon_1}} \\ &\leq \sqrt{2K\pi} + \frac{\pi}{16\sqrt{2K\pi}} \\ &\leq \sqrt{\left(2K + \frac{1}{8}\right)\pi} \end{aligned}$$

provided that  $N \geq N_2(\varepsilon_1)$ . But we know that there are exactly  $2K$  roots of  $\text{erfc}(-\sqrt{N}s)$  inside the disk  $D(0, \sqrt{(2K + 1/4)\pi}/\sqrt{N})$ . Conversely, if  $\theta$  is a root of  $\text{erfc}(-\sqrt{N}s)$  such that

$$|\theta| \leq \sqrt{\frac{2K\pi}{N}} - C_1 \sqrt{\pi} \frac{(\text{Log } N)^{5/2}}{N^{3/2-2\varepsilon_1}} \leq \sqrt{\varepsilon_0 \frac{\text{Log } N}{N}},$$

there is a root  $t$  of  $Q_N(s)$  such that

$$|\theta - t| \leq C_1 \sqrt{\pi} \frac{(\text{Log } N)^{5/2}}{N^{3/2-2\varepsilon_1}},$$

hence  $|t| \leq \sqrt{2K\pi/N}$ ; moreover for  $N \geq N_2(\varepsilon_1)$  we have

$$\sqrt{2K\pi} - C_1 \sqrt{\pi} \frac{(\text{Log } N)^{5/2}}{N^{1-2\varepsilon_1}} > \sqrt{2K\pi} - \frac{\pi}{16\sqrt{2K\pi}} > \sqrt{\left(2K - \frac{1}{8}\right)\pi},$$

so that we have again  $2K$  roots of  $\text{erfc}(-\sqrt{N}s)$  such that

$$|\theta| \leq \sqrt{\frac{2K\pi}{N}} - C_1 \sqrt{\pi} \frac{(\text{Log } N)^{5/2}}{N^{1-2\varepsilon_1}}.$$

Finally, we conclude by noticing that (33) shows us that if  $\text{erfc}(-\sqrt{N}\theta_i) = 0$ ,  $i = 1, 2$ ,  $\theta_1 \neq \theta_2$  and  $|\theta_i| \leq \sqrt{(2K + 1/8)\pi/N}$  then  $|\theta_1 - \theta_2| \geq$

$C_0/\sqrt{KN}$  and  $|\text{Im}\theta_i| \geq C_0\sqrt{K/N}$  for some positive  $C_0$  which doesn't depend on  $K$  nor  $N$ ; hence the balls

$$D\left(\theta_i, C_1\sqrt{\pi} \frac{(\text{Log } N)^{5/2}}{N^{3/2-2\varepsilon_1}}\right)$$

are disjoint and don't meet the real axis (for  $N$  large enough). Thus (31) is proved, if we notice that

$$\frac{(\text{Log } N)^{5/2}}{N^{1-2\varepsilon_1}} < \frac{1}{N^{1-2\varepsilon'_1}}$$

for  $\varepsilon_1 < \varepsilon'_1 < 1/2$  and  $N$  large enough.

### 3. Big roots of $Q_N$ : first estimates.

In this section, we are going to devote our attention to formula (26). A straightforward application of (26) is the following one:

**Result 5.** *For  $N$  large enough, if  $t \neq -1$  and  $Q_N(t) = 0$ , then  $|1-t^2| > 1$ .*

PROOF. If  $Q_N(t) = 0$ , then we have  $\sqrt{N} \int_t^1 (1-s^2)^N ds = \sqrt{\pi} (1 + \eta_N)$  with  $\eta_N = O(1/N^2)$ . Now, since  $\text{Re } t > 0$  (due to Result 1), we may write

$$\begin{aligned} \int_t^1 (1-s^2)^N ds &= \int_0^{1-t^2} \omega^N \frac{d\omega}{2\sqrt{1-\omega}} \\ &= (1-t^2)^{N+1} \int_0^1 \lambda^N \frac{d\lambda}{2\sqrt{1-\lambda(1-t^2)}}. \end{aligned}$$

We write  $\Omega = 1 - t^2$ . If  $|\Omega| \leq 1$  then we will prove that

$$\inf_{\lambda \in [0,1]} |1 - \lambda\Omega| \geq \frac{1}{2} |1 - \Omega|.$$

This is obvious if  $\text{Re } \Omega \leq 0$ : we have  $|1 - \lambda\Omega| \geq 1$  and  $|1 - \Omega| \leq 2$ . If  $\text{Re } \Omega > 0$ ,  $\Omega = \rho e^{i\varphi}$  ( $0 < \rho \leq 1$ ,  $\varphi \in (-\pi/2, \pi/2)$ ), we distinguish the case  $\rho \leq \sin \varphi$  and  $\rho > \sin \varphi$ . If  $\rho \leq \sin \varphi$ , it is easily checked

that  $|1 - \lambda\Omega| \geq |1 - \Omega|$ . If  $\rho > \sin \varphi$ , we have  $|1 - \lambda\Omega| \geq \sin \varphi$  and  $|1 - \Omega| \leq |1 - e^{i\varphi}| = 2|\sin(\varphi/2)|$ ; hence

$$|1 - \lambda\Omega| \geq \left| \cos \frac{\varphi}{2} \right| |1 - \Omega| \geq \frac{\sqrt{2}}{2} |1 - \Omega|.$$

Thus, we have for  $\operatorname{Re} t > 0$  and  $|1 - t^2| \leq 1$

$$\left| \int_t^1 (1 - s^2)^N ds \right| \leq \frac{|1 - t^2|^{N+1}}{N+1} \frac{1}{|t|} \leq \frac{1}{\sqrt{N}} \left( \frac{1}{\sqrt{N}|t|} \right).$$

If  $|t\sqrt{N}| \geq 2/\sqrt{\pi}$ , we get

$$\left| \sqrt{N} \int_t^1 (1 - s^2)^N ds \right| \leq \frac{1}{2} \sqrt{\pi},$$

and thus  $Q_N(t) \neq 0$  (for  $N$  large enough so that  $|\eta_N| < 1/2$ ). If  $\sqrt{N}|t| \leq 2/\sqrt{\pi}$ , then  $t \sim -\bar{\gamma}/\sqrt{N}$  for a root  $\gamma$  of  $\operatorname{erfc}(z)$  such that  $|\gamma| \leq 2/\sqrt{\pi}$ ; but the roots of  $\operatorname{erfc}(z)$  satisfy  $\pi/2 < |\operatorname{Arg} \gamma| < 3\pi/4$  so that (for  $N$  large enough)  $|\operatorname{Arg} t| > \pi/4$  and  $t$  cannot lie inside the lemniscate  $|1 - t^2| \leq 1$ .

We may now enter the core of our computations. We are going to give a precise description of  $\int_t^1 (1 - s^2)^N ds$ . Integration by parts gives us

$$\begin{aligned} \int_t^1 (1 - s^2)^N ds &= \frac{(1 - t^2)^{N+1}}{2t(N+1)} - \int_t^1 \frac{(1 - s^2)^{N+1}}{2s^2(N+1)} ds \\ &= \frac{(1 - t^2)^{N+1}}{2t(N+1)} - \frac{(1 - t^2)^{N+2}}{4(N+1)} \int_0^1 \frac{\lambda^{N+1} d\lambda}{(1 - \lambda(1 - t^2))^{3/2}}. \end{aligned}$$

We then define  $\eta(t)$  as

$$(34) \quad \eta(t) = \frac{|t^2|}{\inf_{\lambda \in [0,1]} |1 - \lambda(1 - t^2)|}.$$

We have

$$(35) \quad \int_t^1 (1 - s^2)^N ds = \frac{(1 - t^2)^{N+1}}{2t(N+1)} \left( 1 + \frac{(1 - t^2)}{2(N+2)t^2} \mu_N(t) \right),$$



for  $\operatorname{Re} t > 0$  with

$$(36) \quad |\mu_N(t)| \leq \eta(t)^{3/2}.$$

Of course, (35) is a good formula if  $\mu_N(t)$  cannot explode. As a matter of fact, we will show that in the neighbourhood of the roots of  $Q_N(s)$  we have  $|\eta(t)| \leq C_0$  where  $C_0$  doesn't depend on  $N$  nor  $t$ ; but we are still far from being able to prove it! The only obvious estimations on  $\eta$  are the following ones: if  $\operatorname{Re} t^2 \geq 1$ , we have of course  $|\eta(t)| = |t^2|$ , while if  $\operatorname{Re} t^2 < 1$  and  $|1 - t^2| > 1$  we have

$$|\eta(t)| = \frac{|t^2|}{|\sin(\operatorname{Arg}(1 - t^2))|}.$$

With help of formula (35) and a careful estimate of  $\eta(t)$  in (36), we are going to prove:

**Result 6.** *Let  $\varphi_{N,k} = (8k - 1)\pi/(8N + 6)$ . Then for  $N$  large enough, the roots  $x_{N,1}, \dots, x_{N,N}$  of  $Q_N$  such that  $x_{N,k} \neq -1$ , ordered by*

- for  $1 \leq k \leq [(N + 1)/2]$ ,  $\operatorname{Re} x_{N,k} \geq 0$  and  $x_{N,N+1-k} = \overline{x_{N,k}}$
- $|x_{N,1}| < |x_{N,2}| < \dots < |x_{N,[(N+1)/2]}|$

satisfy

$$(37) \quad \left| x_{N,k} - \sqrt{2 \sin \varphi_{N,k}} e^{i(\pi/4 - \varphi_{N,k})/2} - \frac{e^{i(3\pi/4 - 3\varphi_{N,k}/2)}}{2N \sqrt{2 \sin \varphi_{N,k}}} \operatorname{Log} (2\sqrt{2N\pi \sin \varphi_{N,k}}) \right| \leq C \frac{1}{\sqrt{N}} \max \left\{ \frac{(1 + \operatorname{Log} k)^2}{k^{3/2}}, \frac{(1 + \operatorname{Log} N + 1 - k)^2}{(N + 1 - k)^{3/2}} \right\},$$

where  $C$  doesn't depend on  $k$  nor  $N$ .

PROOF. Since  $\varphi_{N,N+1-k} = \pi - \varphi_{N,k}$ , it is enough to prove (37), for  $1 \leq k \leq [(N + 1)/2]$ , i.e. for the roots which lie in the upper half-plane. The proof is decomposed in the following steps: one first proves that  $\operatorname{Arg}(1 - x_{N,k}^2)$  cannot be too small, so that we have a first control on  $\mu_N(x_{N,k})$ ; then one gives through (35) a first estimate on  $x_{N,k}$  and on the related error; this gives us a more precise information on  $\operatorname{Arg}(1 - x_{N,k}^2)$  and thus we may conclude with our final estimate.

*Step 1.* We want to estimate  $\text{Arg}(1 - x_{N,k}^2)$ . We fix  $\theta_0 \in (\pi/4, \pi/2)$  so that the sector  $\{z : \pi/2 \leq |\text{Arg } z| \leq \pi - \theta_0\}$  contains no zero of  $\text{erfc}(z)$  (remember that  $\lim_{k \rightarrow +\infty} \text{Arg } \gamma_k = 3\pi/4$ ). We now distinguish the cases  $\text{Arg } x_{N,k} \in [0, \theta_0]$  and  $\text{Arg } x_{N,k} \in ]\theta_0, \pi/2[$ . If  $\text{Re } 1 - x_{N,k}^2 \leq 0$ , we know that  $\eta(x_{N,k}) \leq |x_{N,k}|^2 \leq 4$ . If  $\text{Re } 1 - x_{N,k}^2 > 0$  and  $\text{Arg } x_{N,k} \in [0, \pi/4]$ , then we see that  $|x_{N,k}|^2 \leq |\tan \text{Arg}(1 - x_{N,k}^2)|$  (because  $\omega = 1 - x_{N,k}^2$  satisfies  $\text{Re } \omega \in (0, 1]$  and  $|\omega| > 1$  so that  $|\sin \text{Arg } \omega| \leq |1 - \omega| \leq |\tan \text{Arg } \omega|$ ); moreover we have  $|x_{N,k}|^2 \leq 4$ ; thus if  $|\tan(\text{Arg}(1 - x_{N,k}^2))| \leq 4$ , then we have

$$|\sin(\text{Arg}(1 - x_{N,k}^2))| = \frac{|\tan(\text{Arg}(1 - x_{N,k}^2))|}{\sqrt{1 + \tan^2(\text{Arg}(1 - x_{N,k}^2))}} \geq \frac{|x_{N,k}|^2}{\sqrt{17}}$$

and  $\eta(x_{N,k}) \leq \sqrt{17}$ . On the other hand, if  $|\tan(\text{Arg}(1 - x_{N,k}^2))| \geq 4$ , then we have  $|\text{Arg}(1 - x_{N,k}^2)| \in [\text{Arg } \tan 4, \pi/2]$  and thus

$$|\sin(\text{Arg}(1 - x_{N,k}^2))| \geq \sin \text{Arg } \tan 4 = \frac{4}{\sqrt{17}} \geq \frac{|x_{N,k}|^2}{\sqrt{17}}$$

and  $\eta(x_{N,k}) \leq \sqrt{17}$  again.

If  $\text{Arg}(x_{N,k}) \in [\pi/4, \theta_0]$ , we have

$$|\text{Im}(1 - x_{N,k}^2)| = |x_{N,k}^2| |\sin 2 \text{Arg } x_{N,k}|$$

so that

$$|\text{Im}(1 - x_{N,k}^2)| \geq |x_{N,k}|^2 |\sin 2\theta_0|,$$

while

$$|\sin \text{Arg}(1 - x_{N,k}^2)| = \frac{|\text{Im}(1 - x_{N,k}^2)|}{|1 - x_{N,k}^2|} \geq \frac{1}{3} |\text{Im}(1 - x_{N,k}^2)|,$$

so that

$$\eta(x_{N,k}) \leq \frac{3}{|\sin 2\theta_0|}.$$

The difficult case is when  $\theta_0 \leq \text{Arg } x_{N,k} \leq \pi/2$  (as a matter of fact, we will see in step 3 that this case never occurs when  $N$  is big enough!). For the moment, we will show that we have necessarily for such an  $x_{N,k}$  (and provided  $N$  is large enough) the inequality

$$N |x_{N,k}|^4 \geq \frac{|\cos \theta_0|}{100 C_0^2} = \varepsilon_1,$$

where  $C_0$  is given by

$$C_0 = \max \left\{ \sup_{|\sigma| \leq 1/2} \left| \frac{\sigma^2 + \text{Log}(1 - \sigma^2)}{\sigma^4} \right|, \sup_{|\sigma| \leq 1} \frac{|e^\sigma - 1|}{|\sigma|} \right\}.$$

Indeed, let  $A_0 > 0$  be large enough so that for  $A \geq A_0$ ,  $e^{3A^2 \cos(2\theta_0)/4} (1 + A^2/2) < 1/100$  (remember that  $\cos 2\theta_0 < 0$ ),  $4/(A^2 |\cos 2\theta_0|) < 1/100$  and  $A e^{A^2 \cos(2\theta_0)/4} < 1/100$ . If  $\sqrt{N} |x_{N,k}| \geq A_0$  and  $N |x_{N,k}|^4 \leq \varepsilon_1$ , we write

$$Q_N(x_{N,k}) = \frac{1}{2} + \left( 1 + O\left(\frac{1}{N^2}\right) \right) \sqrt{\frac{N}{\pi}} \int_0^{x_{N,k}} (1 - s^2)^N ds$$

and thus

$$|Q_N(x_{N,k})| \geq \frac{1}{10} \sqrt{N} \left| \int_0^{x_{N,k}} (1 - s^2)^N ds \right| - \frac{1}{2}.$$

We write

$$(1 - s^2)^N = e^{-Ns^2} e^{N(s^2 - \text{Log}(1 - s^2))},$$

since  $|s| \leq \sqrt{\varepsilon_1/N}/4$ , we have  $|s| \leq 1/2$  for  $N$  large enough, thus

$$|N(s^2 - \text{Log}(1 - s^2))| \leq C_0 |N s^4| \leq \frac{1}{100},$$

thus

$$|e^{N(s^2 - \text{Log}(1 - s^2))} - 1| \leq C_0^2 |N s^4|.$$

Thus, writing  $x_{N,k} = \rho_{N,k} e^{i\theta_{N,k}}$ , we get

$$\begin{aligned} |Q_N(x_{N,k})| &\geq \frac{1}{10} \left| \int_0^{\sqrt{N}x_{N,k}} e^{-s^2} ds \right| \\ &\quad - \frac{C_0^2}{10} \int_0^{\sqrt{N}\rho_{N,k}} e^{-s^2 \cos 2\theta_{N,k}} \frac{s^4}{N} ds - \frac{1}{2} \\ &\geq \frac{1}{10} \left| \int_0^{\sqrt{N}x_{N,k}} e^{-s^2} ds \right| \\ &\quad - \frac{C_0^2}{10} \frac{(\sqrt{N}\rho_{N,k})^3}{N |\cos 2\theta_{N,k}|} \end{aligned}$$

$$\begin{aligned}
& \cdot \int_0^{\sqrt{N}\rho_{N,k}} e^{-s^2 \cos 2\theta_{N,k}} s |\cos 2\theta_{N,k}| ds - \frac{1}{2} \\
& \geq \frac{1}{10} \left| \int_0^{\sqrt{N}x_{N,k}} e^{-s^2} ds \right| \\
& \quad - \frac{e^{-N\rho_{N,k}^2 \cos 2\theta_{N,k}}}{2\sqrt{N}\rho_{N,k}} \left( \frac{C_0^2 (\sqrt{N}\rho_{N,k})^4}{10 |\cos 2\theta_0|} \right) - \frac{1}{2}.
\end{aligned}$$

