Periodic-by-Nilpotent Linear Groups

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ABSTRACT - Let $G$ be a linear group of (finite) degree $n$ and characteristic $p \geq 0$. Suppose that for every infinite subset $X$ of $G$ there exist distinct elements $x$ and $y$ of $X$ with $(x, x^n)$ periodic-by-nilpotent. Then $G$ has a periodic normal subgroup $T$ such that if $p > 0$ then $G/T$ is torsion-free abelian and if $p = 0$ then $G/T$ is torsion-free nilpotent of class at most $\max \{1, n-1\}$ and is isomorphic to a linear group of degree $n$ and characteristic zero. We also discuss the structure of periodic-by-nilpotent linear groups.

In [4] Rouabhi and Trabelsi prove that if $G$ is a finitely generated soluble-by-finite group such that for every infinite subset $X$ of $G$ there exist distinct elements $x$ and $y$ of $X$ with $(x, x^n)$ periodic-by-nilpotent, then $G$ is periodic-by-nilpotent, work that ultimately was prompted by very much earlier work of B.H. Neumann, see [4]. Throughout for any positive integer $n$ we set $n' = \max \{1, n-1\}$. Here we prove the following.

THEOREM. Let $G$ be a linear group of (finite) degree $n$ and characteristic $p \geq 0$. Suppose that for every infinite subset $X$ of $G$ there exist distinct elements $x$ and $y$ of $X$ with $(x, x^n)$ periodic-by-nilpotent. Then $G$ is periodic-by-nilpotent. Further $G$ has a periodic normal subgroup $T$ such that if $p > 0$ then $G/T$ is torsion-free abelian and if $p = 0$ then $G/T$ is torsion-free nilpotent of class at most $n' = \max \{1, n-1\}$ and is isomorphic to a linear group of degree $n$ and characteristic zero.

If $T$ is a periodic normal subgroup of some linear group $G$ of degree $n$ and characteristic zero, then $G/T$ is always isomorphic to a linear group of...
characteristic zero (see [9]), but not necessarily of degree \( n \), so the above situation is unusual. As a simple example, \( SL(2, 5) \) has a faithful representation of degree 2 over the complex numbers (indeed over \( \mathbb{Q}(\sqrt{5}, \sqrt{1}) \)), but the least degree of a faithful representation in characteristic zero of its image \( PSL(2, 5) \cong \text{Alt}(5) \) is 3.

Not every torsion-free nilpotent group is isomorphic to a linear group; for example, a direct product \( D \) of infinitely many copies of the full unitriangular group \( \text{Tr}_1(3, Z) \) over the integers \( Z \) is torsion-free nilpotent of class 2, but is not isomorphic to any linear group of any characteristic. Every torsion-free abelian group is isomorphic to a linear group of arbitrary characteristic ([6] 2.2) and any finitely generated torsion-free nilpotent group is isomorphic to a unipotent linear group over the integers (see [6] Page 23), so the counter example \( D \) above is about as small as one can get.

For any group \( G \) denote its hypercentre by \( \zeta(G) \) and the \( i \)-th terms of its upper and lower central series by \( \zeta_i(G) \) and \( \gamma^i G \) respectively (where \( \zeta_0(G) = \langle 1 \rangle \) and \( \gamma^1 G = G \)). Let \( G \) be a group. If \( \gamma^{m+1} G \) is finite, then \( G/\zeta_m(G) \) is finite (if \( G \) is finitely generated even \( G/\zeta_m(G) \) is finite) and if \( G/\zeta_m(G) \) is finite, then \( \gamma^{m+1} G \) is finite, see [3] 4.25, 4.24 and 4.21, Corollary 2. Something similar happens for periodic-by-nilpotent linear groups.

