A Short Proof of the Hölder-Poincaré Duality for
$L_p$-Cohomology

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Abstract - We give a short proof of the duality theorem for the reduced $L_p$-cohomology of a complete oriented Riemannian manifold.

Let $(M, g)$ be an oriented Riemannian manifold. For any $1 \leq p < \infty$ we denote by $L^p (M, \mathcal{A})$ the space of $p$-integrable differential forms on $M$. An element of that space is a measurable differential $k$-form $\omega$ such that

$$\| \omega \|_p := \left( \int_M |\omega|^p d\text{vol}(x) \right)^{1/p} < \infty.$$

Recall that a differential form $\theta \in L^p (M, \mathcal{A}^{k+1})$ is the weak exterior differential of the form $\hat{\phi} \in L^p (M, \mathcal{A}^k)$ if one has

$$\int_M \theta \wedge \omega = (-1)^{k+1} \int_M \hat{\phi} \wedge d\omega$$

for any $\omega \in \mathcal{D}^{n-k} (M)$, where $\mathcal{D}^m (M)$ denotes the vector space of smooth differential $m$-forms with compact support in $M$.

One writes $d\hat{\phi} = \theta$ if $\theta$ is the weak exterior differential of $\hat{\phi}$ and $Z^k_p (M) = \ker d \cap L^p (M, \mathcal{A}^k)$ denotes the set of weakly closed forms in $L^p (M, \mathcal{A}^k)$. It is easy to check that $Z^k_p (M)$ is a closed linear subspace of

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\(L^p(M, \mathcal{A})\), in particular it is a Banach space (see [5, Lemma 2.2]). We then introduce the space
\[
B^k_p(M) = d\left(L^p(M, \mathcal{A}^{k-1})\right) \cap L^p(M, \mathcal{A})
\]
of exact \(L^p\)-forms and we shall denote by \(\overline{B}^k_p(M)\) the closure of \(B^k_p(M)\) in \(L^p(M, \mathcal{A})\). Because \(Z^k_p(M) \subseteq L^p(M, \mathcal{A})\) is a closed subspace and \(d \circ d = 0\), we have \(\overline{B}^k_p(M) \subseteq Z^k_p(M)\). The reduced \(L_p\)-cohomology of \((M, g)\) (where \(1 \leq p < \infty\)) is defined to be the quotient
\[
\overline{H}^k_p(M) = Z^k_p(M) / \overline{B}^k_p(M).
\]

This is a Banach space for the natural (quotient) norm and the goal of this paper is to prove the following Theorem (here and throughout the paper, \(p' = p/(p - 1)\) is the conjugate number of \(p\)).

**Duality Theorem.** Let \((M, g)\) be a complete oriented Riemannian manifold of dimension \(n\) and \(1 < p < \infty\). Then \(\overline{H}^k_p(M)\) is isometric to the dual of \(\overline{H}^{n-k}_p(M)\). The duality is given by the integration pairing:

\[
\overline{H}^k_p(M) \times \overline{H}^{n-k}_p(M) \rightarrow \mathbb{R}
\]
\[
([\omega], [\theta]) \mapsto \int_M \omega \wedge \theta.
\]

**Remark.** By “the dual space” \(X'\) of a Banach space \(X\), we of course mean the topological dual, i.e. the vector space of continuous linear functionals together with its natural norm. The isomorphism between \(\overline{H}^k_p(M)\) and the dual of \(\overline{H}^{n-k}_p(M)\) has first been proved in 1986 by V. M. Gol’dshtein, V.I. Kuz’minov and I.A. Shvedov, see [4]. In fact that paper also describes the dual space to the \(L_p\)-cohomology of non complete manifolds. The proof we present here is simpler and more direct than the proof in [4], although it doesn’t seem to be extendable to the non complete case. Note that this duality theorem is useful to prove vanishing or non vanishing results in \(L_p\)-cohomology, see e.g. [5, 7, 8].

Let us also mention that Gromov deduced the above theorem from the simplicial version of the \(L_p\)-cohomology, see [7]. Gromov’s argument works only for Riemannian manifolds with bounded geometry, while the proof we give here works for any complete manifold. Our proof can also be extended to the more general \(L_{q,p}\)-cohomology, see [6].