We have now to estimate  $\int_0^{\sqrt{N}x_{N,k}} e^{-s^2} ds$ . We write

$$\begin{aligned}
& \int_0^{\sqrt{N}x_{N,k}} e^{-s^2} ds \\
& = e^{i\theta_{N,k}} \left( \int_0^{\sqrt{N}\rho_{N,k}/2} e^{-s^2} e^{2i\theta_{N,k}} ds + \int_{\sqrt{N}\rho_{N,k}/2}^{\sqrt{N}\rho_{N,k}} e^{-s^2} e^{2i\theta_{N,k}} ds \right) \\
& = e^{i\theta_{N,k}} (I_1 + I_2).
\end{aligned}$$

We have  $|I_1| \leq e^{-N\rho_{N,k}^2 \cos(2\theta_{N,k})/4} \rho_{N,k} \sqrt{N}/2$ , while

$$\begin{aligned}
I_2 & = \left[ \frac{e^{-s^2} e^{2i\theta_{N,k}}}{-2s e^{2i\theta_{N,k}}} \right]_{\sqrt{N}\rho_{N,k}/2}^{\sqrt{N}\rho_{N,k}} - \int_{\sqrt{N}\rho_{N,k}/2}^{\sqrt{N}\rho_{N,k}} \frac{e^{-s^2} e^{2i\theta_{N,k}}}{2s^2 e^{2i\theta_{N,k}}} ds \\
& = \frac{e^{-N\rho_{N,k}^2} e^{2i\theta_{N,k}}}{-2\sqrt{N}\rho_{N,k} e^{2i\theta_{N,k}}} - \frac{e^{-N\rho_{N,k}^2} e^{2i\theta_{N,k}/4}}{-\sqrt{N}\rho_{N,k} e^{2i\theta_{N,k}}} - I_3.
\end{aligned}$$

We have

$$\begin{aligned}
|I_3| & \leq \frac{1}{4 \left( \frac{1}{2} \sqrt{N} \rho_{N,k} \right)^3 |\cos 2\theta_{N,k}|} \\
& \quad \cdot \int_{\sqrt{N}\rho_{N,k}/2}^{\sqrt{N}\rho_{N,k}} e^{-s^2 \cos 2\theta_{N,k}} 2s |\cos 2\theta_{N,k}| ds \\
& \leq \frac{e^{-N\rho_{N,k}^2 \cos 2\theta_{N,k}}}{4 \left( \frac{1}{2} \sqrt{N} \rho_{N,k} \right)^3 |\cos 2\theta_0|}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
 |Q_N(x_{N,k})| &\geq \frac{1}{10} \frac{e^{-N\rho_{N,k}^2 \cos 2\theta_{N,k}}}{2\sqrt{N} \rho_{N,k}} \\
 &\quad \cdot \left( 1 - 2 e^{3N\rho_{N,k}^2 \cos 2\theta_{N,k}/4} - \frac{4}{N \rho_{N,k}^2 |\cos 2\theta_0|} \right. \\
 &\quad \left. - N \rho_{N,k}^2 e^{3N\rho_{N,k}^2 \cos 2\theta_{N,k}/4} - \frac{C_0^2 \varepsilon_1}{|\cos 2\theta_0|} \right. \\
 &\quad \left. - 10 \sqrt{N} \rho_{N,k} e^{N\rho_{N,k}^2 \cos 2\theta_{N,k}} \right) \\
 &\geq \frac{1}{10} \frac{e^{-N\rho_{N,k}^2 \cos 2\theta_{N,k}}}{2\sqrt{N} \rho_{N,k}} \left( 1 - \frac{2}{100} - \frac{1}{100} - \frac{1}{100} - \frac{1}{100} - \frac{10}{100} \right) > 0,
 \end{aligned}$$

which contradicts  $Q_N(x_{N,k}) = 0$ . Up to now, we have proved that if  $\arg x_{N,k} > \theta_0$  then either  $\sqrt{N}|x_{N,k}| \leq A_0$  or  $N|x_{N,k}|^4 \geq \varepsilon_1$ . But if  $|x_{N,k}| \leq A_0/\sqrt{N}$  and  $N$  is large enough, Result 4 ensures that  $-\sqrt{N}x_{N,k}$  is close to a zero of  $\operatorname{erfc}(z)$ . This is not possible for  $N$  large enough since the distance between  $\{z : \pi/2 \leq |\operatorname{Arg} z| \leq \pi - \theta_0\}$  and  $\{z : \operatorname{erfc}(z) = 0\}$  is positive.

Thus we must have  $N|x_{N,k}|^4 \geq \varepsilon_1$ . Write again  $x_{N,k} = \rho_{N,k} e^{i\theta_{N,k}}$ ; since  $|x_{N,k} - 1| \leq 1$  by Result 1, we have  $\rho_{N,k} \leq 2 \cos \theta_{N,k}$ ; thus  $2 \cos \theta_{N,k} \geq (\varepsilon_1/N)^{1/4}$  and

$$|\operatorname{Im} x_{N,k}^2| = |x_{N,k}^2| |\sin 2\theta_{N,k}| \geq \sin \theta_0 \left(\frac{\varepsilon_1}{N}\right)^{1/4} |x_{N,k}|^2.$$

We thus have proved

$$\eta(x_{N,k}) = \frac{|x_{N,k}|^2 |1 - x_{N,k}^2|}{|\operatorname{Im} x_{N,k}^2|} \leq \frac{3 N^{1/4}}{(\sin \theta_0) \varepsilon_1^{1/4}} = C_1^{N^{1/4}}.$$

We thus have proved

- if  $\operatorname{Arg} x_{N,k} < \theta_0$ ,

$$|\mu_N(x_{N,k})| \leq \eta(x_{N,k})^{3/2} \leq C_2,$$

- if  $\text{Arg } x_{N,k} > \theta_0$ ,

$$\begin{aligned} |\mu_N(x_{N,k})| &\leq \eta(x_{N,k})^{3/2} \\ &\leq (C_1 N^{1/4})^{3/2} \\ &= C_1^{3/2} \frac{(N |x_{N,k}^2|)^{3/4}}{(N (|x_{N,k}|^4)^{3/8}} \\ &\leq \frac{C_1^{3/2}}{\varepsilon_1^{3/8}} (N |x_{N,k}|^2)^{3/4}. \end{aligned}$$

In any case, we have

$$(38) \quad |\mu_N(x_{N,k})| \leq C (N |x_{N,k}|^2)^{3/4}.$$

(Remember that  $\lim_{N \rightarrow +\infty} \inf_k N |x_{N,k}|^2 = |\gamma_1|^2 > 0$ ).

*Step 2.* We are now able to give an estimate for  $x_{N,k}$ . Let us consider a root  $y \neq -1$  of  $Q_N$  such that  $\text{Im } y \geq 0$ . We have

$$\int_y^1 (1 - s^2)^N ds = 2 \frac{4^N (N!)^2}{(2N + 1)!},$$

hence from (35) and (36),

$$(39) \quad \frac{(1 - y^2)^{N+1}}{2(N + 1)\sqrt{\pi}y} \left(1 + O\left(\frac{\eta(y)^{3/2}}{N|y|^2}\right)\right) = \sqrt{\frac{\pi}{N}} \left(1 + O\left(\frac{1}{N^2}\right)\right),$$

(where  $\alpha = O(\varepsilon(N, y))$  means that  $|\alpha|/\varepsilon(N, y) \leq C$  for a positive constant  $C$  which doesn't depend neither on  $N$  nor on  $y$ ). Taking the  $(N + 1)$ -th root of the modulus of both terms of equality (39), we get

$$\begin{aligned} |1 - y^2| &= 1 + \frac{1}{N + 1} \text{Log}\left(2\sqrt{N\pi} \frac{N + 1}{N} |y|\right) \\ &\quad + O\left(\frac{(\text{Log } N)^2}{N^2}\right) + O\left(\frac{1}{N^3}\right) + O\left(\frac{\eta(y)^{3/2}}{N^2|y|^2}\right) \\ &= 1 + \frac{1}{N} \text{Log}(2\sqrt{N\pi} |y|) + O\left(\frac{(\text{Log } N)^2}{N^2}\right) + O\left(\frac{\eta(y)^{3/2}}{N^2|y|^2}\right). \end{aligned}$$

Now, we write  $1 - y^2 = \rho e^{-i\varphi}$  ( $\varphi \in [0, \pi]$ ,  $\rho > 0$ ), so that  $y = \sqrt{1 - \rho e^{-i\varphi}}$ . We have found

$$\begin{aligned} |1 - \rho| &= O\left(\frac{1}{N} \text{Log}(\sqrt{N}|y|)\right) + O\left(\frac{(\text{Log } N)^2}{N^2}\right) + O\left(\frac{\eta(y)^{3/2}}{N^2|y|^2}\right) \\ &= O\left(\frac{1}{N} \text{Log}(\sqrt{N}|y|)\right), \end{aligned}$$

(since  $1/CN \leq \text{Log}(\sqrt{N}|y|)/N \leq C \text{Log } N/N$ , while  $\eta(y)^{3/2}/(N^2|y|^2) \leq C/(N(N|y|^2)) \leq C'/N$ ). Thus  $1 - \rho e^{-i\varphi} = 1 - e^{-i\varphi} + (1 - \rho) e^{-i\varphi}$  with

$$\left| \frac{(1 - \rho) e^{-i\varphi}}{1 - \rho e^{-i\varphi}} \right| = O\left(\frac{\text{Log}(\sqrt{N}|y|)}{N|y|^2}\right)$$

and we find

$$\begin{aligned} y &= \sqrt{(1 - e^{-i\varphi}) \left(1 + O\left(\frac{\text{Log } \sqrt{N}|y|}{N|y|^2}\right)\right)} \\ &= \sqrt{2 \sin\left(\frac{\varphi}{2}\right)} e^{i(\pi/4 - \varphi/4)} \left(1 + O\left(\frac{\text{Log } \sqrt{N}|y|}{N|y|^2}\right)\right). \end{aligned}$$

We insert this result in (39) and take the phase

$$-(N + 1)\varphi - \frac{\pi}{4} + \frac{\varphi}{4} + O\left(\frac{\text{Log } \sqrt{N}|y|}{N|y|^2}\right) + O\left(\frac{\eta(y)^{3/2}}{N|y|^2}\right) = -2k\pi$$

or

$$(40) \quad \varphi = \frac{8k - 1}{4N + 3} \pi + O\left(\frac{\text{Log } \sqrt{N}|y|}{N^2|y|^2}\right) + O\left(\frac{\eta(y)^{3/2}}{N^2|y|^2}\right).$$

If we assume  $\sqrt{N}|y| \geq A_0$  where  $A_0$  is big enough so that

$$O\left(\frac{\text{Log } A_0}{NA_0^2}\right) + O\left(\frac{1}{NA_0^{1/2}}\right)$$

is less than  $4\pi/(4N + 3)$  ( $A_0$  being chosen independently from  $N$ ), we see that  $0 \leq \varphi \leq \pi$  implies  $0 \leq k \leq [(N + 1)/2]$ ; moreover since

$$|y| = \sqrt{2 \sin\left(\frac{\varphi}{2}\right)} \left(1 + O\left(\frac{\text{Log } \sqrt{N}|y|}{N|y|^2}\right)\right)$$

we must have

$$2 \sin\left(\frac{\varphi}{2}\right) \geq \frac{A_0^2}{N} + O\left(\frac{\text{Log } \sqrt{N} |y|}{N^2 |y|^2}\right).$$

We take  $A_0^2 = \sqrt{2K_0\pi}$ , where  $K_0$  is big enough; we then see that we must have  $k > K_0$ .

If  $\sqrt{N} |y| \leq \sqrt{2K_0\pi}$ , we know that (provided  $N$  is big enough)  $y \sim -\bar{\gamma}_k/\sqrt{N}$  for  $k \in \{1, \dots, K_0\}$ . We have moreover found candidates  $y_{N,k}$  for the remaining roots  $x_{N,k}$ ,  $K_0 < k \leq [(N + 1)/2]$ , which are given by

$$(41) \quad 1 - y_{N,k}^2 = \left(1 + \frac{1}{N} \text{Log } 2 \sqrt{2N\pi \sin \varphi_{N,k}}\right) e^{-2i\varphi_{N,k}},$$

for  $K_0 < k \leq [(N + 1)/2]$  and  $\varphi_{N,k} = (8k - 1)\pi/(8N + 6)$ .

More precisely, we have shown that if  $Q_N(y) = 0$ ,  $\text{Im } y \geq 0$ ,  $y \neq -1$  and  $\sqrt{N} |y| \geq \sqrt{2K_0\pi}$ , then for some  $k \in \{K_0 + 1, \dots, [(N + 1)/2]\}$  we have

$$(42) \quad 1 - y^2 = 1 - y_{N,k}^2 + O\left(\frac{(\text{Log } N)^2}{N^2}\right) + O\left(\frac{\eta(y)^{3/2}}{N^2 |y|^2}\right) + O\left(\frac{\text{Log } \sqrt{N} |y|}{N^2 |y|^2}\right).$$

We are going now to prove that, provided that  $K_0$  is fixed large enough (and provided thereafter that  $N$  is large enough), for each  $y_{N,k}$  there is exactly one root  $y$  satisfying (42). Notice that  $|y_{N,k}^2 - y_{N,k+1}^2| \geq C_0/N$  while

$$\begin{aligned} O\left(\frac{(\text{Log } N)^2}{N^2}\right) + O\left(\frac{\eta(y)^{3/2}}{N^2 |y|^2}\right) + O\left(\frac{\text{Log } \sqrt{N} |y|}{N^2 |y|^2}\right) \\ \leq C \frac{1}{N} \left(\frac{(\text{Log } N)^2}{N} + \frac{1}{(\sqrt{N} |y|)^{1/2}}\right). \end{aligned}$$

Indeed, let's write  $s = \sqrt{y_{N,k}^2 - v}$  where  $|v| = \eta_0/N$ ,  $\eta_0$  small enough.

