Hölder Type Estimates for the $\overline{\partial}$-Equation in Strongly Pseudoconvex Domains.

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ABSTRACT - In this paper we generalize the Hölder space with a majorant function, define its order, and prove the existence and regularity for the solutions of the Cauchy-Riemann equation in the generalized Hölder space over a bounded strongly pseudoconvex domain.

1. Introduction and regular majorant.

If $D$ is a bounded domain in $\mathbb{C}^n$, the Hölder space of order $\alpha$, $A_\alpha(D)$ ($0 < \alpha < 1$), is defined as the set of all functions $g$ on $D$ which satisfy for a constant $C = C_g > 0$ the condition

$$|g(z) - g(\zeta)| \leq C|z - \zeta|^{\alpha}, \quad z, \zeta \in D.$$ 

We first generalize this Hölder space following Dyakonov [Dya97] (also see Pavlović’s book [Pav04]). For this purpose we introduce the notion of a regular majorant. Let $\omega$ be a continuous increasing function on $[0, \infty)$. We assume $\omega(0) = 0$, and suppose that $\omega(t)/t$ is non-increasing.

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and satisfies the inequality

\[ \int_0^\delta \frac{\omega(t)}{t} \, dt + \delta \int_\delta^\infty \frac{\omega(t)}{t^2} \, dt \leq C \omega(\delta), \quad \text{for any } 0 < \delta < 1, \]

for a suitable constant \( C = C(\omega) \). Such a function \( \omega \) is called a regular majorant. Given a regular majorant \( \omega \), the Hölder type space, \( A(\omega, D) \) is defined as the family of all functions \( g \) on \( D \) such that

\[ |g(z) - g(\zeta)| \leq C \omega(|z - \zeta|), \quad z, \zeta \in D. \]

The norm \( \|g\|_\omega \) of \( g \in A(\omega, D) \) is given by \( C_g + \|g\|_\infty \), where \( C_g \geq 0 \) is the smallest constant satisfying (1.2) and \( \|g\|_\infty \) is the \( L^\infty \) norm in \( D \). Note that with this norm \( A(\omega, D) \) is a Banach space and \( A(\omega, D) \subset L^\infty(D) \) (see the chapter 10 of Pavlović’s book [Pav04]). We denote by \( A_q(\omega, D) \) the set of differential forms of type \((0, q)\) whose coefficients are in \( A(\omega, D) \). We define the order of a regular majorant as follows:

**Definition 1.1.** We say that a regular majorant \( \omega \) has order \( \alpha \) \((0 < \alpha < 1)\) if there exist an \( \alpha \) and a positive real number \( t_0 \) such that

\[
\alpha = \sup \left\{ \gamma : \frac{\omega(t)}{t^\gamma} \text{ is increasing } \forall t, \ 0 < t < t_0 \right\} = \inf \left\{ \gamma : \frac{\omega(t)}{t^\gamma} \text{ is decreasing } \forall t, \ 0 < t < t_0 \right\}.
\]

If a regular majorant \( \omega \) has order \( \alpha \), then we let \( \omega = \omega_\alpha \) and call \( A(\omega_\alpha, D) \) the Hölder type space of order \( \alpha \). By definition of the order of a regular majorant, it is uniquely determined, if it exists. Now we state our main result of this paper.

**Theorem 1.2.** Let \( D \subset \subset C^n \quad (n \geq 2) \) be a strongly pseudoconvex domain with \( C^4 \)-boundary and \( 0 < \alpha < 1/2 \). If a regular majorant \( \omega_\alpha \) has order \( \alpha \) and \( f \in A_q(\omega_\alpha, D) \) with \( \overline{\partial} f = 0 \quad (1 \leq q \leq n) \), then there is a solution \( u \in A_{(q-1)}(t^{1/2}\omega_\alpha, D) \) of \( \overline{\partial} u = f \) such that for some constant \( C = C(\omega_\alpha) \)

\[
\|u\|_{t^{1/2}\omega_\alpha} \leq C \|f\|_{\omega_\alpha}.
\]

The above inequality (1.3) generalizes the estimate by Henkin-Romanov [RH71] and Lieb-Range [LR80]

\[
\|u\|_{t^{1/2}} \leq C \|f\|_{t^{1/2}}.
\]
For the proof of the Hölder type estimate (1.3), we need a variant of the Hardy-Littlewood Lemma \cite{HL84}.