**Proposition.** Let \( G \) be a linear group of degree \( n \) and characteristic \( p \geq 0 \).

a) Suppose that \( \gamma^{m+1} G \) is periodic for some integer \( m \geq 0 \) and that \( O_p(G) = \langle 1 \rangle \) if \( p > 0 \). Then \( G/\zeta_m(G) \) (and \( \gamma^{m+1} G \)) are locally finite.

b) If \( G/\zeta(G) \) is periodic, then \( \gamma^{n+1} G \) and \( G/\zeta_n(G) \) are locally finite.

c) If \( G/\zeta_m(G) \) is periodic for some integer \( m \geq 0 \), then \( \gamma^{m+1} G \) and \( G/\zeta_m(G) \) are locally finite.

Of course Part c) only adds to Part b) in the Proposition for \( m < n \) and Part a) for \( m > n \) adds nothing to the case \( m = n \) by the Theorem. Perhaps Part b) is a slight surprise, since if \( G/\zeta(G) \) is finite there is no need for any \( \gamma^G \) to be finite, even if \( G \) is also linear (consider the infinite locally dihedral 2-group). If \( G \) is any group with \( G/\zeta_m(G) \) locally finite, then \( \gamma^{m+1} G \) is easily seen to be locally finite.

If \( G \) is the wreath product of a cyclic group of prime order \( p \) by an infinite cyclic group, then \( G \) is isomorphic to a triangular linear group of degree 2 and characteristic \( p \) with \( \gamma^2 G \) periodic (even elementary abelian) and yet \( \zeta(G) = \langle 1 \rangle \). Thus the extra hypothesis if \( p > 0 \) in Part a) cannot be
removed. There is no obvious analogue to Part b) in the context of Part a); if $G$ is a non-cyclic free group, then $G$ is isomorphic to a linear group of degree 2 in any characteristic and yet $\gamma^m G = (1) = \zeta(G)$. Note that there exist hypercentral linear groups of infinite central height, even periodic ones, see [6] 8.3, so for example in Part b) there is no need for $\zeta(G)$ and $\zeta_n(G)$ to be equal.

Let $G$ be any group. Denote its unique maximal periodic normal subgroup by $\tau(G)$ and its unique maximal normal $\pi$-subgroup for $\pi$ some set of primes by $O_{\pi}(G)$. If $G$ is linear, $G^0$ denotes its connected component containing the identity (relative to the Zariski topology).

**Proof of the Theorem.** To begin with, note that if $G$ is a torsion-free, locally nilpotent group with a normal subgroup $H$ such that $G/H$ is periodic and such that $H$ is nilpotent of class $c$, then $G$ is nilpotent of class $c$. This can be derived either from isolator theory, see [2] 2.3.9, or from the Zariski topology using [5], 5.11 and Point 3 on Page 23.

Suppose $G$ is a subgroup of $GL(n, F)$, where $F$ is an algebraically closed field of characteristic $p \geq 0$. If $G$ is not soluble-by-(locally finite), then $G$ contains a free subgroup on an infinite set $X$ by Tits’ Theorem, see [6] 10.17. Then $\langle x, x^y \rangle$ is free of rank 2 for every pair of distinct elements $x$ and $y$ of $X$. Consequently $G$ is soluble-by-(locally finite). By Rouabhi & Trabelsi’s theorem, see [4], the group $G$ is locally (periodic-by-nilpotent) and hence is locally (periodic-by-(torsion-free nilpotent)). Therefore $G$ is periodic-by-(torsion-free, locally nilpotent).

Set $T = \tau(G)$ and suppose $p$ is positive. Clearly $O_p(G) \leq T$, so $G/T$ is isomorphic to a torsion-free, locally nilpotent, linear group of characteristic $p$ by Corollary 1 of [9]. Then $G/T$ is also abelian-by-finite by [6] 3.6 and consequently $G/T$ is abelian by the remark at the beginning of this proof. This settles the positive characteristic case.