We are going to estimate  $Q_N(s)$ . We know that

$$\int_s^1 (1 - \sigma^2)^N d\sigma = \frac{(1 - s^2)^{N+1}}{2s(N + 1)} \left(1 + O\left(\frac{\eta(s)}{N|s|^2}\right)\right),$$

where  $\eta(s)$  is bounded independently of  $s$  provided that  $|1 - s| < 1$ ,  $|1 - s^2| > 1$  and  $|\text{Arg } s| < \theta_0$  (where  $\theta_0 \in (\pi/4, \pi/2)$ ). Thus, we are



going to estimate  $|1 - s|$ ,  $|1 - s^2|$  and  $|\text{Arg } s|$ . We have obviously from (41)

$$y_{N,k}^2 = 1 - e^{-2i\varphi_{N,k}} + O\left(\frac{\text{Log } k}{N}\right) = (1 - e^{-2i\varphi_{N,k}})\left(1 + O\left(\frac{\text{Log } k}{k}\right)\right)$$

and such an estimate holds as well for  $s^2$ . (We see also from (41) that

$$\begin{aligned} |1 - s^2| &\geq 1 + \frac{1}{N} \text{Log } 2 \sqrt{2N\pi \sin \varphi_{N,k}} - \frac{\eta_0}{N} \\ &\geq 1 + \frac{1}{N} \text{Log } 2 \sqrt{4\pi K_0} - \frac{\eta_0}{N} \\ &> 1 \end{aligned}$$

provided  $\eta_0$  is small enough). Thus we find that

$$\text{Arg } s^2 = \frac{\pi}{2} - \varphi_{N,k} + O\left(\frac{\text{Log } k}{k}\right) < 2\theta_0,$$

if  $K_0$  is large enough (so that  $O(\text{Log } K_0/K_0) < 2\theta_0 - \pi/2$ ) and thus

$$\text{Arg } s = \frac{\pi}{4} - \frac{1}{2} \varphi_{N,k} + O\left(\frac{\text{Log } k}{k}\right) \in (-\theta_0, \theta_0).$$

Moreover,

$$|s| = \sqrt{2 \sin \varphi_{N,k}} \left(1 + O\left(\frac{\text{Log } k}{k}\right)\right)$$

and this latter estimate gives  $|s| < 2 \cos(\text{Arg } s)$ : if  $\varphi_{N,k} > \varepsilon_0$  (where  $\varepsilon_0$  is fixed small enough as we shall see below) and  $K_0$  and  $N$  are large enough we have

$$\sqrt{2 \sin \varphi_{N,k}} \left(1 + O\left(\frac{\text{Log } k}{k}\right)\right) \leq \sqrt{2} \left(1 + C \frac{\text{Log } K_0}{K_0}\right) \leq \sqrt{2} \left(1 + \frac{\varepsilon_0}{100}\right),$$

while

$$\begin{aligned} 2 \cos(\text{Arg } s) &\geq 2 \cos\left(\frac{\pi}{4} - C \frac{\text{Log } K_0}{K_0}\right) \\ &\geq 2 \cos\left(\frac{\pi}{4} - \frac{\varepsilon_0}{3}\right) \\ &\geq \sqrt{2} \left(1 + \frac{2\varepsilon_0}{3\pi} - \frac{\varepsilon_0^2}{2}\right). \end{aligned}$$

On the other hand, if  $\varphi_{N,k} < \varepsilon_0$  we find

$$\sqrt{2 \sin \varphi_{N,k}} \left(1 + O\left(\frac{\text{Log } k}{k}\right)\right) \leq \sqrt{2 \varepsilon_0} \sqrt{1 + C \frac{\text{Log } K_0}{K_0}} \leq C' \sqrt{\varepsilon_0},$$

while  $2 \cos(\text{Arg } s) \geq 2 \cos \theta_0$ ; thus if  $\varepsilon_0$  is small enough to ensure  $\varepsilon_0 < 4/(3\pi) - 1/50$  and  $\varepsilon_0 < 4 \cos^2 \theta_0 / C'^2$  we find  $|s| < 2 \cos(\text{Arg } s)$ . But this latter inequality is equivalent to  $|1 - s| < 1$ . Thus we found

$$Q_N(s) = 1 - \left(1 + O\left(\frac{1}{N^2}\right)\right) \sqrt{\frac{N}{\pi}} \frac{(1 - s^2)^{N+1}}{2s(N+1)} \left(1 + O\left(\frac{1}{|Ns^2|}\right)\right).$$

We have moreover:

$$\begin{aligned} (1 - s^2)^{N+1} &= (1 - y_{N,k}^2)^{N+1} \left(1 + \frac{v}{1 - y_{N,k}^2}\right)^{N+1} \\ &= (1 - y_{N,k}^2)^{N+1} \left(1 + \frac{Nv}{1 - y_{N,k}^2} + O(N^2 v^2)\right) \\ s &= \sqrt{y_{N,k}^2 - v} = y_{N,k} \left(1 - \frac{v}{2y_{N,k}^2} + O\left(\frac{v^2}{y_{N,k}^4}\right)\right). \end{aligned}$$

This gives, since  $|s|$  has  $\sqrt{k/N}$  as order of magnitude

$$\begin{aligned} Q_N(s) &= 1 - \left(1 + O\left(\frac{1}{k}\right)\right) \frac{(1 - y_{N,k}^2)^{N+1}}{2\sqrt{N}\pi y_{N,k}} \\ &\quad \cdot \left(1 + \frac{Nv}{1 - y_{N,k}^2} + \frac{v}{2y_{N,k}^2} + O(N^2 v^2) + O\left(\frac{v^2}{y_{N,k}^4}\right)\right). \end{aligned}$$

Moreover

$$|y_{N,k}| \geq 2 \sqrt{\frac{8k-1}{8N+6}} \left(1 + O\left(\frac{1}{k} \text{Log } k\right)\right)$$

and

$$y_{N,k} = \sqrt{2 \sin\left(\frac{8k-1}{8N+6} \pi\right)} e^{i(\pi/4 - (8k-1)\pi/(16N+12))} \left(1 + O\left(\frac{1}{k} \text{Log } k\right)\right),$$

so that

$$\begin{aligned} & \frac{(1 - y_{N,k}^2)^{N+1}}{2\sqrt{N\pi} y_{N,k}} \\ &= \frac{\left(1 + \frac{1}{N} \operatorname{Log} 2 \sqrt{2N\pi \sin\left(\frac{8k-1}{8N+6}\pi\right)}\right)^N}{2\sqrt{2N\pi \sin\left(\frac{8k-1}{8N+6}\pi\right)}} \left(1 + O\left(\frac{1}{k} \operatorname{Log} k\right)\right) \\ &= \left(1 + O\left(\frac{1}{N^2} (\operatorname{Log} k)^2\right)\right)^N \left(1 + O\left(\frac{1}{k} \operatorname{Log} k\right)\right) \end{aligned}$$

and finally

$$\begin{aligned} Q_N(s) &= 1 - \left(1 + O\left(\frac{1}{k} (\operatorname{Log} k)^2\right)\right) \\ &\quad \cdot \left(1 + \frac{Nv}{1 - y_{N,k}^2} + \frac{v}{2y_{N,k}^2} + O(N^2v^2) + O\left(\frac{v^2}{y_{N,k}^4}\right)\right). \end{aligned}$$

Now, we write

$$R_{N,k}(s) = N \frac{v}{1 - y_{N,k}^2} = N \frac{y_{N,k}^2 - s^2}{1 - y_{N,k}^2}.$$

Since  $|v| = \eta_0/N$ , we have

$$|R_{N,k}(s)| = \eta_0 \left(1 + O\left(\frac{\operatorname{Log} k}{N}\right)\right),$$

while

$$|Q_N(s) - R_{N,k}(s)| = O\left(\frac{(\operatorname{Log} k)^2}{k}\right) + O\left(\frac{\eta_0}{k}\right) + O(\eta_0^2).$$

We choose  $\eta_0$  small enough to ensure that the  $O(\eta_0^2)$  term is smaller than  $\eta_0/2$  (independently of  $N$  and  $k$ ), and then choose  $K_0$  large enough to ensure that  $O((\operatorname{Log} k)^2/k) + O(\eta_0/k)$  is smaller than  $\eta_0/4$  for  $k > K_0$ . For this choice of  $K_0$ , we get

$$|Q_N(s) - R_{N,k}(s)| < \frac{3}{4} \eta_0 < |R_{N,k}(s)|.$$

Thus, by Rouché’s theorem,  $Q_N(s)$  and  $R_{N,k}(s)$  have the same number of roots inside the domain  $\{|y_{N,k}^2 - s^2| \leq \eta_0/N, \operatorname{Re} s > 0\}$ .

*Step 3.* We have thus found a number  $K_0$  so that for  $N$  large enough we may list the roots  $x_{N,1}, \dots, x_{N,[(N+1)/2]}$  of  $Q_N$  with  $x_{N,k} \neq -1$ ,  $\operatorname{Im} x_{N,k} \geq 0$ ,  $|x_{N,k}| < |x_{N,k+1}|$  in the following way:

- for  $k \leq K_0$ ,  $|x_{N,k}| < \sqrt{2K_0\pi/N}$  and  $x_{N,k} \sim -\bar{\gamma}_k/\sqrt{N}$ ,
- for  $k \geq K_0$ ,

$$|x_{N,k}^2 - y_{N,k}^2| = O\left(\frac{\eta(x_{N,k})^{3/2}}{N^2|x_{N,k}|^2}\right) + O\left(\frac{\operatorname{Log}(\sqrt{N}|x_{N,k}|)}{N^2|x_{N,k}|^2}\right),$$

where  $y_{N,k}$  is given by (41).

Moreover, we have seen in step 2 that in that case we must have  $\operatorname{Arg} x_{N,k} < \theta_0$ , hence  $\eta(x_{N,k})$  is bounded independently of  $N$  and  $k$ . Moreover  $x_{N,k}$  is of order of magnitude  $\sqrt{k/N}$ , hence

$$|x_{N,k}^2 - y_{N,k}^2| = O\left(\frac{\operatorname{Log} k}{Nk}\right).$$

Thus we find

$$(43) \quad \begin{aligned} 1 - x_{N,k}^2 &= \left(1 + \frac{1}{N} \operatorname{Log} 2 \sqrt{2N\pi \sin\left(\frac{8k-1}{8N+6}\pi\right)}\right) \\ &\cdot e^{-2i\pi(8k-1)/(8N+6)} + O\left(\frac{\operatorname{Log} k}{Nk}\right) \end{aligned}$$

and thus

$$\begin{aligned} x_{N,k}^2 &= \left(1 - e^{-2i\pi(8k-1)/(8N+6)}\right) \\ &\cdot \left(1 - \frac{e^{-2i\pi(8k-1)/(8N+6)}}{N(1 - e^{-2i\pi(8k-1)/(8N+6)})} \operatorname{Log} 2 \sqrt{2N\pi \sin\left(\frac{8k-1}{8N+6}\pi\right)}\right) \\ &+ O\left(\frac{\operatorname{Log} k}{k^2}\right) \end{aligned}$$

which gives

$$\begin{aligned}
 (44) \quad x_{N,k} &= e^{i(\pi/4 - (8k-1)\pi/(16N+12))} \sqrt{2 \sin\left(\frac{8k-1}{8N+6} \pi\right)} \\
 &\cdot \left( 1 + \frac{e^{i(\pi/2 - (8k-1)\pi/(8N+6))}}{4N \sin\left(\frac{8k-1}{8N+6} \pi\right)} \operatorname{Log} 2 \sqrt{2N\pi \sin\left(\frac{8k-1}{8N+6} \pi\right)} \right. \\
 &\quad \left. + O\left(\frac{\operatorname{Log} k}{k^2}\right) \right),
 \end{aligned}$$

which gives (37) for  $k > K_0$ . For  $k \leq K_0$ , (37) says only that  $x_{N,k}$  is  $O(1/\sqrt{N})$ , which we already know since  $\sqrt{N} |x_{N,k}| \leq \sqrt{2K_0\pi}$ .

Thus we have proved Result 6.

A nice corollary of Result 6 is that we may recover formula (33) on the roots of  $\operatorname{erfc}(z)$ :

**Corollary.** *The  $k$ -th root  $\gamma_k$  of  $\operatorname{erfc}(z)$  such that  $\operatorname{Im} \gamma_k > 0$  is given by*

$$\begin{aligned}
 (45) \quad \gamma_k &= e^{3i\pi/4} \sqrt{\left(2k - \frac{1}{4}\right)\pi} \\
 &\cdot \left( 1 - \frac{i}{2\left(2k - \frac{1}{4}\right)\pi} \operatorname{Log} 2 \sqrt{\pi} \sqrt{\left(2k - \frac{1}{4}\right)\pi} + O\left(\frac{(\operatorname{Log} k)^2}{k^2}\right) \right).
 \end{aligned}$$

PROOF. It is enough to use formula (37) for  $x_{N,k}$  with  $N, k \rightarrow +\infty$  and  $k < \operatorname{Log} N/8$ : we have

$$x_{N,k} = -\frac{-\bar{\gamma}_k}{\sqrt{N}} + O\left(\frac{1}{N}\right) \quad \text{and} \quad \frac{k}{N} = O\left(\frac{\operatorname{Log} N}{N}\right),$$

thus we find  $\gamma_k$ . The only thing to check is the exact number of roots  $\gamma$  such that  $|\gamma| \leq \sqrt{2K_0\pi}$  (since we used formula (33) to give it). But this is an old and classical result of Nevanlinna [9], and thus we may recover formula (33) from formula (37).

**4. Big roots of  $Q_N$ : further estimates.**

Though Result 6 is enough for the proof of theorems 1 to 3 (provided we improve result n° 4 for the smaller roots), we may give even more precise estimations for the roots  $x_{N,k}$ . For instance, we may integrate by parts one step further formula (35) and thus get an  $O((\text{Log } k)^3/Nk^2)$  error instead of  $O(\text{Log } k/Nk)$  for  $1 - x_{N,k}^2$ .

More generally, how far can we compute  $\int_t^1 (1 - s^2)^N ds$ ? We have

$$\int_t^1 (1 - s^2)^N ds = (1 - t^2)^{N+1} \int_0^1 \lambda^N \frac{d\lambda}{2\sqrt{1 - \lambda(1 - t^2)}}.$$

If we write

$$1 - \lambda(1 - t^2) = t^2 \left( 1 + \frac{1 - t^2}{t^2} (1 - \lambda) \right),$$

we see that if  $\text{Re } t^2 > 1/2$  (so that  $|1 - t^2| < t^2$ ), we may develop  $(\sqrt{1 - \lambda(1 - t^2)})^{-1}$  as a Taylor series in  $(1 - \lambda)$  and find (for  $\text{Re } t^2 > 1/2$ )

$$\frac{1}{\sqrt{1 - \lambda(1 - t^2)}} = \frac{1}{t} \sum_{k=0}^{+\infty} (-1)^k \frac{2k!}{4^k (k!)^2} \left( \frac{(1 - \lambda)(1 - t^2)}{t^2} \right)^k,$$

which gives

$$(46) \left\{ \begin{array}{l} \text{for } \text{Re } t > 0 \text{ and } \text{Re } t^2 > \frac{1}{2}, \\ \int_t^1 (1 - s^2)^N ds \\ = \frac{(1 - t^2)^{N+1}}{2t} \sum_{k=0}^{+\infty} (-1)^k \frac{(2k)!}{4^k (k!)^2} \frac{N! k!}{(N + k + 1)!} \left( \frac{1 - t^2}{t^2} \right)^k. \end{array} \right.$$

Unfortunately, we are mostly interested in small  $t$ 's (remember that  $x_{N,k} = O(\sqrt{k/N})$ ). (46) has to be replaced by an asymptotic formula (which is obtained by repeatedly integrating by parts)

$$(47) \left\{ \begin{array}{l} \text{for } \text{Re } t > 0 \text{ and } M \in \mathbb{N}, \\ \int_t^1 (1 - s^2)^N ds \\ = \frac{(1 - t^2)^{N+1}}{2t} \sum_{k=0}^M (-1)^k \frac{(2k)!}{4^k (k!)^2} \frac{N! k!}{(N + k + 1)!} \left( \frac{1 - t^2}{t^2} \right)^k \\ + R_{M,N}(t), \end{array} \right.$$

where the remainder

$$R_{M,N}(t) = (-1)^{M+1}(1 - t^2)^{N+M+2} \frac{(2M + 2)!}{4^{M+1}((M + 1)!)^2} \cdot \frac{N!(M + 1)!}{(N + M + 2)!} \int_0^1 \frac{\lambda^{N+M+1} d\lambda}{(1 - \lambda(1 - t^2))^{1/2+M+1}}$$

may be estimated by

$$(48) \quad |R_{M,N}(t)| \leq \left| \frac{(1 - t^2)^{N+1}}{2t} \right| \frac{(2M + 2)!}{4^{M+1}((M + 1)!)^2} \frac{(M + 1)! N!}{(N + M + 2)!} \cdot \left| \frac{1 - t^2}{t^2} \right|^{M+1} \eta(t)^{1/2+M+1}.$$

$M = 0$  gave Result 6.  $M = 1$  gives the following result:

**Result 7.** *Writing  $\varphi_{N,k} = (8k - 1)\pi/(8N + 6)$  and*

$$\lambda_k = \text{Log } 2 \sqrt{2N\pi \sin \varphi_{N,k}},$$

*we have more precisely for all  $k \in \{1, \dots, N\}$*

$$(49) \quad 1 - x_{N,k}^2 = e^{-2i\varphi_{N,k}} \cdot \left( 1 + \frac{1}{N} \lambda_k + \frac{1}{N^2} + \frac{\lambda_k}{N^2} + \frac{\lambda_k^2}{2N^2} + \frac{i e^{-i\varphi_{N,k}}}{4N^2 \sin \varphi_{N,k}} (\lambda_k - 1) \right) + \varepsilon_{N,k},$$

where

$$|\varepsilon_{N,k}| \leq C \max \left\{ \frac{1 + (\text{Log } k)^3}{Nk^2}, \frac{1 + \text{Log } (N + 1 - k)^3}{N(N + 1 - k)^2} \right\}$$

and  $C$  doesn't depend neither on  $N$  nor on  $K$ .