**Lemma 1.3.** Let $D \subset \subset \mathbb{R}^n$ be a bounded domain with $C^1$ boundary. If $g$ is a $C^1(D)$-function and $\omega_\gamma$ is a regular majorant of order $\gamma$, $0 < \gamma < 1$ such that for some constant $c_g$ depending on $g$,

$$|dg(x)| \leq c_g \frac{\omega_\gamma(|\rho(x)|)}{|\rho(x)|}, \quad x \in D,$$

then we have

$$|g(x) - g(y)| \leq c_g \omega_\gamma(|x - y|).$$

As convention we use the notation $A \leq B$ or $A \geq B$ if there are constants $c_1, c_2$, independent of the quantities under consideration, satisfying $A \leq c_1B$ and $A \geq c_2B$, respectively.

Before proving our theorem, we discuss some properties of a regular majorant and some examples.

**Example 1.4.** (i) The most typical example is a function $\omega(t) = t^\alpha$ ($0 < \alpha < 1$). Clearly, $\omega$ is a regular majorant and has order $\alpha$.

(ii) A non-trivial example is the function, $\omega(t) = t^\alpha \log t^\beta$ on $[0, t_0]$ extended continuously for $t > t_0$ to be a regular majorant. Here $0 < \alpha < 1$, $-\infty < \beta < \infty$ and $t_0$ must be chosen sufficiently small so that the function $\omega$ should be a regular majorant ($t_0$ depends on $\alpha, \beta$). Since $\lim_{t^\varepsilon} t^\epsilon \log t^\beta = 0$ for any $\varepsilon > 0$, it follows that $\omega(t) = t^\alpha \log t^\beta$ has order $\alpha$ for any choice of $\beta$.

(iii) Define the function $m(t) = 1/|\log t|, \beta > 0$ for $0 < t < t_0$ and $m(0) = 0$. Then $m(t)$ is continuous and increasing near 0, but it is not a regular majorant.

We end this section by describing useful properties of a regular majorant.

**Remark 1.5.** (i) If $\omega, m$ are two regular majorants and have orders $\alpha, \beta$ respectively with $0 < \alpha < \beta < 1$, then letting $\omega_\alpha, m_\beta$, there exist $t_0 > 0$ and $c$ such that $m_\beta(t) \leq c\omega_\alpha(t), 0 \leq t \leq t_0$. Hence we have the inclusion $\Lambda(m_\beta, D) \subset \Lambda(\omega_\alpha, D)$. Note that if two regular majorants, $\omega, m$ have the same order $\alpha$, then generally there is no inclusion relation between $\Lambda(\omega, D)$ and $\Lambda(m, D)$.

(ii) In our Theorem 1.2, for a general regular majorant of order $1/2$, the
estimate \( \|u\|_{\omega_{1/2}} \lesssim \|f\|_{\infty} \) does not hold. In fact, the celebrated Henkin’s theorem [RH71] holds only for the special regular majorant \( \omega_{1/2}(t) = |t|^{1/2} \) and this number 1/2 is the sharp bound [Ran86]. But there is a regular majorant \( m_{1/2}(t) = |t|^{1/2} \log t \) near the origin of order 1/2, which is strictly bigger than \( |t|^{1/2} \).

**Remark 1.6.** Let \( \omega \) be a regular majorant of order \( \alpha (0 < \alpha < 1/2) \), say \( \omega = \omega_{z} \). Then \( t^{1/2}\omega_{z} \) is also a regular majorant of order \( (\alpha + 1/2) \). In fact, \( t^{1/2}\omega_{z} \) is increasing and \( (t^{1/2}\omega_{z})/t \) is non-increasing, since \( \omega_{z}/t^{\gamma} \) is decreasing. Here we use the fact that \( \omega_{z} \) has order \( \alpha \). It remains to show that \( t^{1/2}\omega_{z} \) also satisfies (1.1). Since \( (t^{1/2}\omega_{z})/t \) is non-increasing, we have for any \( \delta, (0 < \delta < 1) \),

\[
\int_{0}^{\delta} \frac{s^{1/2}\omega_{z}(s)}{s} \, ds \lesssim \delta^{1/2} \int_{0}^{\delta} \frac{\omega_{z}(s)}{s} \, ds \lesssim \delta^{1/2} \omega_{z}(\delta).
\]