From now on assume that $p = 0$. Set $C = C_G(T)$. Then $G/CT$ is finite by [5] 5.1.6. Also $C$ is locally nilpotent, so $C$ has a Jordan decomposition

$$C \leq C_n \times C_d = C_n C = CC_d = GL(n, F),$$

see [6] Chapter 7, especially 7.14 and 7.13 (recall $F$ here is algebraically closed). Here $C_n$ is unipotent, torsion-free and nilpotent of class less than $n$. Set $P = \tau(C_n)$. Then $C_d/P$ is torsion-free, locally nilpotent and abelian-by-finite ([6] 7.7 & 3.5). Therefore $C_d/P$ is abelian. Consequently $P = \tau(C_n C_d)$, $C \cap T = C \cap P$ and $CT/T \cong C/(C \cap P)$ is nilpotent of class at most $\max \{n - 1, 1\} = n'$. But then $G/T$ is torsion-free, locally nilpotent and has a nilpotent subgroup $CT/T$ of finite index and class at most $n'$. Therefore $G$
is torsion-free and nilpotent of class at most $n'$, again by our remark at the beginning.

Finally $G/T$ is isomorphic to a linear group over $F$ of $n$-bounded degree by the theorem of [9], but we need to ensure it is actually isomorphic to a linear group of degree $n$ and characteristic zero. If $n = 1$, then $G$ is abelian and clearly $G/T$ embeds into $F^* = GL(1, F)$, since $F^*$ is divisible and splits over $\tau(F^*)$. Suppose $n > 1$. If $K$ is an extension field of $F$, then the centre of the unitriangular group $\text{Tr}_1(n, K)$ is isomorphic to the additive group of $K$ and hence is equal to $Z \times R$ for $Z$ the centre of $\text{Tr}_1(n, F)$ and $R$ a direct sum of copies of the additive group of the rationals. Further $C_n$ is isomorphic to a subgroup of $\text{Tr}_1(n, F)$ and $C_d/P$ is embeddable in $R$ for a suitably large $K$. In which case $C_nC_d/P$ is isomorphic to a subgroup of $\text{Tr}_1(n, K)$ and hence so too is $CT/T \cong CP/P \leq C_nC_d/P$. Now $\text{Tr}_1(n, K)$ is torsion-free, nilpotent and divisible. Thus $\text{Tr}_1(n, K)$ contains a divisible completion $D$ of $CT/T$. Since $G/T$ is torsion-free, nilpotent and of finite index over $CT/T$, so $D$ contains a copy of $G/T$. (See [2] Chap. 2, especially 2.1.1, for divisible completions of nilpotent groups.). Therefore $G/T$ is isomorphic to a linear group of degree $n$ and characteristic zero. The proof is complete.

As an example of an application of this theorem, we have the following.

**COROLLARY.** Let $G$ be a soluble-by-finite group with finite Hirsch number. Suppose that for every infinite subset $X$ of $G$ there exist distinct elements $x$ and $y$ of $X$ with $(x, x^2)$ periodic-by-nilpotent. Then $G$ is periodic-by-nilpotent.

**PROOF.** For if $T = \tau(G)$, then $G/T$ has a torsion-free soluble normal subgroup of finite rank and index. Then $G/T$ is isomorphic to a linear group over the rationals (e.g. [7] 1.2) and applying the theorem to $G/T$ yields the corollary. Alternatively it follows from [4] as follows. By [4] the group $G$ is locally periodic-by-nilpotent, so $G$ is periodic-by-torsion-free and locally nilpotent of finite rank). Consequently $G$ is periodic-by-nilpotent by a theorem of Mal’cev ([3] 6.36).

The Proposition follows at once from the following three lemmas.

**LEMMA 1.** Let $G$ be a linear group of degree $n$ and characteristic $p \geq 0$ such that $O_p(G) = \{1\}$ if $p > 0$. If there exists an integer $m \geq 0$ such that $\gamma^{m+1}G$ is periodic, then $G/\zeta_m(G)$ is locally finite.

Note that here $\gamma^{m+1}G$ is locally finite by [6] 4.9.
PROOF. Let $X$ be a finitely generated subgroup of $G$. Suppose first that $p = 0$. Then $\gamma^{m+1}X$ is finite by [6] 4.8 and therefore $X/\zeta_m(X)$ is finite by [3] 4.24. Also $\zeta_m(X)$ is closed in $X$ by [6] 5.10, so $X^0 \leq \zeta_m(X)$. If $Y$ is a finitely generated subgroup of $G$ containing $X$, then

$$X^0 \leq X \cap Y^0 \leq \zeta_m(Y).$$

Thus $X^0 \leq \zeta_m(G)$. Set $G^* = \bigcup_X X^0$. Then $G^*$ is a normal subgroup of $G$ with $G/G^*$ locally finite and $G^* \leq \zeta_m(G)$. The case $p = 0$ follows.