PROOF. We assume  $k \leq [(N + 1)/2]$ . We write  $1 - x_{N,k}^2 = 1 - y_{N,k}^2 + v$  and the problem is to estimate  $v$ . We already know  $v = O(\text{Log } k/(Nk))$ . Furthermore, we know that

$$\int_{x_{N,k}}^1 (1 - s^2)^N ds = \frac{2 \cdot 4^N (N!)^2}{(2N + 1)!} = \sqrt{\frac{\pi}{N}} \left( 1 + O\left(\frac{1}{N^2}\right) \right)$$

and

$$\int_{x_{N,k}}^1 (1-s^2)^N ds = \frac{(1-x_{N,k}^2)^{N+1}}{2(N+1)x_{N,k}} \left( 1 - \frac{1-x_{N,k}^2}{2(N+2)x_{N,k}^2} + O\left(\frac{1}{N^2x_{N,k}^4}\right) \right).$$

Now, write

$$\begin{aligned} \frac{1-x_{N,k}^2}{2(N+2)x_{N,k}^2} &= \frac{1-y_{N,k}^2}{2(N+2)y_{N,k}^2} + O\left(\frac{v}{k}\right) + O\left(\frac{Nv}{k^2}\right) \\ &= \frac{1-y_{N,k}^2}{2(N+2)y_{N,k}^2} + O\left(\frac{\text{Log } k}{k^3}\right) \end{aligned}$$

and

$$\begin{aligned} \frac{1-y_{N,k}^2}{2(N+2)y_{N,k}^2} &= \frac{e^{-2i\varphi_{N,k}}}{2(N+2)y_{N,k}^2} + O\left(\frac{\text{Log } k}{Nk}\right) \\ &= \frac{e^{-2i\varphi_{N,k}}}{2N(1-e^{-2i\varphi_{N,k}})} + O\left(\frac{\text{Log } k}{k^2}\right), \end{aligned}$$

so that

$$1 - \frac{1-x_{N,k}^2}{2(N+2)x_{N,k}^2} + O\left(\frac{1}{N^2x_{N,k}^4}\right) = 1 + \frac{ie^{-i\varphi_{N,k}}}{4N \sin \varphi_{N,k}} + O\left(\frac{\text{Log } k}{k^2}\right).$$

We now turn our attention to  $(1-x_{N,k}^2)^{N+1}/(2(N+1)x_{N,k})$ . We have

$$\begin{aligned} &2(N+1)x_{N,k} \sqrt{\frac{\pi}{N}} \\ &= 2\left(1 + \frac{1}{N}\right) \sqrt{N\pi} \sqrt{y_{N,k}^2 - v} \\ &= 2\left(1 + \frac{1}{N}\right) \sqrt{N\pi} \sqrt{1 - e^{-2i\varphi_{N,k}} - \frac{e^{-2i\varphi_{N,k}}}{N} \lambda_{N,k} + O\left(\frac{\text{Log } k}{Nk}\right)} \\ &= 2\left(1 + \frac{1}{N}\right) \sqrt{N\pi} \sqrt{2 \sin \varphi_{N,k}} e^{i(\pi/4 - \varphi_{N,k}/2)} \\ &\quad \cdot \left(1 + \frac{ie^{-i\varphi_{N,k}}}{4N \sin \varphi_{N,k}} \lambda_{N,k} + O\left(\frac{(\text{Log } k)^2}{k^2}\right)\right) \end{aligned}$$



and

$$\begin{aligned} (1 - x_{N,k}^2)^{N+1} &= (1 - y_{N,k}^2)^{N+1} \left(1 + \frac{v}{1 - y_{N,k}^2}\right)^{N+1} \\ &= (1 - y_{N,k}^2)^{N+1} \left(1 + \frac{(N + 1)v}{1 - y_{N,k}^2} + O\left(\frac{(\text{Log } k)^2}{k^2}\right)\right) \\ &= (1 - y_{N,k}^2)^{N+1} \left(1 + Nv e^{2i\varphi_{N,k}} + O\left(\frac{(\text{Log } k)^2}{k^2}\right)\right). \end{aligned}$$

Finally we have

$$\begin{aligned} &\frac{(1 - y_{N,k}^2)^{N+1}}{2\sqrt{2N\pi} \sin \varphi_{N,k} e^{i(\pi/4 - \varphi_{N,k}/2)}} \\ &= \left(\frac{1 + \frac{1}{N} \lambda_{N,k}}{1 + \frac{1}{N+1} \lambda_{N,k} + \frac{1}{2(N+1)^2} \lambda_{N,k}^2 + O\left(\frac{(\text{Log } k)^3}{N^3}\right)}\right)^{N+1} \\ &= 1 - \frac{1}{N} \lambda_{N,k} - \frac{1}{2N} \lambda_{N,k}^2 + O\left(\frac{(\text{Log } k)^3}{N^2}\right). \end{aligned}$$

We have thus obtained

$$\begin{aligned} &\left(1 + \frac{1}{N}\right) \left(1 + O\left(\frac{1}{N^2}\right)\right) \\ &= \frac{(1 - x_{N,k}^2)^{N+1}}{2\sqrt{N\pi} x_{N,k}} \left(1 - \frac{1 - x_{N,k}^2}{2(N+2)x_{N,k}^2} + O\left(\frac{1}{N^2 x_{N,k}^4}\right)\right) \\ &= 1 - \frac{\lambda_{N,k}}{N} - \frac{1}{2N} \lambda_{N,k}^2 - \frac{i e^{-i\varphi_{N,k}}}{4N \sin \varphi_{N,k}} \lambda_{N,k} + Nv e^{2i\varphi_{N,k}} \\ &\quad + \frac{i e^{-i\varphi_{N,k}}}{4N \sin \varphi_{N,k}} + O\left(\frac{(\text{Log } k)^3}{k^2}\right) \end{aligned}$$

which gives the value of  $v$  with an  $O((\text{Log } k)^3/(Nk^2))$  error.

As a corollary, we find a further development of  $\gamma_k$ , which is exactly the formula given in [3]:

**Corollary.** *If  $\mu_k = (2k - 1/4)\pi$ , then*

$$(50) \quad \begin{aligned} \gamma_k = e^{-3i\pi/4} \sqrt{\mu_k} & \left( 1 - \frac{i}{2\mu_k} \operatorname{Log} 2 \sqrt{\pi\mu_k} - \frac{1}{4\mu_k^2} \operatorname{Log} 2 \sqrt{\pi\mu_k} + \frac{1}{4\mu_k^2} \right. \\ & \left. + \frac{1}{8\mu_k^2} (\operatorname{Log} 2 \sqrt{\pi\mu_k})^2 + O\left(\frac{(\operatorname{Log} k)^3}{k^3}\right) \right). \end{aligned}$$

PROOF. From (31) and (49), we get

$$\begin{aligned} 1 - \frac{\bar{\gamma}_k^2}{N} &= \left(1 - i \frac{\mu_k}{N}\right) \left(1 + \frac{1}{N} \operatorname{Log} 2 \sqrt{\pi} \sqrt{\mu_k} + \frac{i}{2N\mu_k} (\operatorname{Log} 2 \sqrt{\pi} \sqrt{\mu_k} - 1)\right) \\ &+ O\left(\frac{(\operatorname{Log} k)^3}{Nk^2}\right), \end{aligned}$$

hence

$$\gamma_k^2 = -i \mu_k - \operatorname{Log} 2 \sqrt{\pi\mu_k} + \frac{i}{2\mu_k} \operatorname{Log} 2 \sqrt{\pi\mu_k} - \frac{i}{2\mu_k} + O\left(\frac{(\operatorname{Log} k)^3}{k^2}\right)$$

and

$$\begin{aligned} \gamma_k &= \sqrt{-i \mu_k} \left(1 - \frac{i}{2\mu_k} \operatorname{Log} 2 \sqrt{\pi\mu_k} - \frac{1}{4\mu_k^2} \operatorname{Log} 2 \sqrt{\pi\mu_k} \right. \\ &+ \frac{1}{4\mu_k^2} + \frac{1}{8\mu_k^2} (\operatorname{Log} 2 \sqrt{\pi\mu_k})^2 + O\left(\frac{(\operatorname{Log} k)^3}{k^3}\right) \Big) \end{aligned}$$

and the corollary is proved.

### 5. Small roots of $Q_N$ : further estimates.

We are now able to give a much better estimate for the small roots of  $Q_N$ . Indeed, we used the rough estimate  $|e^{-Nx_{N,k}^2}| \leq e^{N|x_{N,k}^2|}$  which is far from being good since  $x_{N,k}$  accumulates on the line  $x = y$  for  $k$  big (and  $k^2 = O(N)$ ), so that  $e^{-Nx_{N,k}^2}$  is much smaller than  $e^{N|x_{N,k}|^2}$ : indeed if  $k^2 = O(N)$  we find that

$$x_{N,k}^2 = -\frac{1}{N} \operatorname{Log} 2 \sqrt{\pi \left(2k - \frac{1}{4}\right) \pi} + \frac{i}{N} \left(2k - \frac{1}{4}\right) \pi + O\left(\frac{\operatorname{Log} k}{Nk}\right),$$

hence

$$\begin{aligned} |e^{-Nx_{N,k}^2}| &= e^{\text{Log } 2 \sqrt{\pi(2k-1/4)\pi}} e^{O(\text{Log } k/k)} \\ &= 2\sqrt{\pi} \sqrt{\left(2k - \frac{1}{4}\right)\pi} \left(1 + O\left(\frac{\text{Log } k}{k}\right)\right), \end{aligned}$$

while

$$e^{N|x_{N,k}|^2} \geq e^{(2k-1/4)\pi} \left(1 + O\left(\frac{\text{Log } k}{k}\right)\right).$$

Thus, we may improve Result 4 in an impressive manner: for a much bigger set of indexes  $k$ ,  $-\bar{\gamma}_k/\sqrt{N}$  provides a very precise approximation of  $x_{N,k}$ :

**Result 8.** *There exist  $\eta_0 > 0$  and  $C_0 > 0$  so that for  $N$  large enough and  $k \leq \eta_0 N^{1/5}/(\text{Log } N)^{2/5}$  we have*

$$(51) \quad \left|x_{N,k} + \frac{\bar{\gamma}_k}{\sqrt{N}}\right| \leq C_0 \frac{1}{N\sqrt{N}} \left(\frac{k^{5/2}}{1 + \text{Log } k}\right).$$

PROOF. We write

$$\tilde{Q}_N(t) = 4\sqrt{\frac{N}{\pi}} \frac{4^N(N!)^2}{(2N+1)!} Q_N(t) = 1 + O\left(\frac{1}{N^2}\right) + 2\sqrt{\frac{N}{\pi}} \int_0^t (1-s^2)^N ds$$

and approximate  $(1-s^2)^N$  by  $e^{-Ns^2}$  (provided that  $Nt^4$  remains bounded:  $|Nt^4| \leq A_0$ )

$$(1-s^2)^4 = e^{N \text{Log}(1-s^2)} = e^{-Ns^2} (1 + O(Ns^4)).$$

Thus

$$\tilde{Q}_N(t) = \text{erfc}(-\sqrt{N}t) + O\left(\frac{1}{N^2}\right) + \sqrt{N} \int_0^t e^{-Ns^2} O(Ns^4) ds.$$

Let  $\theta = \text{Arg } t$  and assume  $\theta \in (\pi/4, \pi/2)$ . Then we have

$$\begin{aligned} \left|\sqrt{N} \int_0^t e^{-Ns^2} O(Ns^4) ds\right| &\leq C N \sqrt{N} |t|^3 \int_0^{|t|} e^{-N\lambda^2 \cos 2\theta} \lambda d\lambda \\ &\leq C \frac{|e^{-Nt^2}| \sqrt{N} |t|^3}{2 |\cos 2\theta|}. \end{aligned}$$

We have thus proved that for  $|Nt^4| \leq A_0$  and  $\text{Arg } t \in (\pi/4, \pi/2)$  we have

$$|\tilde{Q}_N(t) - \text{erfc}(-\sqrt{N}t)| \leq C \left( \frac{1}{N^2} + \sqrt{N}|t|^3 \frac{|e^{-Nt^2}|}{2|\cos 2 \text{Arg } t|} \right).$$

Now, we write  $t = x_{N,k} + \delta$ ,  $|\delta| \leq \delta_0/N$ . Remember that we have

$$|x_{N,k}| \approx \sqrt{\frac{\left(2k - \frac{1}{4}\right)\pi}{N}}$$

(hence we will look at  $k \leq \sqrt{A_0 N / (2\pi)}$ ) and

$$\begin{aligned} \text{Arg } x_{N,k} &= \frac{\pi}{4} - \frac{1}{2} \varphi_{N,k} + \text{Arg} \left( 1 + \frac{i e^{-i\varphi_{N,k}}}{4N \sin \varphi_{N,k}} \text{Log} (2\sqrt{2N\pi \sin \varphi_{N,k}}) \right) \\ &\quad + O\left(\frac{(\text{Log } k)}{k^2}\right) \\ &= \frac{\pi}{4} + \frac{\text{Log} \left( 2\sqrt{\pi} \sqrt{\left(2k - \frac{1}{4}\right)\pi} \right)}{2\left(2k - \frac{1}{4}\right)\pi} + O\left(\frac{(\text{Log } k)^2}{k^2}\right) + O\left(\frac{k}{N}\right), \end{aligned}$$

hence if  $k \geq k_0$  where  $k_0$  is large enough so that

$$O\left(\frac{(\text{Log } k)^2}{k^2}\right) + O\left(\frac{k}{N}\right) = O\left(\frac{(\text{Log } k)^2}{k^2}\right) + O\left(\frac{1}{k}\right)$$

is smaller than

$$\frac{1}{2} \frac{\text{Log } 2 \sqrt{\pi} \sqrt{\left(2k - \frac{1}{4}\right)\pi}}{2\left(2k - \frac{1}{4}\right)\pi},$$

we find that  $\text{Arg } x_{N,k} \in (\pi/4, \pi/2)$ . (This is also true for  $k \leq k_0$ , if  $N$  is large enough, since  $x_{N,k} \sim -\bar{\gamma}_k/\sqrt{N}$ ).

Moreover

$$\begin{aligned} \cos(2 \operatorname{Arg} x_{N,k}) &= -\sin\left(\frac{\operatorname{Log}\left(2\sqrt{\pi}\sqrt{\left(2k-\frac{1}{4}\right)\pi}\right)}{\left(2k-\frac{1}{4}\right)\pi} + O\left(\frac{(\operatorname{Log} k)^2}{k^2}\right) + O\left(\frac{k}{N}\right)\right) \\ &= -\frac{\operatorname{Log}\left(2\sqrt{\pi}\sqrt{\left(2k-\frac{1}{4}\right)\pi}\right)}{\left(2k-\frac{1}{4}\right)\pi} + O\left(\frac{(\operatorname{Log} k)^2}{k^2}\right) + O\left(\frac{k}{N}\right), \end{aligned}$$

hence  $\cos(2 \operatorname{Arg} x_{N,k})$  has order of magnitude  $\operatorname{Log} k/k$ . Thus we obtain for  $\delta_0$  small enough

- $t = x_{N,k}\left(1 + O\left(\frac{1}{\sqrt{Nk}}\right)\right)$ ,
- $e^{-Nt^2} = e^{-Nx_{N,k}^2}\left(1 + O\left(\sqrt{\frac{k}{N}}\right) + O\left(\frac{1}{N}\right)\right)$ ,
- $\operatorname{Arg} t = \operatorname{Arg} x_{N,k} + O\left(\frac{1}{\sqrt{Nk}}\right) = \operatorname{Arg} x_{N,k} + O\left(\frac{1}{k\sqrt{k}}\right)$ ,

thus we have

$$\begin{aligned} |\tilde{Q}_N(t) - \operatorname{erfc}(-\sqrt{N}t)| &\leq C\left(\frac{1}{N^2} + \sqrt{N}\left(\frac{k}{N}\right)^{3/2} \frac{\sqrt{k}}{(\operatorname{Log} k)/k}\right) \\ &\leq C' \frac{k^3}{N \operatorname{Log} k}. \end{aligned}$$

On the other hand we have

$$\begin{aligned} &|\operatorname{erfc}(-\sqrt{N}t) - \operatorname{erfc}(-\sqrt{N}x_{N,k})| \\ &= \left|2\sqrt{\frac{N}{\pi}} \int_{x_{N,k}}^t e^{-Ns^2} ds\right| \\ &= |e^{-Nx_{N,k}^2}| 2\sqrt{\frac{N}{\pi}} \left|\int_0^\delta e^{-2Nx_{N,k}s - Ns^2} ds\right|. \end{aligned}$$

We notice that

$$|2Nx_{N,k}s + Ns^2| \leq 2|x_{N,k}|\delta_0 + \frac{\delta_0^2}{N} \leq C \frac{\delta_0}{\sqrt{N}},$$

so that if  $N$  is large enough,

$$|e^{-2Nx_{N,k}s - Ns^2} - 1| \leq \frac{1}{2},$$

which gives

$$|\operatorname{erfc}(-\sqrt{N}t) - \operatorname{erfc}(-\sqrt{N}x_{N,k})| \geq 2\sqrt{\frac{N}{\pi}} |e^{-Nx_{N,k}^2}| \frac{1}{2} |\delta| \geq C\sqrt{Nk} |\delta|.$$

Thus

$$(52) \quad \begin{cases} |\operatorname{erfc}(-\sqrt{N}t)| \geq C_1\sqrt{N}k\delta - C_2 \frac{k^3}{N \operatorname{Log} k}, \\ |\operatorname{erfc}(-\sqrt{N}t) - \tilde{Q}_N(t)| \leq C_2 \frac{k^3}{\sqrt{N} \operatorname{Log} k}. \end{cases}$$

Now choose

$$\delta_{N,k} = \frac{3C_2}{C_1} \frac{k^{5/2}}{N^{3/2} \operatorname{Log} k}$$

(we have  $\delta_{N,k} < \delta_0/N$  if  $k^{5/2}/\operatorname{Log} k < \delta_0 C_1 \sqrt{N}/(3C_2)$ ); we obtain that

$$\sup_{|t-x_{N,k}|=\delta_{N,k}} |\operatorname{erfc}(-\sqrt{N}t) - \tilde{Q}_N(t)| \leq \frac{1}{2} \inf_{|t-x_{N,k}|=\delta_{N,k}} |\operatorname{erfc}(-\sqrt{N}t)|,$$

hence by Rouché's theorem we find that  $\tilde{Q}_N$  and  $\operatorname{erfc}(-\sqrt{N}t)$  have the same number of roots in the disk  $|t - x_{N,k}| < \delta_{N,k}$ . Since

$$|x_{N,k} - x_{N,k+1}| \approx \sqrt{\frac{\pi}{2kN}}$$

and

$$\sqrt{kN} \delta_{N,k} = O\left(\frac{k^3}{N \operatorname{Log} k}\right) = O\left(\frac{1}{N^{2/5}(\operatorname{Log} N)^{7/5}}\right) = o(1)$$

(if  $k \leq CN^{1/5}/(\operatorname{Log} N)^{2/5}$ ), we find: for  $k \leq \eta_0 N^{1/5}/(\operatorname{Log} N)^{2/5}$  ( $\eta_0$  small enough)

$$|x_{N,k} + \frac{\bar{\gamma}_k}{\sqrt{N}}| \leq C \frac{1}{N\sqrt{N}} \left(\frac{k^{5/2}}{\operatorname{Log} k}\right).$$

Result 8 is proved.