On the other hand, for a given \( 0 < \alpha < 1/2 \), we can choose a sufficiently small \( \varepsilon \) such that \( \alpha < 1/2 - \varepsilon \). It follows from the order of \( \omega_{z} \) that \( \omega_{z}/t^{(1/2-\varepsilon)} \) is decreasing. Hence we obtain

\[
\delta \int_{\delta}^{\infty} \frac{s^{1/2}\omega_{z}(s)}{s^{2}} \, ds = \delta \int_{\delta}^{\infty} \omega_{z}(s) \frac{1}{s^{1/2-\varepsilon} s^{1+\varepsilon}} \, ds 
\]

\[
\lesssim \delta \cdot \omega_{z}(\delta) \delta^{1/2-\varepsilon} \int_{\delta}^{\infty} \frac{1}{s^{1+\varepsilon}} \, ds \lesssim \delta^{1/2} \omega_{z}(\delta).
\]

By (1.4) and (1.5), \( t^{1/2}\omega_{z} \) is a regular majorant of order \( (\alpha + 1/2) \).

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## 2. Henkin’s solution operator of the \( \bar{\partial} \)-equation.

In this section, we introduce the Henkin’s solution operator [HL84] of the \( \bar{\partial} \)-equation and prove the integral estimates for the solution operator in a strongly pseudoconvex domain in \( \mathbb{C}^{n} \). Let \( D \) be defined by a function \( \rho \), i.e., \( D = \{ z \in \mathbb{C}^{n} : \rho(z) < 0 \} \), where \( \rho \in C^{4} \) and \( \nabla \rho \neq 0 \) on \( bD \).

To construct the integral formula for solutions of the \( \bar{\partial} \)-equation in a strongly pseudoconvex domain, we need a support function (see [HL84]). For
the global support function, we follow Fornaess construction [For76]. He showed that there exist a neighborhood $U$ of $\overline{D}$ and a function $\phi(\cdot, \cdot) \in C^3(U \times U)$ such that for all $\zeta \in U$, $\phi(\zeta, \cdot)$ is holomorphic in $U$ and $\phi(\zeta, z) = \langle \Phi, \zeta - z \rangle$, where we define $\Phi = \Phi(\zeta, z) = (\phi_1(\zeta, z), \ldots, \phi_n(\zeta, z))$ and $\langle \Phi, \zeta - z \rangle = \sum_{j=1}^n \phi_j(\zeta, z)(\zeta_j - z_j)$. In [For76], Fornaess also showed that $\phi_j \in C^3(U \times U)$ is holomorphic in $z$ and there is a constant $c$ such that for all $z \in \overline{D}$ and $\zeta \in D$ we have

\begin{equation}
2 \text{Re} \phi(\zeta, z) \geq \rho(\zeta) - \rho(z) + c|\zeta - z|^2
\end{equation}

and $d_{\bar{z}}\phi(\zeta, z)|_{z=\zeta} = \partial \rho(\zeta)$. Suppose that $f \in \mathcal{A}_q(\omega_D, D) (1 \leq q \leq n)$ and $\partial f = 0$. Then $f$ is uniformly continuous in $D$. Using the above global support function $\phi$, we define Henkin kernel $H(\zeta, z)$ as follows:

$$H(\zeta, z) = \frac{1}{(2\pi i)^n} \frac{\langle \zeta - \bar{z}, d\zeta \rangle}{|\zeta - z|^2} \wedge \frac{\langle \Phi, d\zeta \rangle}{\langle \Phi, \zeta - z \rangle} \wedge \sum_{k+l=n-2} \left( \frac{\langle \zeta - \bar{z}, d\zeta \rangle}{|\zeta - z|^2} \right)^k \left( \frac{\langle d\zeta - \bar{d}z, d\zeta \rangle}{|\zeta - z|^2} \right)^l,$$

where $\partial_{\bar{\zeta}} \Phi = \bar{\partial}_{\zeta} \Phi$ and $d\zeta = (d\zeta_1, \ldots, d\zeta_n)$. Note that $\Phi$ is holomorphic in $z$. We also define the Bochner–Martinelli kernel:

$$K(\zeta, z) = \frac{1}{(2\pi i)^n} \frac{\langle \zeta - \bar{z}, d\zeta \rangle}{|\zeta - z|^2} \wedge \left( \frac{\langle d\zeta - \bar{d}z, d\zeta \rangle}{|\zeta - z|^2} \right)^{n-1}.$$