The proof will rest on a few auxiliary facts. Recall first that a pairing between two Banach spaces \(X_0\) and \(X_1\) is simply a continuous bilinear map
$I : X_0 \times X_1 \to \mathbb{R}$. Such a pairing defines two continuous linear maps $\lambda : X_0 \to X_1'$, and $\mu : X_1 \to X_0'$ defined by

$$\lambda_\xi(\eta) = \mu_\eta(\xi) = I(\xi, \eta),$$

for any $\xi \in X_0$ and $\eta \in X_1$.

**Definition 1.** An *isometric duality* between two Banach spaces $X_0$ and $X_1$ is a pairing $I : X_0 \times X_1 \to \mathbb{R}$ such that the associated maps $\lambda : X_0 \to X_1'$, and $\mu : X_1 \to X_0'$ are bijective isometries.

Observe that if an isometric duality exists between two Banach spaces, then these spaces are reflexive. The classic $L^p-L^{p'}$ duality for function spaces extends to the case of differential forms, see [4]:

**Proposition 2.** If $1 < p < \infty$, then the pairing $L^p(M, \mathbb{A}^k) \times L^{p'}(M, \mathbb{A}^{k-\ell}) \to \mathbb{R}$ defined by

$$(1) \quad \langle \omega, \varphi \rangle = \int_M \omega \wedge \varphi$$

is an isometric duality. In particular, $L^p(M, \mathbb{A}^k)$ is a reflexive Banach space.

We will also need the following density result whose proof is based on regularization methods, see e.g. [3, 5]:

**Proposition 3.** Let $\theta \in L^p(M, \mathbb{A}^{k-1})$ be a $(k-1)$-form whose weak exterior differential is $p$-integrable, $d\theta \in L^p(M, \mathbb{A}^k)$. Then there exists a sequence $\theta_j \in C^\infty(M, \mathbb{A}^{k-1})$ such that $\theta = \lim_{j \to \infty} \theta_j$ and $d\theta = \lim_{j \to \infty} d\theta_j$ in $L^p(M)$.

The next lemma is the place where the completeness hypothesis enters:

**Lemma 4.** If $(M, g)$ is complete, then $d\mathcal{D}^{k-1}(M)$ is dense in $B^k_p(M)$.

**Proof.** Because $M$ is complete, one can find a sequence of smooth functions with compact support $\{\eta_j\} \subseteq C_0^\infty(M)$ such that $0 \leq \eta_j \leq 1$, $\lim_{j \to \infty} \sup_{M} |d\eta_j| = 0$ and $\eta_j \to 1$ uniformly on every compact subset of $M$. Let $\omega \in B^k_p(M)$, then there exists $\theta \in L^p(M, \mathbb{A}^{k-1})$ such that $d\theta = \omega$. Choose a sequence $\{\theta_j\} \subseteq C^\infty(M, \mathbb{A}^{k-1})$ as in Proposition 3, i.e. $\theta_j \to \theta$ and $d\theta_j \to d\theta = \omega$. Then

$$d\mathcal{D}^{k-1}(M) \subseteq B^k_p(M).$$
\[ \omega \in L^p(M) \text{ and set } \tilde{\theta}_j = \eta_j \theta_j \in \mathcal{D}^{k-1}(M). \text{ We first claim that } (\tilde{\theta}_j - \theta_j) \to 0 \text{ in } L^p(\mathcal{M}, \mathcal{A}^{k-1}). \text{ Indeed, fix } \varepsilon > 0 \text{ and choose a compact set } Q \text{ such that } \|\tilde{\theta}\|_{L^p(\mathcal{M}, Q)} < \varepsilon. \text{ Since } |\eta_j - 1| < 1, \text{ we have }
\begin{align*}
\|\tilde{\theta}_j - \theta_j\|_{L^p(M)} &\leq \|(\eta_j - 1)\theta_j\|_{L^p(Q)} + \|\theta_j\|_{L^p(\mathcal{M}, Q)} \\
&\leq \|(\eta_j - 1)\theta_j\|_{L^p(Q)} + \|\theta_j - \theta\|_{L^p(\mathcal{M}, Q)} + \|\theta\|_{L^p(\mathcal{M}, Q)}.
\end{align*}
\]

The first term converges to zero because \( \eta_j \to 1 \) uniformly on \( Q \) and \( \{\|\eta_j\|_{L^p}\} \) is bounded. The second term converges to zero because \( \theta_j \to \theta \) in \( L^p(M, \mathcal{A}^{k-1}) \) and the last term is bounded by \( \varepsilon \), hence
\[
\limsup_{j \to \infty} \|\tilde{\theta}_j - \theta_j\|_{L^p(M)} \leq \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, the limit is zero and we obtain
\[
\lim_{j \to \infty} \|\tilde{\theta}_j - \theta\|_{L^p(M)} \leq \lim_{j \to \infty} \|\tilde{\theta}_j - \theta_j\|_{L^p(M)} + \lim_{j \to \infty} \|\theta_j - \theta\|_{L^p(M)} = 0.
\]