Result 8 is enough for what we want to prove. But, of course, we may develop a bit further  $(1 - s^2)^N$  and get a better approximation for  $x_{N,k}$ :

**Result 9.** For  $k \leq \eta_0 N^{1/5} / (\text{Log } N)^{2/5}$  we have more precisely

$$x_{N,k} = -\frac{\bar{\gamma}_k}{\sqrt{N}} + \frac{1}{N\sqrt{N}} \left( \frac{1}{2} \bar{\gamma}_k^3 + \frac{3}{8} \bar{\gamma}_k + O(\sqrt{\text{Log } k}) \right).$$

PROOF. We write  $\text{Log}(1 - s^2) = -s^2 - s^4/2 + O(s^6)$ . Hence we have

$$(1 - s^2)^N = e^{-Ns^2} \left( 1 - N \frac{s^4}{2} + O(Ns^6) + O(N^2s^8) \right),$$

provided that  $|s| \leq A_0/N^{1/4}$ .

Thus we have for  $|t| \leq A_0/N^{1/4}$  and  $\text{Arg } t \in (\pi/4, \pi/2)$

$$\begin{aligned} & \left| \tilde{Q}_N(t) - \text{erfc}(-\sqrt{N}t) + 2\sqrt{\frac{N}{\pi}} N \int_0^t e^{-Ns^2} s^4 ds \right| \\ & \leq C \left( \frac{1}{N^2} + \sqrt{N} \left| \frac{t^5 e^{-Nt^2}}{\cos(2 \text{Arg } t)} \right| + N\sqrt{N} \left| \frac{t^7 e^{-Nt^2}}{\cos(2 \text{Arg } t)} \right| \right). \end{aligned}$$

Moreover we have

$$\begin{aligned} N \int_0^t e^{-Ns^2} s^4 ds &= \left[ \frac{-e^{-Ns^2} s^3}{2} \right]_0^t + \frac{3}{2} \int_0^t e^{-Ns^2} s^2 ds \\ &= \frac{-e^{-Nt^2} t^3}{2} - \frac{3}{4N} e^{-Nt^2} t + \frac{3}{4N} \int_0^t e^{-Ns^2} ds. \end{aligned}$$

Now, we write  $\eta = 1/\sqrt{2N|\cos(2 \text{Arg } t)|}$  (if  $t \approx x_{N,k}$ , we have  $\eta \approx \sqrt{4k/(N \text{Log } k)} < |t|$ ) and we write

$$\begin{aligned} \left| \int_0^t e^{-Ns^2} ds \right| &\leq \int_0^\eta |e^{-Ns^2}| ds + \int_\eta^{|t|} e^{-Ns^2 \cos(2 \text{Arg } t)} \frac{s ds}{\eta} \\ &\leq \eta |e^{-Nt^2}| + \frac{|e^{-Nt^2}|}{2N|\cos(2 \text{Arg } t)|\eta} \\ &= \frac{2|e^{-Nt^2}|}{\sqrt{2N|\cos(2 \text{Arg } t)|}}. \end{aligned}$$

Finally we get

$$\begin{aligned} \operatorname{erfc}(-\sqrt{N} x_{N,k}) &= e^{-Nx_{N,k}^2} \sqrt{\frac{N}{\pi}} x_{N,k}^3 + e^{-Nx_{N,k}^2} \frac{3}{4\sqrt{N}\pi} x_{N,k} \\ &\quad + O\left(\frac{1}{N^2}\right) + O\left(\frac{k^4}{N^2 \log k}\right) + O\left(\frac{k^5}{N^2 \log k}\right) \\ &\quad + O\left(\frac{\sqrt{\log k}}{N}\right) \end{aligned}$$

and, assuming again  $k < \eta_0 N^{1/5}/(\operatorname{Log} N)^{2/5}$ ,

$$\operatorname{erfc}(-\sqrt{N} x_{N,k}) = e^{-Nx_{N,k}^2} \sqrt{\frac{N}{\pi}} x_{N,k}^3 \left(1 + O\left(\frac{1}{k}\right)\right)$$

On the other hand, we have  $x_{N,k} = -\bar{\gamma}_k/\sqrt{N} + s$  with

$$s = O\left(\frac{1}{N\sqrt{N}} \frac{k^{5/2}}{\operatorname{Log} k}\right)$$

and we want a better estimate for  $s$ . We have

$$\sqrt{N} s \bar{\gamma}_k = O\left(\frac{1}{N} \frac{k^3}{\operatorname{Log} k}\right) = O\left(\frac{1}{N^{2/5}}\right)$$

and thus we may develop

$$\begin{aligned} \operatorname{erfc}(\bar{\gamma}_k - \sqrt{N} s) &= e^{-\bar{\gamma}_k^2} \frac{2}{\sqrt{\pi}} \int_0^{-\sqrt{N} s} e^{-2\bar{\gamma}_k u - u^2} du \\ &= -\frac{2}{\sqrt{\pi}} e^{-\bar{\gamma}_k^2} \sqrt{N} s (1 + O(\sqrt{N} s \bar{\gamma}_k) + O(Ns^2)). \end{aligned}$$

Hence we find

$$-\frac{2}{\sqrt{\pi}} e^{-\bar{\gamma}_k^2} \sqrt{N} s \sim \sqrt{\frac{N}{\pi}} x_{N,k}^3 e^{-Nx_{N,k}^2}$$

and therefore

$$s \sim -\frac{1}{2} x_{N,k}^3 = O\left(\frac{k^{3/2}}{N^{3/2}}\right),$$



so that

$$\begin{aligned}
 & - e^{-\bar{\gamma}_k^2} \frac{2}{\sqrt{\pi}} \sqrt{N} s \left( 1 + O\left(\frac{k^2}{N}\right) + O\left(\frac{k^3}{N^2}\right) \right) \\
 & = \sqrt{\frac{N}{\pi}} x_{N,k}^3 e^{-Nx_{N,k}^2} + \frac{3}{4\sqrt{N}\pi} x_{N,k} e^{-Nx_{N,k}^2} + O\left(\frac{\sqrt{\text{Log } k}}{N}\right),
 \end{aligned}$$

so that (since  $e^{-Nx_{N,k}^2 + \bar{\gamma}_k^2} = 1 + O(\sqrt{N} s \bar{\gamma}_k) = 1 + O(k^2/N)$ )

$$\begin{aligned}
 s & = -\frac{1}{2} x_{N,k}^3 - \frac{3}{8N} x_{N,k} + O\left(\frac{\sqrt{\text{Log } k}}{N\sqrt{N}}\right) \\
 & = \frac{1}{2} \frac{\bar{\gamma}_k^3}{N\sqrt{N}} + \frac{3\bar{\gamma}_k}{8N\sqrt{N}} + O\left(\frac{\sqrt{\text{Log } k}}{N\sqrt{N}}\right)
 \end{aligned}$$

and Result 9 is proved.

### 6. The phase of a general Daubechies filter.

We have now almost achieved the proof of Theorem 1. Indeed, we have given estimates for  $x_{N,k}$ , hence for  $z_{N,k}$ , which is the solution of  $x_{N,k} = (z_{N,k} + 1/z_{N,k})/2$  with  $\text{Re } z_{N,k} > 0$ , hence which is given by  $z_{N,k} = x_{N,k} + \sqrt{x_{N,k}^2 - 1}$ . We thus have proved:

**Proposition 3.** *Let  $P_N$  be the  $N$ -th polynomial of I. Daubechies*

$$(54) \quad P_N(z) = \left(\frac{1+z}{2}\right)^{2N+2} \sum_{k=0}^N (-1)^k \binom{N+k}{k} \left(\frac{1-z}{2}\right)^{2k}$$

which is related to  $Q_N$  by

$$(55) \quad e^{i(2N+1)\xi} P_N(e^{-i\xi}) = Q_N(\cos \xi)$$

or equivalently

$$(56) \quad P_N(z) = z^{2N+1} Q_N\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right).$$

Then the roots of  $P_N$  are precisely given as the following ones:

- $z = -1$  with multiplicity  $2N + 2$ ,

- $2N$  roots with multiplicity 1 which can be decomposed into

$$\left\{ z_{N,k}, \overline{z_{N,k}}, \frac{1}{z_{N,k}}, \frac{1}{\overline{z_{N,k}}} \right\}_{1 \leq k \leq [N/2]},$$

(together with  $\{z_{N,(N+1)/2}, 1/z_{N,(N+1)/2}\}$  if  $N$  is odd), where  $\text{Im } z_{N,k} \geq 0$ ,  $\text{Re } z_{N,k} \geq 0$ ,  $|z_{N,k}| > 1$ ,  $\text{Im } z_{N,k} > 0$  for  $k < [(N + 1)/2]$  and  $\text{Im } z_{N,(N+1)/2} = 0$ .

Moreover we have, for  $N$  large enough:

- if  $k \leq \eta_0 N^{1/5} / (\text{Log } N)^{2/5}$  (where  $\eta_0$  is fixed independently of  $N$  and is small enough)

$$(57) \quad z_{N,k} = i - \frac{\overline{\gamma}_k}{\sqrt{N}} + O\left(\frac{k}{N}\right),$$

where  $\gamma_k$  is the  $k$ -th zero  $\gamma$  of  $\text{erfc}(z)$  with  $\text{Im } \gamma > 0$

- for all  $k$

$$(58) \quad z_{N,k} = y_{N,k} + \sqrt{y_{N,k}^2 - 1} + O\left(\frac{1 + \text{Log } k}{k\sqrt{Nk}}\right),$$

where

$$y_{N,k} = \left( 1 - e^{-2i(8k-1)\pi/(8N+6)} - \frac{1}{N} e^{-2i(8k-1)\pi/(8N+6)} \text{Log } 2 \sqrt{2N\pi \sin\left(\frac{8k-1}{8N+6}\pi\right)} \right)^{1/2}.$$

PROOF. Just write  $z_{N,k} = x_{N,k} + \sqrt{x_{N,k}^2 - 1}$  and apply results 6 and 8.

Of course, we could give better estimates using results 7 and 9, but we won't need them. We have easy estimates for  $1/z_{N,k}$  as well since  $1/z_{N,k} = x_{N,k} - \sqrt{x_{N,k}^2 - 1}$ .

We are now going to use proposition 3 in the estimation of the phase of a Daubechies filter. We want to approximate for  $\xi \in [-\pi, \pi]$ ,  $1/(e^{-i\xi} - \lambda_{N,k})$  where

$$\lambda_{N,k} \in \left\{ z_{N,k}, \frac{1}{z_{N,k}}, \overline{z_{N,k}}, \frac{1}{\overline{z_{N,k}}} \right\}.$$

A direct consequence of Proposition 3 is the following proposition:

**Proposition 4.** *Let  $\xi \in [-\pi, \pi]$  and let  $z_{N,k}$ ,  $1 \leq k \leq [(N+1)/2]$  be the roots of  $P_N$  described in Proposition 3. Let  $\lambda_{N,k} \in \{z_{N,k}, 1/z_{N,k}, \bar{z}_{N,k}, 1/\bar{z}_{N,k}\}$ . Then*

i) *for  $1 \leq k \leq \eta_0 N^{1/5}/(\text{Log } N)^{2/5}$  we have, writing  $\widetilde{z}_{N,k} = i - \bar{\gamma}_k/\sqrt{N}$ ,*

$$(59) \quad \left| \frac{1}{e^{-i\xi} - \lambda_{N,k}} - \frac{1}{e^{-i\xi} - \widetilde{\lambda}_{N,k}} \right| \leq C \frac{k}{N} \frac{1}{\frac{k}{N} + |\cos \xi|^2},$$

where  $C$  doesn't depend neither on  $N$  nor on  $k$  nor on  $\xi$  (and where  $\widetilde{\lambda}_{N,k} = \widetilde{z}_{N,k}$  if  $\lambda_{N,k} = z_{N,k}$ ,  $1/\widetilde{z}_{N,k}$ , if  $\lambda_{N,k} = 1/z_{N,k}$  and so on ...).

ii) *for  $k \geq k_0$  ( $k_0$  large enough independently of  $N$ ) we have, writing  $\widehat{z}_{N,k} = y_{N,k} + \sqrt{y_{N,k}^2 - 1}$  as in formula (58),*

$$(60) \quad \left| \frac{1}{e^{-i\xi} - \lambda_{N,k}} - \frac{1}{e^{-i\xi} - \widehat{\lambda}_{N,k}} \right| \leq C \frac{\text{Log } k}{k\sqrt{Nk}} \frac{1}{\frac{k}{N} + |\cos \xi|^2}.$$

PROOF. Of course, we may assume  $\xi \in [0, \pi]$ . If  $\xi \in [\pi/2, \pi]$ , the estimation is easy since  $\text{Re } e^{-i\xi} < 0$  and  $\text{Re } \lambda_{N,k} > 0$  (as well  $\text{Re } \widehat{\lambda}_{N,k}$  and  $\text{Re } \widetilde{\lambda}_{N,k}$ ). Thus,

$$|e^{-i\xi} - \lambda_{N,k}| \geq \text{Re}(-e^{-i\xi} + \lambda_{N,k}) \geq C\sqrt{\frac{k}{N}} + |\cos \xi|$$

and the same for  $|e^{-i\xi} - \widehat{\lambda}_{N,k}|$  and  $|e^{-i\xi} - \widetilde{\lambda}_{N,k}|$ . Of course, we must prove that  $\min\{\text{Re } \lambda_{N,k}, \text{Re } \widehat{\lambda}_{N,k}, \text{Re } \widetilde{\lambda}_{N,k}\} \geq C\sqrt{k/N}$ . For  $\text{Re } \widetilde{\lambda}_{N,k}$ , it is obvious, since

$$\text{Re } \widetilde{\lambda}_{N,k} \geq \frac{-\text{Re } \gamma_k}{\sqrt{N} \left| i - \frac{\bar{\gamma}_k}{\sqrt{N}} \right|^2} \approx \sqrt{\frac{k\pi}{N}}.$$

For  $\text{Re } \lambda_{N,k}$ , if  $k < \eta_0 N^{1/5}/(\text{Log } N)^{2/5}$ , we deduce that  $\text{Re } \lambda_{N,k} \geq C\sqrt{k/N}$  since

$$|\lambda_{N,k} - \widetilde{\lambda}_{N,k}| \leq |z_{N,k} - \widetilde{z}_{N,k}| \leq C \frac{k}{N} \leq \sqrt{\frac{k}{N}} C' N^{-2/5}.$$