For the construction of the above kernels, see [Ran86] or [CS01]. We have the Henkin’s solution operator $Sf = Kf - Hf$ of the $\overline{\partial}$-equation, where

$$Hf(z) = \int_{\zeta \in bD} f(\zeta) \wedge H(\zeta, z), \quad Kf(z) = \int_{\zeta \in D} f(\zeta) \wedge K(\zeta, z).$$

We remark that the fact that the support function $\phi(\zeta, z)$ is holomorphic in $z$ is very crucial in the construction of the solution operator $Sf$ of the $\overline{\partial}$-equation.

To prove the Hölder type estimate (1.3) of the main Theorem 1.2, we use Lemma 1.3. Hence, we have to estimate the differential of the Henkin solution operator, $d_z Sf$. Using the fact that $|\zeta - z|^2 \leq |\phi(\zeta, z)|$ for $(\zeta, z) \in bD \times \overline{D}$, straightforward computations give the kernel estimate (for the details, see [Ran86])

$$|d_z H(\zeta, z)| \leq \frac{1}{|\phi(\zeta, z)|^2 |\zeta - z|^{2n-3}}, \quad (\zeta, z) \in bD \times \overline{D}.$$
Remark 2.1. Without loss of generality, we assume that the differential of Henkin kernel, $d_z H(\zeta, z)$ has the following form:

$$d_z H(\zeta, z) = \frac{A(\zeta, z)}{(\phi(\zeta, z))^2 |\zeta - z|^{2n-2}},$$

where $A(\cdot, z)$ belongs to $C^1(\overline{D})$ and satisfies $|A(\zeta, z)| \lesssim |\zeta - z|$. Actually, $d_z H(\zeta, z)$ contains more terms whose singularity order is lower than that of (2.7) and so we can ignore other terms (refer to § 3. of chapter 4 in [Ran86]).

Generally, the Bochner-Martinelli integral, $\mathbb{K} f$, has a good regularity, so $\mathbb{K}$ is a bounded operator from $L^\infty$-forms to $A_2$-forms for any $0 < \alpha < 1$. This kind of regularity still holds for a regular majorant of order $\alpha$ ($0 < \alpha < 1$). Hence, we only prove the estimate for the differential of Henkin kernel, $d_z \mathbb{H} f$, which is the main part of this paper.

Proposition 2.2. For any $\alpha$ with $0 < \alpha < 1/2$, there exists a constant $C_\alpha > 0$ such that

$$|d_z \mathbb{H} f(z)| \leq C_\alpha \| f \|_{A_\alpha} \omega_{2}(\rho(z)) \left| \frac{\omega_{2}(\rho(z))}{\rho(z)^{1/2}} \right| \text{ for } z \in D.$$  

Proof. Since the singularities of the Henkin kernel are located in the diagonal $bD \times bD$, to show the inequality (2.8), it suffices to estimate the integral of (2.8) near boundary points. Fix a point $z \in D$ which is sufficiently close to the boundary of $D$ and choose a ball $B(z, r)$ with $B(z, r) \cap bD \neq \emptyset$, in which we have a $C^1$ coordinates system $(t_1, \ldots, t_{2n}) = t = t(\zeta, z)$ such that $t_1 = -\rho(\zeta), t_2 = \text{Im} \phi(\zeta, z)$, $(t(z, z) = (-\rho(z), \ldots, 0)$, and $|t(\zeta, z)| < 1$ for $\zeta \in B(z, r)$. (For the detail, see [HL84].) Moreover, this coordinate system $t$ satisfies

$$|t| \lesssim |\zeta - z| \lesssim |t|, \quad \zeta \in B(z, r) \cap bD.$$  

Also, note that the new coordinate system satisfies $t(\zeta, z) = (0, t')$ for $\zeta \in B(z, r) \cap bD$, where $t' = (t_2, \ldots, t_{2n})$. By Remark 2.1, we have to show that

$$I(z) = \int_{bD \cap B(z, r)} \frac{f(\zeta)\chi(\zeta)A(\zeta, z)}{(\phi(\zeta, z))^2 |\zeta - z|^{2n-2}} dV(\zeta) \lesssim \| f \|_{A_\alpha} \frac{\omega_{2}(\delta(z))}{\delta(z)^{1/2}},$$