Now assume that $p > 0$. Here [6] 4.8 only yields that $\gamma^{m+1}X$ is a finite extension of a $p$-group. Define $Z_i(X)$ by

$$Z_i(X)/O_p(X) = \zeta_i(X/O_p(X)).$$

Then $X/Z_m(X)$ is finite by [3] 4.24 and $O_p(X)$ and $Z_m(X)$ are closed in $X$, so $X^0 = Z_m(X)$. Thus $X^0 \leq \bigcap_{Y \geq X} X \cap Z_m(Y)$ and a simple localizing argument (cf. the previous paragraph) yields that $[G^*, mG]$ is a $p$-group. Clearly it is normal in $G$ and $O_p(G) = (1)$. Therefore $G^* \leq \zeta_m(G)$ and the lemma follows.

LEMMA 2. Let $G$ be a linear group of degree $n$ with $G/\zeta_m(G)$ periodic for some integer $m \geq 0$. Then $\gamma^{m+1}G$ and $G/\zeta_m(G)$ are locally finite.

PROOF. Now $\zeta_m(G)$ is closed in $G$, so $G/\zeta_m(G)$ is isomorphic to a periodic linear group ([6] 6.4) and so is locally finite ([6] 4.9). Then [3] 4.21, Corollary 2, yields that $\gamma^{m+1}X$ is finite for every finitely generated subgroup $X$ of $G$. Consequently $\gamma^{m+1}G$, which equals $\bigcup_X \gamma^{m+1}X$, is locally finite.

LEMMA 3. Let $G$ be a linear group of degree $n$ and characteristic $p \geq 0$ such that $G/\zeta(G)$ is periodic. Then $\gamma^{m+1}G$ and $G/\zeta(G)$ are locally finite.

PROOF. We may assume that the ground field $F$ of $G$ is algebraically closed. If $g \in GL(n, F)$, let $g = g_u g_d = g_u g_d$ be its Jordan decomposition (see [6] Chapter 7). Set

$$K = \langle g_u, g_d : g \in \zeta(G) \rangle \leq GL(n, F)$$

Then $K = \zeta(G) \cap N = \zeta(G)$ and $K = K_u \times K_d$, where $K_u$ is unipotent and $K_d$ is a $d$-subgroup, see [6] 7.17, 7.14 and 7.13. Also $GK/K \cong G/\zeta(G)$, which is periodic.

By the theorem of [8] we have $K_u \leq \zeta_n(GK)$. Set $D = (K_d)^0$. Then $D$ is a diagonalizable normal subgroup of $GK$ by [6] 7.7 and 5.8. Let $n$ denote the finite set of all primes not exceeding $n$. Then $O_n(D)$ has finite rank (at most $n$), so $D$ splits over $O_n(D)$ by [1] 21.2 and 27.5, say $D = O_n(D) \times E$. Also
GK/C_{GK}(D) is a finite \(\pi\)-group by [6] 1.12. Then \(H = \bigcap_{y \in GK} E^y\) is a normal subgroup of \(GK\) with \(O_\pi(H) = (1)\) and \(D/H\) a periodic \(\pi\)-group. Also \([H, GK]\) is a \(\pi\)-group by [6] 4.14. Therefore \([H, GK] = (1)\). We have now shown that \(K_u \times H \cong \zeta^u(GK)\). Since \(GK/K\) and \(D/H\) are periodic and \(K_u/D\) is finite, so \(GK/\zeta^u(GK)\) is periodic. It follows that \(G/\zeta^u(G)\) is periodic. Finally \(\gamma^{u+1}G\) and \(G/\zeta^u(G)\) are locally finite by Lemma 2.

REFERENCES


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