We thus turn our attention to  $\operatorname{Re} \widehat{\lambda}_{N,k} \geq \operatorname{Re} \widehat{z}_{N,k} / |\widehat{z}_{N,k}|^2$  and  $\operatorname{Re} \lambda_{N,k} \geq \operatorname{Re} z_{N,k} / |z_{N,k}|^2$  for large  $k$ 's. We define  $\mu_{N,k} = \sqrt{1 - e^{-2i(8k-1)\pi/(8N+6)}}$  and  $\xi_{N,k} = \mu_{N,k} + \sqrt{\mu_{N,k}^2 - 1}$ . We have

$$\begin{aligned} \xi_{N,k} &= \sqrt{2 \sin\left(\frac{8k-1}{8N+6}\right)} e^{i(\pi/4 - (8k-1)\pi/(2(8N+6)))} \\ &\quad + e^{i(\pi/2 - (8k-1)\pi/(8N+6))} \\ &= 1 + \sqrt{2} e^{i(\pi/4 - (8k-1)\pi/(2(8N+6)) + \arcsin \sqrt{2} \sin(\pi/4 - (8k-1)\pi/2(8N+6)))} \end{aligned}$$

and thus we study  $1 + \sqrt{2} e^{i(\omega + \arcsin \sqrt{2} \sin \omega)}$  for  $\omega \in [0, \pi/4]$ . We have

$$\begin{aligned} \operatorname{Re}(1 + \sqrt{2} e^{i(\omega + \arcsin \sqrt{2} \sin \omega)}) &= \sqrt{1 - 2 \sin^2 \omega} (\sqrt{1 - 2 \sin^2 \omega} + \sqrt{2 \cos^2 \omega}) \\ &= \sqrt{\cos 2\omega} (\sqrt{2 \cos^2 \omega} + \sqrt{1 - 2 \sin^2 \omega}) \geq \sqrt{\frac{4}{\pi} \left(\frac{\pi}{4} - \omega\right)}, \end{aligned}$$

which gives

$$\operatorname{Re} \xi_{N,k} \geq \sqrt{2 \frac{8k-1}{8N+6}} \geq \sqrt{\frac{k}{N}}.$$

Now we have

$$|\widehat{z}_{N,k} - \xi_{N,k}| \leq C \sqrt{\frac{k}{N}} \frac{\operatorname{Log} k}{k},$$

so that if  $k$  is large enough we have

$$\operatorname{Re} \widehat{z}_{N,k} \geq C' \sqrt{\frac{k}{N}}.$$

Moreover

$$|z_{N,k} - \widehat{z}_{N,k}| \leq C \sqrt{\frac{k}{N}} \frac{\operatorname{Log} k}{k^2}$$

and thus

$$\operatorname{Re} z_{N,k} \geq C'' \sqrt{\frac{k}{N}}.$$

Finally, we control  $|z_{N,k}|$  and  $|\widehat{z}_{N,k}|$  by

$$|z_{N,k}| + |\widehat{z}_{N,k}| \leq 1 + \sqrt{2} + O\left(\sqrt{\frac{k}{N}} \frac{\operatorname{Log} k}{k}\right) \leq C.$$

Thus we obtain

$$\operatorname{Re} \lambda_{N,k} \geq C \sqrt{\frac{k}{N}} \quad \text{and} \quad \operatorname{Re} \widehat{\lambda}_{N,k} \geq C \sqrt{\frac{k}{N}}.$$

We are going to prove that

$$|e^{-i\xi} - \lambda_{N,k}| \geq C \left( \sqrt{\frac{k}{N}} + |\cos \xi| \right)$$

and

$$|e^{-i\xi} - \widehat{\lambda}_{N,k}| \geq C \left( \sqrt{\frac{k}{N}} + |\cos \xi| \right)$$

holds for  $\xi \in [0, \pi/2]$  as well. Notice that if  $|\lambda_{N,k}| < 1$ , we have

$$|\lambda_{N,k} - e^{-i\xi}| = \left| \frac{1}{z_{N,k}} \right| \left| e^{-i\xi} - \frac{1}{\lambda_{N,k}} \right| \geq \frac{1}{C'} \left| e^{-i\xi} - \frac{1}{\lambda_{N,k}} \right|$$

(and the same for  $|e^{-i\xi} - \widehat{\lambda}_{N,k}|$ ) so that we may assume  $|\lambda_{N,k}| > 1$ . If  $\lambda_{N,k} = z_{N,k}$ , our equality is obvious: for  $\xi_{N,k}$  we have either  $\operatorname{Im} \xi_{N,k} \geq 1$  or  $\operatorname{Re} \xi_{N,k} \geq 2$  and, since  $\operatorname{Im} e^{-i\xi} < 0$ , we find  $|e^{-i\xi} - \xi_{N,k}| \geq 1$ , hence (for  $k$  large),  $|e^{-i\xi} - z_{N,k}| \geq 1/2$  and  $|e^{-i\xi} - \widehat{z}_{N,k}| \geq 1/2$ , while

$$\frac{1}{2} \geq \frac{1}{4} \left( \sqrt{\frac{k}{N}} + |\cos \xi| \right).$$

Now if  $\lambda_{N,k}$  is the conjugate of  $z_{N,k}$  or  $\widehat{z}_{N,k}$ , we are going to show that

$$|e^{-i\xi} - \bar{\xi}_{N,k}| \geq C \left( \sqrt{\frac{k}{N}} + |\cos \xi| \right),$$

which gives the control over  $|e^{-i\xi} - \lambda_{N,k}|$  for large  $k$ 's. Thus we are led to show that

$$(61) \quad \begin{cases} \text{for } \xi \in \left[0, \frac{\pi}{2}\right] \text{ and } \omega \in \left[0, \frac{\pi}{4}\right], \\ |e^{-i\xi} - 1 - \sqrt{2} e^{-i(\omega + \arcsin \sqrt{2} \sin \omega)}| \\ \geq C \left( |\cos \xi| + \sqrt{\frac{\pi}{4} - \omega} \right). \end{cases}$$

We compute easily  $\mu(\xi, \omega) = |e^{-i\xi} - 1 - \sqrt{2} e^{-i(\omega + \arcsin \sqrt{2} \sin \omega)}|^2$

$$\begin{aligned}
\mu(\xi, \omega) &= \left( \cos \xi - \sqrt{1 - 2 \sin^2 \omega} (\sqrt{2} \cos \omega + \sqrt{1 - 2 \sin^2 \omega}) \right)^2 \\
&\quad + \left( \sin \xi - \sqrt{2} \sin \omega (\sqrt{2} \cos \omega + \sqrt{1 - 2 \sin^2 \omega}) \right)^2 \\
&= 1 + (\sqrt{2} \cos \omega + \sqrt{1 - 2 \sin^2 \omega})^2 \\
&\quad - 2(\sqrt{2} \cos \omega + \sqrt{1 - 2 \sin^2 \omega}) \\
&\quad \cdot (\cos \xi \sqrt{1 - 2 \sin^2 \omega} + \sin \xi \sqrt{2} \sin \omega) \\
&= (\sqrt{2} \cos \omega - 1 + \sqrt{1 - 2 \sin^2 \omega})^2 \\
&\quad + 2(\sqrt{2} \cos \omega + \sqrt{1 - 2 \sin^2 \omega}) \\
&\quad \cdot (1 - \cos(\xi - \arcsin(\sqrt{2} \sin \omega))) \\
&\geq 1 - 2 \sin^2 \omega + 2(1 - \cos(\xi - \arcsin(\sqrt{2} \sin \omega))).
\end{aligned}$$

We have

$$1 - 2 \sin^2 \omega = \cos 2\omega \geq \frac{2}{\pi} \left( \frac{\pi}{2} - 2\omega \right).$$

On the other hand, we have

$$\begin{aligned}
1 - \cos(\xi - \arcsin \sqrt{2} \sin \omega) &= 2 \sin^2 \left( \frac{\xi}{2} - \frac{1}{2} \arcsin \sqrt{2} \sin \omega \right) \\
&\geq \frac{2}{\pi^2} |\xi - \arcsin \sqrt{2} \sin \omega|^2.
\end{aligned}$$

Moreover we have

$$\frac{\pi}{2} - \arcsin \sqrt{2} \sin \omega = \arcsin \sqrt{\cos 2\omega} \leq \frac{\pi}{2} \sqrt{\cos 2\omega},$$

hence we have (using  $|a + b|^2 \geq a^2/3 - b^2/2$ )

$$\begin{aligned}
\mu(\xi, \omega)^2 &\geq \cos 2\omega + \frac{4}{\pi^2} \left| \xi - \frac{\pi}{2} + \frac{\pi}{2} - \arcsin \sqrt{2} \sin \omega \right|^2 \\
&\geq \cos 2\omega + \frac{4}{3\pi^2} \left| \xi - \frac{\pi}{2} \right|^2 - \frac{2}{\pi^2} \left| \frac{\pi}{2} - \arcsin \sqrt{2} \sin \omega \right|^2 \\
&\geq \frac{1}{2} \cos 2\omega + \frac{4}{3\pi^2} \cos^2 \xi \\
&\geq \frac{4}{3\pi^2} \left( \cos^2 \xi + \left| \frac{\pi}{4} - \omega \right| \right)
\end{aligned}$$

and thus (61) is proved.

Proposition 4 is then obvious since

$$\left| \frac{1}{e^{-i\xi} - \lambda_{N,k}} - \frac{1}{e^{-i\xi} - \tilde{\lambda}_{N,k}} \right| = \frac{|\lambda_{N,k} - \tilde{\lambda}_{N,k}|}{|e^{-i\xi} - \lambda_{N,k}| |e^{-i\xi} - \tilde{\lambda}_{N,k}|}$$

and since we control each term due to (61) or to Proposition 3.

We may now obtain Theorem 1 as a corollary of Proposition 4:

**Corollary.** *With the same notation as in Proposition 4, if  $k_0 \leq k_N \leq \eta_0 N^{1/5} / (\text{Log } N)^{2/5}$  then*

$$(62) \quad \int_0^{2\pi} \left| \sum_{k=1}^{[(N+1)/2]} \frac{i e^{-i\xi}}{e^{-i\xi} - \lambda_{N,k}} - \sum_{k=1}^{k_N} \frac{i e^{-i\xi}}{e^{-i\xi} - \tilde{\lambda}_{N,k}} - \sum_{k_{N+1}}^{[(N+1)/2]} \frac{i e^{-i\xi}}{e^{-i\xi} - \hat{\lambda}_{N,k}} \right| d\xi \leq C \left( \frac{k_N^{3/2}}{\sqrt{N}} + \frac{\text{Log } k_N}{k_N} \right).$$

PROOF. Using Proposition 4, and writing  $I_N(\xi)$  for

$$I_N(\xi) = \sum_{k=1}^N \frac{i e^{-i\xi}}{e^{-i\xi} - \lambda_{N,k}} - \sum_{k=1}^{k_N} \frac{i e^{-i\xi}}{e^{-i\xi} - \tilde{\lambda}_{N,k}} - \sum_{k_{N+1}}^{[(N+1)/2]} \frac{i e^{-i\xi}}{e^{-i\xi} - \tilde{\lambda}_{N,k}},$$

we get

$$I_N(\xi) \leq \sum_{k=1}^{k_N} C \frac{k}{N} \frac{1}{\frac{k}{N} + |\cos \xi|^2} + \sum_{k_{N+1}}^{[(N+1)/2]} C \frac{\text{Log } k}{k \sqrt{Nk}} \frac{1}{\frac{k}{N} + |\cos \xi|^2}.$$

Thus we have to estimate

$$\begin{aligned} \int_0^{2\pi} \frac{d\xi}{k + N |\cos \xi|^2} &\leq 4 \int_0^{\arccos \sqrt{k/N}} \frac{d\xi}{N \cos^2 \xi} + 4 \int_{\arccos \sqrt{k/N}}^{\pi/2} \frac{d\xi}{k} \\ &= \frac{4}{N} \tan \left( \arccos \sqrt{\frac{k}{N}} \right) + \frac{4}{k} \left( \frac{\pi}{2} - \arccos \sqrt{\frac{k}{N}} \right) \\ &\leq \frac{4}{\sqrt{Nk}} + \frac{2\pi}{\sqrt{Nk}}, \end{aligned}$$

so that

$$\int_0^{2\pi} I_N(\xi) d\xi \leq C' \left( \sum_{k=1}^{k_N} \sqrt{\frac{k}{N}} + \sum_{k_{N+1}}^{[(N+1)/2]} \frac{\text{Log } k}{k^2} \right) \leq C'' \left( \frac{k_N^{3/2}}{\sqrt{N}} + \frac{\text{Log } k_N}{k_N} \right).$$

Now Theorem 1 is proved with  $k_N = [N^{1/5}/\text{Log } N]$ . At least, we have proved it for  $\xi \in [0, 2\pi]$ . But  $\omega(z_{N,1}^{\varepsilon_1}, \dots, z_{N,N}^{\varepsilon_N}) - \omega(Z_{N,1}^{\varepsilon_1}, \dots, Z_{N,N}^{\varepsilon_N})$  is  $2\pi$ -periodical, since  $\omega(Z_1, \dots, Z_N)(\xi + 2\pi) - \omega(Z_1, \dots, Z_N)(\xi) = 2i\pi M$  where  $M$  is the number of  $Z_k$ 's which lie inside the open disk  $|Z| < 1$ .

### 7. Minimum-phased Daubechies filters.

This section is devoted to the proof of Theorem 2.

**Result 10.** *We have the following inequality*

$$(63) \quad \left| \frac{d}{d\xi} \omega(z_{N,1}, \dots, z_{N,N})(\xi) - \frac{N}{2\pi} \text{Im} \int_{-\pi}^{\pi} \frac{i e^{-i\xi}}{e^{-i\xi} - \xi(\omega)} d\omega \right| \leq C\sqrt{N},$$

where  $\xi(\omega) = \sqrt{e^{-i\omega}} + \sqrt{1 + e^{-i\omega}}$ .

PROOF. We approximate  $z_{N,k}$  by  $Z_{N,k} = Z((8k - 1)\pi/(8N + 6))$ , ( $1 \leq k \leq N$ ) where

$$Z(\omega) = \sqrt{2 \sin \omega} e^{i(\pi/4 - \omega/2)} + e^{i(\pi/2 - \omega)}.$$

We have shown that for  $k_0 \leq k \leq [(N + 1)/2]$ , ( $k_0$  large enough) we have

$$\left| \frac{1}{e^{-i\xi} - z_{N,k}} - \frac{1}{e^{-i\xi} - Z_{N,k}} \right| \leq C \frac{\text{Log } k}{\sqrt{Nk}} \frac{1}{\frac{k}{N} + \cos^2 \xi}$$

and

$$\left| \frac{1}{e^{-i\xi} - \bar{z}_{N,k}} - \frac{1}{e^{-i\xi} - \bar{Z}_{N,k}} \right| \leq C \frac{\text{Log } k}{\sqrt{Nk}} \frac{1}{\frac{k}{N} + \cos^2 \xi}$$

(notice that  $z_{N,N+1-k} = \bar{z}_{N,k}$  and  $Z_{N,N+1-k} = \bar{Z}_{N,k}$ ). If  $k < k_0$ , we have to prove similarly

$$\left| \frac{1}{e^{-i\xi} - z_{N,k}} - \frac{1}{e^{-i\xi} - Z_{N,k}} \right| \leq C \frac{1}{\sqrt{N}} \frac{1}{\frac{1}{N} + \cos^2 \xi}$$



and

$$\left| \frac{1}{e^{-i\xi} - \bar{z}_{N,k}} - \frac{1}{e^{-i\xi} - \bar{Z}_{N,k}} \right| \leq C \frac{1}{\sqrt{N}} \frac{1}{\frac{1}{N} + \cos^2 \xi}.$$

We have of course

$$|z_{N,k} - Z_{N,k}| \leq |z_{N,k}| + |Z_{N,k}| \leq \frac{C}{\sqrt{N}},$$

so that we only have to check that

$$|e^{-i\xi} - Z_{N,k}| \geq \frac{1}{C} \left( \frac{1}{\sqrt{N}} + |\cos \xi| \right)$$

(which is an easy consequence of (61)) and that

$$|e^{-i\xi} - z_{N,k}| \geq \frac{1}{C} \left( \frac{1}{\sqrt{N}} + |\cos \xi| \right).$$

If  $|\xi + \pi/2| \geq 3|\gamma_{k_0}|/\sqrt{N}$  and  $\xi \in [-2\pi, 0]$ , we find

$$e^{-i\xi} - z_{N,k} = 2e^{-i(\xi/2 + \pi/4)} \sin\left(\frac{\xi}{2} + \frac{\pi}{4}\right) - \frac{\bar{\gamma}_k}{\sqrt{N}} + O\left(\frac{1}{N}\right),$$

hence

$$\begin{aligned} |e^{-i\xi} - z_{N,k}| &\geq \left| \sin\left(\frac{\xi}{2} + \frac{\pi}{4}\right) \right| - \frac{|\gamma_{k_0}|}{\sqrt{N}} + O\left(\frac{1}{N}\right) \\ &\geq \frac{1}{2} \left| \sin\left(\frac{\xi}{2} + \frac{\pi}{4}\right) \right| \\ &\geq \max\left\{ \frac{1}{4} |\cos \xi|, \frac{6}{\pi} \frac{|\gamma_{k_0}|}{\sqrt{N}} \right\}. \end{aligned}$$

On the other hand, if  $|\xi + \pi/2| \leq 3|\gamma_{k_0}|/\sqrt{N}$ , we have

$$e^{-i\xi} - z_{N,k} = -\left(\frac{\xi}{2} + \frac{\pi}{4}\right) - \frac{\bar{\gamma}_k}{\sqrt{N}} + O\left(\frac{1}{N}\right),$$

hence

$$|e^{-i\xi} - z_{N,k}| \geq \frac{1}{2} \frac{\inf \operatorname{Im} \gamma_k}{\sqrt{N}} = \frac{c_0}{\sqrt{N}} \geq C_0 \max\left\{ \frac{1}{\sqrt{N}}, \frac{1}{6|\gamma_{k_0}|} |\cos \xi| \right\}.$$