where $\chi$ is a compactly supported cut-off function in $B(z, r)$. For this kind of estimate of Hölder type, we choose $\zeta' \in B(z, r) \cap bD$ satisfying $t(\zeta', z) = (0, 0, t_3, \ldots, t_{2n})$. This gives the obvious estimate, $I(z) \leq I_1(z) + \ldots + I_{2n}(z)$.
\[ + I_2(z), \text{ where} \]
\[ I_1(z) = \left| \int_{bD \cap B(z, r)} \frac{(f(\zeta) - f(\zeta'))\chi(\zeta)A(\zeta, z)}{(\phi(\zeta, z))^2|\zeta - z|^{2n-2}} dV(\zeta) \right|, \]
\[ I_2(z) = \left| \int_{bD \cap B(z, r)} \frac{(f(\zeta'))\chi(\zeta)A(\zeta, z)}{(\phi(\zeta, z))^2|\zeta - z|^{2n-2}} dV(\zeta) \right|. \]

It follows from the definition of \( \| \cdot \|_{\omega_2} \) and the inequality \( |A(\zeta, z)| \leq |\zeta - z| \), that
\[ (2.9) \]
\[ I_1(z) \leq \|f\|_{\omega_2} \int_{bD \cap B(z, r)} \frac{\omega_2(\zeta - \zeta')}{|\phi(\zeta, z)|^2|\zeta - z|^{2n-3}} dV(\zeta). \]

To estimate the integral of the right hand side of (2.9), we use the coordinate system \( t \), the inequality (2.6), and introduce polar coordinates in \( t'' = (t_3, \ldots, t_{2n}) \in \mathbb{R}^{2n-2} \), and also set \( r = |t''| \). Then we have
\[ I_1(z) \leq \|f\|_{\omega_2} \int_{|t'|<1} \frac{\omega_2(|t|)}{(|t_2| + |t'|^2 + |\rho(z)|^2|t'|^{2n-3}} dV(t') \]
\[ \leq \|f\|_{\omega_2} \int_{|t'|<1} \omega_2(|t|) \left[ \int_0^1 \frac{r^{2n-3}dr}{(|t_2| + r^2 + |\rho(z)|^2) r^{2n-3}} \right] dt_2 \]
\[ \leq \|f\|_{\omega_2} \int_0^1 \frac{\omega_2(t_2)}{(t_2 + |\rho(z)|)^{3/2}} dt_2. \]

We may assume that \( 0 < |\rho(z)| < 1 \), since \( z \in D \) is close to the boundary. We decompose the integral as follows:
\[ \int_0^1 \frac{\omega_2(t_2)}{(t_2 + |\rho(z)|)^{3/2}} dt_2 = \int_{|\rho(z)|}^{(t_2 + |\rho(z)|)^{3/2}} \frac{\omega_2(t_2)}{(t_2 + |\rho(z)|)^{3/2}} dt_2 + \int_0^1 \frac{\omega_2(t_2)}{(t_2 + |\rho(z)|)^{3/2}} dt_2. \]

Since \( \omega_2 \) is a regular majorant, by the first term of the left hand side of (1.1), we have
\[ \int_0^{(t_2 + |\rho(z)|)^{3/2}} \frac{\omega_2(t_2)}{(t_2 + |\rho(z)|)^{3/2}} dt_2 \leq \frac{1}{|\rho(z)|^{1/2}} \int_0^{(t_2 + |\rho(z)|)^{3/2}} \frac{\omega_2(t_2)}{t_2} dt_2 \lesssim \frac{\omega_2(|\rho(z)|)}{|\rho(z)|^{1/2}}. \]

Similarly, since \( s^{1/2}\omega_2(s) \) is also a regular majorant, by the second term of
the left hand side of (1.1), it follows that

\[
\int_{|\rho(z)|}^{1} \frac{\omega_2(t_2)}{(t_2 + |\rho(z)|)^{3/2}} dt_2 = \int_{|\rho(z)|}^{t_2^{1/2}} \frac{\omega_2(t_2)}{t_2^{3/2}} \frac{t_2^{3/2}}{t_2} \frac{1}{(t_2 + |\rho(z)|)^{3/2}} dt_2 \\
\leq \int_{|\rho(z)|}^{1} \frac{t_2^{1/2 \omega_2(t_2)}}{t_2^{3/2}} dt_2 \\
\leq \frac{\rho(z)^{1/2 \omega_2(|\rho(z)|)}}{|\rho(z)|} = \frac{\omega_2(|\rho(z)|)}{|\rho(z)|^{1/2}}.
\]

These inequalities imply \( I_1(z) \leq \| f \|_{\omega_2} \omega_2(|\rho(z)|)/|\rho(z)|^{1/2} \).