We similarly have
\[
\|d\tilde{\theta}_j - d\theta_j\|_{p} \leq \|(\eta_j - 1)d\theta_j\|_{p} + \|d\eta_j \land \theta_j\|_{p} \\
\leq \|(\eta_j - 1)d\theta_j\|_{p} + \sup |d\eta_j| \cdot \|\theta_j\|_{p} \to 0.
\]

This implies that \( \omega = \lim_{j \to \infty} d\tilde{\theta}_j \) in \( L^p \). \( \square \)

**Definition 5.** Given an isometric duality \( I : X_0 \times X_1 \to \mathbb{R} \) and a nonempty subset \( B \) of \( X_0 \), we define the annihilator \( B^\perp \subseteq X_1 \) of \( B \) to be the set of all elements \( \eta \in X_1 \) such that \( I(\xi, \eta) = 0 \) for all \( \xi \in B \).

For any \( B \subseteq X_0 \) the annihilator \( B^\perp \) is a closed linear subspace of \( X_1 \). The Hahn-Banach Theorem implies that if \( B \) is a linear subspace of \( X_0 \) then \( (B^\perp)^\perp = \overline{B} \).

For these and further facts on the notion of annihilator, we refer to the books [1, 2].

The proof of the duality Theorem is based on the following lemma about annihilators:

**Lemma 6.** Let \( I : X_0 \times X_1 \to \mathbb{R} \) be an isometric duality between two Banach spaces. Let \( B_0, A_0, B_1, A_1 \) be linear subspaces such that 
\[
B_0 \subseteq A_0 = B_1^\perp \subseteq X_0 \quad \text{and} \quad B_1 \subseteq A_1 = B_0^\perp \subseteq X_1.
\]

Then the pairing \( I : \overline{B_0} \times \overline{A_1} \to \mathbb{R} \) of \( \overline{B_0} := A_0/B_0 \) and \( \overline{A_1} := A_1/B_1 \) is well defined and induces an isometric duality between \( \overline{B_0} \) and \( \overline{A_1} \).
PROOF. Observe first that $A_i \subseteq X_i$ is a closed subspace since the annihilator of any subset of a Banach space is always a closed linear subspace.

The bounded bilinear map $I : A_0 \times A_1 \rightarrow \mathbb{R}$ is defined by restriction. It gives rise to a well defined bounded bilinear map $I : A_0/\mathcal{B}_0 \times A_1/\mathcal{B}_1 \rightarrow \mathbb{R}$ because we have the inclusions $B_0 \subseteq B_1$ and $B_1 \subseteq B_0$.

We denote by $\lambda : X_0 \rightarrow X_1$ the isometry induced by the pairing $I$, by $\mathcal{F} : \mathcal{H}_0 \rightarrow \mathcal{H}_1$ the map defined by the pairing $I$ and by $\pi_i : A_i \rightarrow \mathcal{H}_i$ ($i = 1, 2$) the canonical projections.

We first prove that $\|\lambda \xi\|_{\mathcal{H}_1} \leq \|\xi\|_{\mathcal{H}_0}$ for any $\xi \in \mathcal{H}_0$. Indeed, let us choose $\tilde{\xi} \in A_0$ such that $\pi_0(\tilde{\xi}) = \xi$, we have

$$\|\lambda \xi\|_{\mathcal{H}_1} = \sup \{ I(\tilde{\xi}, \eta) \mid \eta \in \mathcal{H}_1, \|\eta\|_{\mathcal{H}_1} \leq 1 \} \leq \sup \{ I(\tilde{\xi}, \eta) \mid \eta \in A_1, \|\eta\|_{\mathcal{A}_1} \leq 1 \} \leq \sup \{ I(\tilde{\xi}, \eta) \mid \eta \in X_1, \|\eta\|_{X_1} \leq 1 \} = \|\tilde{\xi}\|_{X_1}.$$ 