Thus we have obtained

$$\begin{aligned} \left| \frac{d}{d\xi} \omega(z_{N,1}, \dots, z_{N,N})(\xi) - \sum_{k=1}^N \operatorname{Im} \frac{i e^{-i\xi}}{e^{-i\xi} - Z_{N,k}} \right| \\ \leq C \sum_{k=1}^N \frac{(1 + \operatorname{Log} k)}{\sqrt{Nk}} \frac{1}{\frac{k}{N} + \cos^2 \xi} \\ \leq C \sqrt{N} \sum_1^\infty \frac{1 + \operatorname{Log} k}{k \sqrt{k}}. \end{aligned}$$

Now we look at

$$S_N(\xi) = \operatorname{Im} \sum_{k=1}^N \frac{i e^{-i\xi}}{e^{-i\xi} - Z_{N,k}}$$

as at a Riemann sum: we have

$$\frac{\pi}{N} S_N(\xi) \xrightarrow{N \rightarrow \infty} \operatorname{Im} \int_0^\pi \frac{i e^{-i\xi} d\omega}{e^{-i\xi} - Z(\omega)}.$$

If  $\xi \neq \pm\pi/2$ , we have a proper Riemann integral; if  $\xi = \pm\pi/2$ , the integrand is unbounded at 0 ( $\xi = -\pi/2$ ) or  $\pi$  ( $\xi = \pi/2$ ); but for  $\xi = -\pi/2$  we have  $e^{-i\xi} - Z(\omega) = e^{i\pi/4} \sqrt{2\omega} + O(\omega)$  near  $\omega = 0$  and thus

$$\int_0^\pi \frac{1}{|i - Z(\omega)|} d\omega < +\infty.$$

It is easy to evaluate the distance between  $\pi S_N/N$  and the integral. We have

$$\begin{aligned} \left| \int_0^{7\pi/(8N+6)} \frac{d\omega}{e^{-i\xi} - Z(\omega)} \right| &\leq C \int_0^{7\pi/(8N+6)} \frac{d\omega}{\sqrt{\omega}} \leq C' \frac{1}{\sqrt{N}}, \\ \left| \int_{(8N-1)\pi/(8N+6)}^\pi \frac{d\omega}{e^{-i\xi} - Z(\omega)} \right| &\leq C \int_{(8N-1)\pi/(8N+6)}^\pi \frac{d\omega}{\sqrt{\pi - \omega}} \\ &\leq C' \frac{1}{\sqrt{N}}, \end{aligned}$$

$$\frac{1}{N} \left| \frac{1}{e^{-i\xi} - Z\left(\frac{8N-1}{8N+6} \pi\right)} \right| \leq C' \frac{\sqrt{N}}{N},$$

and finally for  $1 \leq k < N$

$$\begin{aligned} & \left| \int_{(8k-1)\pi/(8N+6)}^{(8k+7)\pi/(8N+6)} \frac{1}{e^{-i\xi} - Z(\omega)} d\omega - \frac{8\pi}{8N+6} \frac{1}{e^{-i\xi} - Z\left(\frac{8k-1}{8N+6}\pi\right)} \right| \\ & \leq C \int_{(8k-1)\pi/(8N+6)}^{(8k+7)\pi/(8N+6)} \frac{\left| Z(\omega) - Z\left(\frac{8k-1}{8N+6}\pi\right) \right|}{\left| e^{-i\xi} - Z(\omega) \right| \left| e^{-i\xi} - Z\left(\frac{8k-1}{8N+6}\pi\right) \right|} d\omega \\ & \leq C' \int_{(8k-1)\pi/(8N+6)}^{(8k+7)\pi/(8N+6)} \frac{1}{\frac{\sqrt{Nk}}{\sqrt{k}} \frac{\sqrt{k}}{\sqrt{N}}} d\omega \\ & \leq C'' \frac{1}{k^{3/2}\sqrt{N}} \end{aligned}$$

and thus

$$\left| \frac{\pi}{N} S_N(\xi) - \text{Im} \int_0^\pi i e^{-i\xi} \frac{d\omega}{e^{-i\xi} - Z(\omega)} \right| \leq C \frac{1}{\sqrt{N}}.$$

Thus, Result 10 is proved since writing  $-e^{-2i\omega} = e^{-i\sigma}$  gives

$$\begin{aligned} & \int_0^\pi i e^{-i\xi} \frac{d\omega}{e^{-i\xi} - \sqrt{2} \sin \omega} \frac{1}{e^{i(\pi/4-\omega/2)} - e^{i(\pi/2-\omega)}} \\ & = \frac{1}{2} \int_{-\pi}^\pi i e^{-i\xi} \frac{d\sigma}{e^{-i\xi} - \sqrt{e^{-i\sigma} - \sqrt{1 + e^{-i\sigma}}}}. \end{aligned}$$

We will easily prove Theorem 2 if we know the value of  $I(\xi) = \int_{-\pi}^\pi i e^{-i\xi} d\sigma / (e^{-i\xi} - \xi(\sigma))$ :

**Result 11.** Let  $\xi(\sigma) = \sqrt{e^{-i\xi}} + \sqrt{1 + e^{-i\sigma}}$  and  $\xi \in [-\pi, \pi]$ . Then

$$(64) \quad \begin{aligned} & \int_{-\pi}^\pi i e^{-i\xi} \frac{d\sigma}{e^{-i\xi} - \xi(\sigma)} \\ & = \begin{cases} -\pi \tan\left(\frac{\xi}{2}\right) + i \frac{\cos \xi}{\sin \xi} \text{Log}\left(\frac{1 - \sin \xi}{1 + \sin \xi}\right), & \text{if } |\xi| \leq \frac{\pi}{2}, \\ -\pi \cotan\left(\frac{\xi}{2}\right) + i \frac{\cos \xi}{\sin \xi} \text{Log}\left(\frac{1 - \sin \xi}{1 + \sin \xi}\right), & \text{if } |\xi| \geq \frac{\pi}{2}. \end{cases} \end{aligned}$$

We find that  $I(\xi)$  is continuous, which is obvious since by (61)

$$|e^{-i\xi} - \xi(\sigma)| \geq C\sqrt{\pi^2 - \sigma^2},$$

so that we may apply Lebesgue's dominated convergence theorem.

PROOF. Since  $\xi(\sigma) = \bar{\xi}(-\sigma)$ , we find that

$$I(-\xi) = -\int_{-\pi}^{\pi} \frac{i e^{-i\xi}}{e^{-i\xi} - \bar{\xi}(\sigma)} d\sigma = -\overline{I(\xi)},$$

so that it is enough to compute  $I(\xi)$  for  $\xi \in [0, \pi]$ .

Writing  $e^{-i\sigma} = u$ , we may write

$$I(\xi) = \int_{-1+i0}^{-1-i0} \frac{e^{-i\xi}}{\sqrt{u} + \sqrt{1+u} - e^{-i\xi}} \frac{du}{u},$$

where  $u$  runs clockwise on the circle  $|u| = 1$ . The function

$$f(z) = \frac{e^{-i\xi}}{z(\sqrt{z} + \sqrt{1+z} - e^{-i\xi})}$$

is analytical on  $\mathbb{C} \setminus (-\infty, 0]$  and may be extended continuously to  $(-\infty, 0] + i0$  and  $(-\infty, 0] - i0$  but at three points:  $z = 0$  (both a pole and a branching point),  $z = -1$  (a branching point) and if  $\xi \in [0, \pi/2]$  at  $-\sin^2 \xi - i0 = z_\xi$ . Thus we may write:

- for  $\xi \in [\pi/2, \pi]$

$$\begin{aligned} I(\xi) &= \lim_{\varepsilon \rightarrow 0} \int_{-1}^{-1-\varepsilon} \frac{e^{-i\xi}}{\sqrt{u+i0} + \sqrt{1+u} - e^{-i\xi}} \frac{du}{u} \\ &\quad + \int_{-1-\varepsilon}^{-1} \frac{e^{-i\xi}}{\sqrt{u-i0} + \sqrt{1+u} - e^{-i\xi}} \frac{du}{u} \\ &\quad + \int_{-1-\varepsilon+i0}^{-1-\varepsilon-i0} \frac{e^{-i\xi}}{\sqrt{u} + \sqrt{1+u} - e^{-i\xi}} \frac{du}{u} \\ &= 2i \int_0^1 \frac{dt}{\cos \xi - \sqrt{1-t^2}} - 2i\pi \frac{e^{-i\xi}}{1 - e^{-i\xi}} \\ &= 2i \int_0^{\pi/2} \frac{\cos \alpha}{\cos \xi - \cos \alpha} d\alpha - \pi \cotan\left(\frac{\xi}{2}\right) + \pi i. \end{aligned}$$

• if  $\xi \in (0, \pi/2)$  we have, writing  $t_\varepsilon^+ = \sqrt{\sin^2 \xi + \varepsilon}$  and  $t_\varepsilon^- = \sqrt{\sin^2 \xi - \varepsilon}$

$$I(\xi) = \lim_{\varepsilon \rightarrow 0} A_\varepsilon + B_\varepsilon + C_\varepsilon,$$

where

$$\begin{aligned} A_\varepsilon &= \int_{-1}^{-(t_\varepsilon^+)^2} + \int_{-(t_\varepsilon^-)^2}^{-\varepsilon} \frac{e^{-i\xi}}{\sqrt{u+i0} + \sqrt{1+u} - e^{-i\xi}} \frac{du}{u} \\ &\quad + \int_{-\varepsilon}^{-(t_\varepsilon^-)^2} + \int_{-(t_\varepsilon^+)^2}^{-1} \frac{e^{-i\xi}}{\sqrt{u-i0} + \sqrt{1+u} - e^{-i\xi}} \frac{du}{u} \\ &= 2i \int_{\sqrt{\varepsilon}}^{t_\varepsilon^-} + \int_{t_\varepsilon^+}^1 \frac{dt}{\cos \xi - \sqrt{1-t^2}} \\ B_\varepsilon &= \int_{-\varepsilon+i0}^{-\varepsilon-i0} \frac{e^{-i\xi}}{\sqrt{u} + \sqrt{1+u} - e^{-i\xi}} \frac{du}{u} \\ &= -2i\pi \frac{e^{-i\xi}}{1 - e^{-i\xi}} + O(\sqrt{\varepsilon}) \\ &= -\pi \cotan\left(\frac{\xi}{2}\right) + i\pi + O(\sqrt{\varepsilon}), \\ C_\varepsilon &= \int_{-(t_\varepsilon^+)^2}^{-(t_\varepsilon^-)^2} \frac{e^{-i\xi}}{\sqrt{u+i0} + \sqrt{1+u} - e^{-i\xi}} \frac{du}{u} \\ &\quad + \int_{z_\xi+\varepsilon}^{z_\xi-\varepsilon} \frac{e^{-i\xi} du}{(\sqrt{u} + \sqrt{1+u} - e^{-i\xi}) u} \\ &= -i\pi 2i \cotan \xi + O(\varepsilon) \\ &= 2\pi \cotan \xi + O(\varepsilon), \end{aligned}$$

since the residue of

$$f(\xi) = \frac{e^{-i\xi}}{\sqrt{u} + \sqrt{1+u} - e^{-i\xi}} \frac{1}{u}$$

at  $z_\xi = -\sin^2 \xi - i0$  is equal to

$$\frac{1}{2} \frac{1}{\sqrt{z_\xi}} + \frac{1}{2} \frac{1}{\sqrt{1+z_\xi}} \frac{1}{z_\xi} = \frac{2\sqrt{z_\xi} \sqrt{1+z_\xi}}{z_\xi} = 2i \cotan \xi.$$

Hence we have

$$\begin{aligned}
 I(\xi) &= \pi \left( 2 \cotan \xi - \cotan \left( \frac{\xi}{2} \right) \right) \\
 &\quad + i \pi + 2i \lim_{\varepsilon \rightarrow 0} \int_0^{t_\varepsilon^-} + \int_{t_\varepsilon^+}^1 \frac{dt}{\cos \xi - \sqrt{1-t^2}} \\
 &= -\pi \tan \left( \frac{\xi}{2} \right) + i \pi + 2i \lim_{\varepsilon \rightarrow 0} \int_0^{\alpha_\varepsilon^-} + \int_{\alpha_\varepsilon^+}^{\pi/2} \frac{\cos \alpha \, d\alpha}{\cos \xi - \cos \alpha},
 \end{aligned}$$

where  $\alpha_\varepsilon^- = \arcsin t_\varepsilon^-$  and  $\alpha_\varepsilon^+ = \arcsin t_\varepsilon^+$ .

Thus, for proving Result 11, we just have to estimate for  $\xi \in (0, \pi)$ ,  $\xi \neq \pi/2$

$$A(\xi) = \lim_{\varepsilon \rightarrow 0} \int_0^{\alpha_\varepsilon^-} + \int_{\alpha_\varepsilon^+}^{\pi/2} \frac{\cos \alpha \, d\alpha}{\cos \xi - \cos \alpha}$$

with  $\alpha_\varepsilon^- = \arcsin \sqrt{\sin^2 \xi - \varepsilon}$  and  $\alpha_\varepsilon^+ = \arcsin \sqrt{\sin^2 \xi + \varepsilon}$ . We do the usual change of variable  $\beta = \tan(\alpha/2)$ . Then

$$A(\xi) = \lim_{\varepsilon \rightarrow 0} \int_0^{\beta_\varepsilon^-} + \int_{\beta_\varepsilon^+}^1 \frac{2(1-\beta^2)}{(1+\beta^2)((1+\beta^2)\cos \xi - (1-\beta^2))} d\beta.$$

We write

$$\begin{aligned}
 (1+\beta^2)\cos \xi - (1-\beta^2) &= \beta^2(1+\cos \xi) - (1-\cos \xi) \\
 &= 2\beta^2 \cos^2 \left( \frac{\xi}{2} \right) - 2\sin^2 \left( \frac{\xi}{2} \right),
 \end{aligned}$$

hence

$$\begin{aligned}
 A(\xi) &= \frac{1}{\cos^2 \left( \frac{\xi}{2} \right)} \lim_{\varepsilon \rightarrow 0} \int_0^{\beta_\varepsilon^-} + \int_{\beta_\varepsilon^+}^1 \frac{1-\beta^2}{(1+\beta^2)(\beta^2 - \tan^2 \left( \frac{\xi}{2} \right))} d\beta \\
 &= \frac{1}{\cos^2 \left( \frac{\xi}{2} \right)} \lim_{\varepsilon \rightarrow 0} \int_0^{\beta_\varepsilon^-} + \int_{\beta_\varepsilon^+}^1 \left( \frac{-2}{1+\tan^2 \left( \frac{\xi}{2} \right)} \frac{1}{1+\beta^2} \right. \\
 &\quad \left. + \frac{1-\tan^2 \left( \frac{\xi}{2} \right)}{1+\tan^2 \left( \frac{\xi}{2} \right)} \frac{1}{\beta^2 - \tan^2 \left( \frac{\xi}{2} \right)} d\beta \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{\varepsilon \rightarrow 0} \int_0^{\beta_\varepsilon^-} + \int_{\beta_\varepsilon^+}^1 \left( \frac{-2}{1 + \beta^2} \right. \\
 &\qquad \qquad \qquad \left. + \frac{\cos \xi}{\sin \xi} \left( \frac{1}{\beta - \tan \left( \frac{\xi}{2} \right)} - \frac{1}{\beta + \tan \left( \frac{\xi}{2} \right)} \right) \right) d\beta \\
 &= \lim_{\varepsilon \rightarrow 0} -\frac{\pi}{2} + \frac{\cos \xi}{\sin \xi} \operatorname{Log} \left| \frac{1 - \tan \left( \frac{\xi}{2} \right)}{1 + \tan \left( \frac{\xi}{2} \right)} \right| \\
 &\qquad - \frac{\cos \xi}{\sin \xi} \operatorname{Log} \left| \frac{\beta_\varepsilon^+ - \tan \left( \frac{\xi}{2} \right)}{\beta_\varepsilon^+ + \tan \left( \frac{\xi}{2} \right)} \right| + \frac{\cos \xi}{\sin \xi} \operatorname{Log} \left| \frac{\beta_\varepsilon^- - \tan \left( \frac{\xi}{2} \right)}{\beta_\varepsilon^- + \tan \left( \frac{\xi}{2} \right)} \right| \\
 &= -\frac{\pi}{2} + \frac{\cos \xi}{2 \sin \xi} \operatorname{Log} \left( \frac{1 - \tan \left( \frac{\xi}{2} \right)}{1 + \tan \left( \frac{\xi}{2} \right)} \right)^2 \\
 &\qquad + \frac{\cos \xi}{\sin \xi} \lim_{\varepsilon \rightarrow 0} \operatorname{Log} \left| \frac{\beta_\varepsilon^- - \tan \left( \frac{\xi}{2} \right)}{\beta_\varepsilon^+ - \tan \left( \frac{\xi}{2} \right)} \right|.
 \end{aligned}$$

Now we have

$$\begin{aligned}
 \left( \frac{1 - \tan \left( \frac{\xi}{2} \right)}{1 + \tan \left( \frac{\xi}{2} \right)} \right)^2 &= \frac{\cos^2 \left( \frac{\xi}{2} \right) - 2 \sin \left( \frac{\xi}{2} \right) \cos \left( \frac{\xi}{2} \right) + \sin^2 \left( \frac{\xi}{2} \right)}{\cos^2 \left( \frac{\xi}{2} \right) + 2 \sin \left( \frac{\xi}{2} \right) \cos \left( \frac{\xi}{2} \right) + \sin^2 \left( \frac{\xi}{2} \right)} \\
 &= \frac{1 - \sin \xi}{1 + \sin \xi},
 \end{aligned}$$

while we have for  $\xi \in (0, \pi/2)$

$$\begin{aligned}
 \beta_\varepsilon^- - \tan \left( \frac{\xi}{2} \right) &\sim \frac{1}{2} \left( 1 + \tan^2 \left( \frac{\xi}{2} \right) \right) (\alpha_\varepsilon^- - \xi) \\
 &\sim \frac{1}{2} \left( 1 + \tan^2 \left( \frac{\xi}{2} \right) \right) \frac{\sqrt{\sin^2 \xi - \varepsilon} - \sin \xi}{\cos \xi} \\
 &\sim \frac{-\varepsilon \left( 1 + \tan^2 \left( \frac{\xi}{2} \right) \right)}{4 \sin \xi \cos \xi}
 \end{aligned}$$

and

$$\beta_\varepsilon^+ - \tan\left(\frac{\xi}{2}\right) \sim \frac{+\varepsilon\left(1 + \tan^2\left(\frac{\xi}{2}\right)\right)}{4 \sin \xi \cos \xi} \sim -\left(\beta_\varepsilon^- - \tan\left(\frac{\xi}{2}\right)\right).$$

Thus

$$A(\xi) = -\frac{\pi}{2} + \frac{\cos \xi}{2 \sin \xi} \operatorname{Log} \frac{1 - \sin \xi}{1 + \sin \xi}$$

and Result 11 is proved.