For \( I_2(z) \), we need a somewhat different method. The integration by parts allows one to lower the singularity order of the Henkin kernel. This kind of method was used in [Ran92].

We see that

\[
\frac{1}{\varphi^2} = - \left( \frac{\partial \varphi}{\partial t_2} \right)^{-1} \left( \frac{\partial}{\partial t_2} \left( \frac{1}{\varphi} \right) \right).
\]

Therefore, by integration by parts, we have

\[
(2.10) \quad I_2(z) \leq \left| \int_{|t'| \leq 1} \left[ - \left( \frac{\partial \varphi}{\partial t_2} \right)^{-1} \frac{\partial}{\partial t_2} \left( \frac{1}{\varphi} \right) \right] f(0, 0, t') \frac{\chi(t')A(t', z)}{|t'|^{2n-2}} dt' \right|
\]

\[
= \left| \int_{|t'| \leq 1} f(0, 0, t') \frac{1}{\varphi} \left( \frac{\partial \varphi}{\partial t_2} \right)^{-2} \chi(t')A(t', z) \frac{|t'|^{2n-2}}{|t'|^{2n-2}} dt' \right|
\]

\[
= \left| \int_{|t'| \leq 1} f(0, 0, t') \frac{1}{\varphi} \left( \frac{\partial \varphi}{\partial t_2} \right)^{-2} B(t', z)dt' \right|
\]

where

\[
B(t', z) = - \frac{\partial^2 \varphi}{\partial t_2^2} \chi(t')A(t', z) + \left( \frac{\partial \varphi}{\partial t_2} \right) \frac{\partial}{\partial t_2} \left( \frac{\chi(t')A(t', z)}{|t'|^{2n-2}} \right).
\]

In the second equality of (2.10), we use the fact that \( f(0, 0, t'') \) does not depend on \( t_2 \). Since \( t_2 = \text{Im} \varphi \), we have \( |\partial \varphi/\partial t_2| \geq 1 \). Therefore, we have

\[
I_2(z) \leq \| f \|_{\infty} \int_{|t'| \leq 1} \frac{dt'}{|\varphi||t'|^{2n-2}}.
\]
For the moment, we assume that for any $\varepsilon > 0$,

\begin{equation}
J(z) = \int_{|t'| \leq 1} \frac{dt'}{\phi|t'|^{2n-2}} \leq |\rho(z)|^{-\varepsilon},
\end{equation}

which will be proved later as an independent lemma. Since (2.11) holds for arbitrary $\varepsilon > 0$, one can choose $\varepsilon > 0$ so that $0 < 1/2 - \varepsilon < \alpha$. Moreover, $\omega_\alpha$, $0 < \alpha < 1/2$, is a regular majorant and so $\omega_\alpha(t)/t^{1/2-\varepsilon}$ is increasing, or equivalently, $|\rho(z)|^{-\varepsilon} \leq \omega_\alpha(|\rho(z)|)/|\rho(z)|^{1/2}$. It follows that

\begin{align*}
I_2(z) \leq \|f\|_{\infty} J(z) \leq \|f\|_{\omega_\alpha} \frac{\omega_\alpha(|\rho(z)|)}{|\rho(z)|^{1/2}}.
\end{align*}

These two estimates for $I_1(z)$ and $I_2(z)$ complete the proof. \hfill \Box

We end this section with the proof of (2.11).