By hypothesis, we have $\|\tilde{\xi}\|_{X_0} = \|\tilde{\xi}\|_{X_1}$, therefore

$$\|\xi\|_{\mathcal{H}_0} = \inf_{\tilde{\xi} \in \mathcal{C}_{\xi}(\mathcal{A})} \|\tilde{\xi}\|_{X_0} = \inf_{\tilde{\xi} \in \mathcal{C}_{\xi}(\mathcal{A})} \|\tilde{\xi}\|_{X_1} \geq \|\lambda \xi\|_{\mathcal{H}_1}.$$ 

We then prove that for any $\theta \in \mathcal{H}_1$, there exists an element $\xi \in \mathcal{H}_0$ such that $\theta = \lambda \xi$ and $\|\theta\|_{\mathcal{H}_1} \geq \|\xi\|_{\mathcal{H}_1}$. This implies that $\mathcal{F}$ is surjective and $\|\lambda \xi\|_{\mathcal{H}_1} \geq \|\xi\|_{\mathcal{H}_0}$.

Indeed, for any $\theta \in \mathcal{H}_1$, the linear form $\hat{\theta} = \theta \circ \pi_1 : A_1 \rightarrow \mathbb{R}$ satisfies $\hat{\theta}(b) = 0$ for any $b \in B_1$ and $\|\hat{\theta}\|_{\mathcal{A}_1} = \|\theta\|_{\mathcal{H}_1}$. By the Hahn-Banach Theorem, there exists a continuous extension $\hat{\phi} : X_1 \rightarrow \mathbb{R}$ of $\hat{\theta}$ such that $\|\hat{\phi}\|_{X_1} = \|\hat{\theta}\|_{\mathcal{A}_1}$. Since $\lambda : X_0 \rightarrow X_1$ is an isometry, one can find $\tilde{\xi} \in X_0$ such that $\lambda \tilde{\xi} = \hat{\phi}$ and

$$\|\tilde{\xi}\|_{X_0} = \|\hat{\phi}\|_{X_1} = \|\hat{\theta}\|_{\mathcal{A}_1} = \|\theta\|_{\mathcal{H}_1}.$$ 

For any $b \in B_1$, we have $I(\tilde{\xi}, b) = \lambda \tilde{\xi}(b) = \hat{\theta}(b) = 0$, thus $\tilde{\xi} \in B_1 = A_0$. Let us set $\xi = \pi_0(\tilde{\xi})$, we have

$$I(\xi, \eta) = I(\tilde{\xi}, \eta) = \theta(\eta) = \theta(\eta)$$

for any $\eta \in \mathcal{H}_1$ and $\eta \in A_1$, that is $\theta = \lambda \xi$. We also have

$$\|\xi\|_{\mathcal{H}_0} \leq \|\tilde{\xi}\|_{X_0} = \|\theta\|_{\mathcal{H}_1} = \|\lambda \xi\|_{\mathcal{H}_1}.$$ 

In conclusion, we have have proved that $\lambda : \mathcal{H}_0 \rightarrow \mathcal{H}_1$ is norm preserving and surjective: it is an isometry. The proof that $\pi : \mathcal{H}_1 \rightarrow \mathcal{H}_0$ is also an isometry is the same.
PROOF OF THE DUALITY THEOREM. Let $\hat{\phi} \in L^p(M, \mathcal{A}^k)$, then $d\hat{\phi} = 0$ in the weak sense if and only if $\int_M \hat{\phi} \wedge d\omega = 0$ for any $\omega \in \mathcal{D}^{n-k-1}(M)$. This precisely means that $Z^k_p(M) \subseteq L^p(M, \mathcal{A}^k)$ is the annihilator of $d\mathcal{D}^{n-k-1}(M) \subseteq L^p(M, \mathcal{A}^{n-k})$ for the pairing (1):

$$Z^k_p(M) = (d\mathcal{D}^{n-k-1})^\perp(M).$$

By lemma 1, $d\mathcal{D}^{n-k-1}(M)$ and $B^{n-k}_p$ have the same annihilator, thus

$$B^k_p \subseteq Z^k_p = (B^{n-k}_p)^\perp \subseteq L^p(M, \mathcal{A}^k).$$

Similarly, we also have

$$B^{n-k}_p \subseteq Z^{n-k}_p = (B^k_p)^\perp \subseteq L^p(M, \mathcal{A}^{n-k}),$$

and Lemma 1 says that the duality (1) induces an isometric duality between $Z^{n-k}_p/B^{n-k}_p$ and $Z^k_p/B^k_p$. \qed

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REFERENCES


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