Now, (63) gives

$$\left| \frac{d}{d\xi} \omega(z_{N,1}, \dots, z_{N,N})(\xi) - \frac{N}{2\pi} \frac{\cos \xi}{\sin \xi} \operatorname{Log} \frac{1 - \sin \xi}{1 + \sin \xi} \right| \leq C\sqrt{N}.$$

Integrating this for  $\xi \in [-\pi, \pi]$  we get

$$\left| \omega(z_{N,1}, \dots, z_{N,N})(\xi) - \frac{N}{2\pi} (\operatorname{Li}_2(-\sin \xi) - \operatorname{Li}_2(\sin \xi)) \right| \leq C\sqrt{N}.$$

Since both functions are  $2\pi$ -periodical, this inequality can be extended to all  $\xi \in \mathbb{R}$  and Theorem 2 is proved.

## 8. Almost linear-phased Daubechies filters.

In this section, we prove Theorem 3. The proof is very easy. Indeed, we want to estimate for  $N = 4q$ ,  $\omega(z_{N,1}^{\varepsilon_{N,1}}, \dots, z_{N,N}^{\varepsilon_{N,N}})(\xi)$  with  $\varepsilon_{N,k} = 1$  if  $k = 0 \pmod 4$  or  $k = 1 \pmod 4$ , and  $\varepsilon_{N,k} = -1$  otherwise.

We have (writing  $\omega_N$  for  $\omega(z_{N,1}^{\varepsilon_{N,1}}, \dots, z_{N,N}^{\varepsilon_{N,N}})$ ,  $K_N$  for  $\{k \in \mathbb{N} : 1 \leq k \leq N, \varepsilon_{N,k} = 1\}$  and  $\tilde{K}_N$  for  $\{k \in \mathbb{N} : 1 \leq k \leq N, \varepsilon_{N,k} = -1\}$ )

$$\frac{d\omega_N}{d\xi} = \operatorname{Im} \sum_{k \in K_N} \frac{i e^{-i\xi}}{e^{-i\xi} - z_{N,k}} + \sum_{k \in \tilde{K}_N} \frac{i e^{-i\xi}}{e^{-i\xi} - \frac{1}{\bar{z}_{N,k}}}$$

(we have used that for  $k \in \tilde{K}_N$ ,  $N+1-k \in \tilde{K}_N$  and  $z_{N,k} = \bar{z}_{N,N+1-k}$ ).

Hence we have

$$\begin{aligned} \frac{d\omega_N}{d\xi} &= \operatorname{Im} \left( \sum_{k \in K_N} \frac{i e^{-i\xi}}{e^{-i\xi} - z_{N,k}} - \sum_{k \in \tilde{K}_N} \frac{i e^{-i\xi}}{e^{-i\xi} - z_{N,k}} \right) \\ &\quad + \operatorname{Im} \left( \sum_{k \in \tilde{K}_N} \frac{i e^{-i\xi}}{e^{-i\xi} - z_{N,k}} + \frac{i e^{-i\xi}}{e^{-i\xi} - \frac{1}{\bar{z}_{N,k}}} \right). \end{aligned}$$



But we have

$$\begin{aligned}
 \frac{ie^{-i\xi}}{e^{-i\xi} - Z} + \frac{ie^{-i\xi}}{e^{-i\xi} - \frac{1}{\bar{Z}}} &= \frac{ie^{-i\xi}}{e^{-i\xi} - Z} + \frac{i\bar{Z}}{\bar{Z} - e^{+i\xi}} \\
 &= \frac{ie^{-i\xi}(e^{i\xi} - \bar{Z}) + i\bar{Z}(-e^{-i\xi} + Z)}{|e^{-i\xi} - Z|^2} \\
 &= \frac{i(1 - 2\bar{Z}e^{-i\xi} + |Z|^2)}{|Z - e^{-i\xi}|^2} \\
 &= i + \frac{i(Ze^{i\xi} - \bar{Z}e^{-i\xi})}{|Z - e^{-i\xi}|^2},
 \end{aligned}$$

hence

$$\operatorname{Im} \left( \frac{ie^{-i\xi}}{e^{-i\xi} - Z} + \frac{ie^{-i\xi}}{e^{-i\xi} - \frac{1}{\bar{Z}}} \right) = 1.$$

Thus, we have obtained

$$\begin{aligned}
 \frac{d\omega_N}{d\xi} &= \frac{N}{2} + \operatorname{Im} \sum_{k=1}^q ie^{-i\xi} \left( \frac{1}{e^{-i\xi} - z_{N,4k-3}} - \frac{1}{e^{-i\xi} - z_{N,4k-2}} \right. \\
 &\quad \left. - \frac{1}{e^{-i\xi} - z_{N,4k-1}} + \frac{1}{e^{-i\xi} - z_{N,4k}} \right).
 \end{aligned}$$

Now we write, for  $r \in \{1, 2, 3\}$

$$\begin{aligned}
 \frac{1}{e^{-i\xi} - z_{N,4k-r}} &= \frac{1}{e^{-i\xi} - z_{N,4k}} + \frac{z_{N,4k-r} - z_{N,4k}}{(e^{-i\xi} - z_{N,4k})(e^{-i\xi} - z_{N,4k-r})} \\
 &= \frac{1}{e^{-i\xi} - z_{N,4k}} + \frac{z_{N,4k-r} - z_{N,4k}}{(e^{-i\xi} - z_{N,4k})^2} \\
 &\quad + \frac{(z_{N,4k-r} - z_{N,4k})^2}{(e^{-i\xi} - z_{N,4k})^2(e^{-i\xi} - z_{N,4k-r})}.
 \end{aligned}$$

We have, writing  $\tilde{k} = \min\{k, q+1-k\}$

$$\left| \frac{(z_{N,4k-r} - z_{N,4k})^2}{(e^{-i\xi} - z_{N,4k})^2(e^{-i\xi} - z_{N,4k-r})} \right| \leq C \frac{\frac{1}{N\tilde{k}}}{\left(\frac{\tilde{k}}{N} + \cos^2 \xi\right)^{3/2}} \leq \frac{C \frac{1}{\tilde{k}} \sqrt{\frac{1}{N\tilde{k}}}}{\frac{\tilde{k}}{N} + \cos^2 \xi}$$

and

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{d\xi}{\frac{\tilde{k}}{N} + \cos^2 \xi} &\leq 4 \int_0^{\arccos \sqrt{\tilde{k}/N}} \frac{d\xi}{\cos^2 \xi} + \frac{4N}{\tilde{k}} \int_{\arccos \sqrt{\tilde{k}/N}}^{\pi/2} d\xi \\ &= 4\sqrt{\frac{N}{\tilde{k}}} \sin \left( \arccos \sqrt{\frac{\tilde{k}}{N}} \right) + \frac{4N}{\tilde{k}} \arcsin \sqrt{\frac{\tilde{k}}{N}} \\ &\leq 4\sqrt{\frac{N}{\tilde{k}}} + 2\pi\sqrt{\frac{N}{\tilde{k}}}, \end{aligned}$$

so that

$$\begin{aligned} \int_{-\pi}^{\pi} \left| \frac{d\omega_N}{d\xi} - \frac{N}{2} - \operatorname{Im} \sum_{k=1}^q \frac{z_{N,4k-3} - z_{N,4k-2} - z_{N,4k-1} + z_{N,4k}}{(e^{-i\xi} - z_{N,4k})^2} \right| \\ \leq C \sum_{k=1}^{\infty} \frac{1}{k^2} = C' < +\infty \end{aligned}$$

and

$$\begin{aligned} \int_{-\pi}^{\pi} \left| \frac{d\omega_N}{d\xi} - \frac{N}{2} \right| d\xi \\ \leq C' + C \sum_{k=1}^q \sqrt{\frac{N}{\tilde{k}}} |z_{N,4k-3} - z_{N,4k-2} - z_{N,4k-1} + z_{N,4k}|. \end{aligned}$$

When  $\tilde{k} \leq k_0$ , we write

$$|z_{N,4k-r} - z_{N,4k+1-r}| = O\left(\frac{1}{\sqrt{N\tilde{k}}}\right)$$

and obtain

$$\sum_{\tilde{k} \leq k_0} \sqrt{\frac{N}{\tilde{k}}} |z_{N,4k-3} - z_{N,4k-2} - z_{N,4k-1} + z_{N,4k}| \leq C \operatorname{Log} k_0.$$

When  $\tilde{k} \geq k_0$ , we may write as in formula (58)

$$\begin{aligned} z_{N,4k-r} &= y_{N,4k-r} + \sqrt{y_{N,4k-r}^2 - 1} + O\left(\frac{\operatorname{Log} \tilde{k}}{\tilde{k}\sqrt{N\tilde{k}}}\right) \\ &= \sqrt{\omega_{N,4k-r}} + \sqrt{\omega_{N,4k-r} + 1} + O\left(\frac{\operatorname{Log} \tilde{k}}{\tilde{k}\sqrt{N\tilde{k}}}\right), \end{aligned}$$

where

$$\begin{aligned} \omega_{N,\ell} &= -e^{-2i\pi(8\ell-1)/(8N+6)} \\ &\quad - \frac{1}{N} e^{-2i\pi(8\ell-1)/(8N+6)} \operatorname{Log} \left( 2\sqrt{2N\pi \sin \left( \frac{8\ell-1}{8N+6} \pi \right)} \right). \end{aligned}$$

We write

$$\sqrt{\alpha + \beta} = \sqrt{\alpha} + \frac{\beta}{\sqrt{\alpha} + \sqrt{\alpha + \beta}} = \sqrt{\alpha} + \frac{\beta}{2\sqrt{\alpha}} - \frac{\beta^2}{2\sqrt{\alpha}(\sqrt{\alpha} + \sqrt{\alpha + \beta})^2}.$$

Now, we have:  $\omega_{N,\ell}$  is order of magnitude 1,  $\omega_{N,\ell} + 1$  is of order of magnitude  $\min \{ \sqrt{\ell/N}, \sqrt{(N+1-\ell)/N} \}$  and  $\omega_{N,\ell+1} - \omega_{N,\ell}$  is of order of magnitude  $1/N$ . Thus, we may write

$$\begin{aligned} \sqrt{\omega_{N,4k-r}} &= \sqrt{\omega_{N,4k}} + O\left(\frac{1}{N}\right) \\ \sqrt{1 + \omega_{N,4k-r}} &= \sqrt{1 + \omega_{N,4k}} + \frac{\omega_{N,4k-r} - \omega_{N,4k}}{2\sqrt{1 + \omega_{N,4k}}} + O\left(\frac{1}{\tilde{k}\sqrt{N\tilde{k}}}\right) \\ &= \sqrt{1 + \omega_{N,4k}} + \frac{e^{-2i\pi(32k-1)/(8N+6)}(1 - e^{2i8r\pi/(8N+6)})}{2\sqrt{1 + \omega_{N,4k}}} \\ &\quad + O\left(\frac{\operatorname{Log} \tilde{k}}{N^2}\right) + O\left(\frac{1}{\tilde{k}\sqrt{N\tilde{k}}}\right) \end{aligned}$$

and finally

$$\begin{aligned} &\frac{\sqrt{N}}{\tilde{k}} |z_{N,4k-3} - z_{N,4k-2} - z_{N,4k-1} + z_{N,4k}| \\ &= \sqrt{\frac{N}{\tilde{k}}} \left| \frac{e^{2i24\pi/(8N+6)} - e^{2i16\pi/(8N+6)} - e^{2i8\pi/(8N+6)} + 1}{2\sqrt{1 + \omega_{N,4k}}} \right| \\ &\quad + O\left(\frac{\operatorname{Log} \tilde{k}}{\tilde{k}^2}\right) + O\left(\frac{1}{\sqrt{N\tilde{k}}}\right) + O\left(\frac{\operatorname{Log} \tilde{k}}{N\sqrt{N\tilde{k}}}\right) \\ &= O\left(\frac{1}{N\tilde{k}}\right) + O\left(\frac{\operatorname{Log} \tilde{k}}{\tilde{k}^2}\right) + O\left(\frac{1}{\sqrt{N\tilde{k}}}\right) + O\left(\frac{\operatorname{Log} \tilde{k}}{N\sqrt{N\tilde{k}}}\right). \end{aligned}$$

We thus have proved Theorem 3, since

$$\sum_1^N \frac{1}{N\tilde{k}} \leq C \frac{\text{Log } N}{N} = o(1),$$

$$\sum_1^N \frac{1}{\sqrt{N\tilde{k}}} \leq C \frac{\sqrt{N}}{\sqrt{N}} = C < +\infty,$$

$$\sum_1^{\infty} \frac{\text{Log } \tilde{k}}{\tilde{k}^2} < +\infty,$$

$$\sum_1^N \frac{\text{Log } \tilde{k}}{N\sqrt{N\tilde{k}}} \leq C \frac{1}{N\sqrt{N}} \sqrt{N} \text{Log } N = o(1).$$

## References.

- [1] Caglar, H., Akansu, A. N., A generalized parametric PR-QMF design technique based on Bernstein polynomial approximation. *IEEE Trans. Signal Processing* **41** (1993), 2314-2321.
- [2] Daubechies, I., Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.* **41** (1988), 909-996.
- [3] Fettis, H. E., Caslin, J. C., Cramer, K. R., Complex zeros of the error function and of the complementary error function. *Math. Comp.* **27** (1973), 401-407.
- [4] Herrmann, O., On the approximation problem in nonrecursive digital filter design. *IEEE Trans. on Circuit Theory* (1971), 411-413.
- [5] Kantorovich, L. V., Sur la convergence de la suite des polynômes de S. Bernstein en dehors de l'intervalle fondamental. *Bull. Acad. Sci. URSS* (1931), 1103-1115.
- [6] Lemarié-Rieusset, P. G., Polynômes de Bernstein en théorie des ondelettes. *C. R. Acad. Sci. Paris* **319** (1994), 21-24.
- [7] Lorentz, G. G., *Bernstein polynomials*. Chelsea Publ., 1986.
- [8] Meyer, Y., *Ondelettes et opérateurs*. vol. I. Hermann, 1990.
- [9] Nevanlinna, R., *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*. Gauthier-Villars, 1929.
- [10] Pólya, G., Szegő, G., *Problems and theorems in Analysis, I*. Springer-Verlag, 1986.

- [11] Strichartz, R., How to make wavelets. *Amer. Math. Monthly* **100** (1993), 539-556.
- [12] Varga, R. S., Edrei, A., Saff, E. B., Zeros of sections of power series. *Lecture Notes in Math.* **1002** (1983), Springer-Verlag.

*Recibido:* 20 de febrero de 1.995

*Revisado:* 10 de noviembre de 1.995

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