**Lemma 2.3.** For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

\begin{equation}
J(z) \leq C_\varepsilon |\rho(z)|^{-\varepsilon}.
\end{equation}

**Proof.** We have

\begin{align*}
J(z) &\leq \int_{|t'| < 1} \frac{dt'}{(|t_2| + |\rho(z)| + |t'|^2)|t'|^{2n-2}} \\
&\leq \int_{|(t_2, t_3, t_4)| < 1} \frac{dt_2dt_3dt_4}{(|t_2| + |\rho(z)|)(t_3^2 + t_4^2)}.
\end{align*}

Again, using polar coordinates in $(t_3, t_4)$, say $x = |(t_3, t_4)|$, one obtains

\begin{align*}
J(z) &\leq \int_{|t_2| < 1} \frac{1}{|t_2| + |\rho(z)|} \left( \int_0^1 \frac{x dx}{t_2^2 + x^2} \right) dt_2 \\
&\leq \int_0^1 \frac{|\log t_2|}{(t_2 + |\rho(z)|)} dt_2 \\
&\leq C_\varepsilon \int_0^1 \frac{t_2^{-\varepsilon}}{(t_2 + |\rho(z)|)} dt_2.
\end{align*}
By the change of variable \( s = t_2/|\rho(z)| \), we have

\[
J(z) \lesssim \int_0^1 \frac{|t_2|^{-\varepsilon}}{(|t_2| + |\rho(z)|)} \, dt_2
\]

\[
\lesssim |\rho(z)|^{-\varepsilon} \int_0^\infty \frac{ds}{(1 + s)s^\varepsilon} \leq C_\varepsilon |\rho(z)|^{-\varepsilon}.
\]

\[\square\]

3. **Proof of Theorem 1.2.**

In this section, we complete the proof of our main Theorem 1.2 using Proposition 2.2 and Lemma 1.3.

The inequality (2.8) in Proposition 2.2 implies that

\[
|d^H f(z)| \leq c_z \|f\|_\infty \frac{1/2 \omega_x(|\rho(z)|)}{|\rho(z)|}.
\]

Therefore by Lemma 1.3 and regularities of the operator \( Hf \) in the Hölder type spaces we can prove the inequality (1.3) of Theorem 1.2.

Finally, we include a brief sketch of the proof of Lemma 1.3.

**PROOF.** Because \( D \) is compact, by the local coordinate change argument, it suffices to show the following in the special domain \( D(k) = \{ (x_1, x') \in \mathbb{R}^n : 0 < x_1 < k, |x'| < k \} \): if

\[
|dg(x)| \leq c_g \frac{\omega_x(x_1)}{x_1}
\]

for \( x, y \in D(k/2) \) with \( |x - y| \leq k/2 \), then we have

\[
|g(x) - g(y)| \leq c \cdot c_g \omega_x(|x - y|).
\]

To show this, fix two points \( x, y \in D(k/2) \) with \( |x - y| \leq k/2 \) and let \( d = |x - y| \). Here we may assume that \( k \leq 1/2 \) and by symmetry we may also suppose \( x_1 \leq y_1 \).

First it follows from (3.12) that

\[
|g(x_1, x') - g(x_1 + d, x')| \leq \int_{x_1}^{x_1 + d} \left| \frac{\partial g}{\partial x_1}(t, x') \right| \, dt
\]

\[
\leq c_g \int_{x_1}^{x_1 + d} \frac{\omega_x(t)}{t} \, dt \leq c \cdot c_g \omega_x(d)
\]
In fact, if $0 < d \leq x_1$, then

$$
\int_{x_1}^{x_1+d} \frac{\omega_r(t)}{t} \, dt \leq d \frac{\omega_r(x_1)}{x_1} \leq \omega_r(d),
$$

since $\omega_r(t)/t$ is decreasing. If $0 < x_1 \leq d$, then

(3.15) $$
\int_{x_1}^{x_1+d} \frac{\omega_r(t)}{t} \, dt \leq \int_{0}^{d} \frac{\omega_r(t)}{t} \, dt \leq \omega_r(d).
$$

Since $\omega_r(t)/t$ is decreasing, the first inequality of (3.15) holds and by (1.1) the second inequality of (3.15) is also true.

Next, by the Mean Value Theorem and (3.12), since $\omega_r(t)/t$ is decreasing, we have

(3.16) $$
|g(x_1 + d, x') - g(y_1 + d, y')| \leq c_g \frac{\omega_r(a_1)}{a_1} \leq c_g \omega_r(d)
$$

for some $a_1$ in the line segment between $x_1 + d$ and $y_1 + d$. Since

$$
|g(x) - g(y)| \leq |g(x_1, x') - g(x_1 + d, x')| + |g(x_1 + d, x') - g(y_1 + d, y')| + |g(y_1 + d, y') - g(y_1, y')|,
$$

(3.13) follows from the estimates (3.14) and (3.16). \qed

REFERENCES


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