The Asymptotic Behavior of the Solutions of \((\mathcal{A} + \lambda) u = 0\) in a Domain with the Unbounded Boundary

By

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1. Introduction

We shall consider the equation

\[(1.1) \quad (\mathcal{A} + \lambda) u = 0\]

in an unbounded domain \(\Omega\) in the Euclidean \(n\)-space \(E^n (n \geq 2)\), with the boundary condition

\[(1.2) \quad u \mid _\Gamma = 0,\]

where \(\Gamma\) is the boundary of \(\Omega\), and \(\lambda\) is a positive constant. Let \(\Omega(L) = \Omega \cap \{ (x_1, \ldots, x_n) \in E^n : x_1 > L \}\). We shall assume that \(\Gamma\) is smooth \((C^1)\), and that there are positive numbers \(C, N\) and \(l(l \leq 1)\) such that the following (1.3) and (1.4) hold for at least one of the connected components of \(\Omega(N)\), say \(\Omega_1(N)\).

\[(1.3) \quad \Omega_1(N) \subset \{ x_1, \ldots, x_n \} \in E^n : (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}} < Cx_1 \} \]

\[(1.4) \quad \mathbf{n}(p) \cdot \mathbf{a}(p) \leq 0 \quad \text{for} \quad p \in \Gamma \cap \partial \Omega_1(N)\]

where \(\mathbf{n}(p)\) is the outer unit normal to \(\Gamma\) at \(p = (x_1, \ldots, x_n)\) and \(\mathbf{a}(p)\) is the vector \(\mathbf{a}(p) = (x_1, lx_2, \ldots, lx_n)\). Our purpose in this paper is to prove the following.

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**Theorem 1.1.** Let $\Omega$ and $\lambda$ be as above. If $u$ is a non-trivial solution of (1.1) and (1.2), then

\[
(1.5) \quad \lim_{t \to \infty} t^\epsilon \int_{P_t} (u^2 + |F u|^2) dS = \infty
\]

for any $\epsilon > 0$, where $P_t$ is the section of $\Omega_1(N)$ by the hyperplane $x_1 = t$.

If $\Omega$ lies in the half-space $x_1 > 1$, and (1.4) holds on the whole of $\Gamma$ with $l = 0$. (1.5) is a part of the well known results by Rellich [1]. Jones [2] (Theorem 9) has treated the problem in the case of $l = 1$. We can find in Agmon [3] (Theorem 11) an extension of Jones' result, and, when $l = 1$, our Theorem 1.1 is also included in Agmon's theorem. So the proof of Theorem 1.1 must be carried out for $0 < l < 1$, and it will be done in the framework developed by Roze [4] and Eidus [5].

In §2, introducing a curvilinear coordinate system for the convenience of calculations, we shall give some preliminary lemmas. In §3, it will be shown that a solution which does not satisfy (1.5) decreases, in a sense, like an exponential function in $\Omega_1(N)$, and in §4, it will turn out that such solution is the trivial solution.

In consequence of Theorem 1.1 it is easy to see that the self-adjoint realization of $-\mathcal{A}$ in $L^2(\Omega)$ with the Dirichlet boundary condition has no positive point eigenvalues. A short remark on the spectrum will be given in the final §5.

## 2. Preliminaries

In the sequel the conditions of the Theorem 1.1 are always assumed. Let us introduce a curvilinear coordinate system $(X_1, \ldots, X_n)$ in $E_n = \{(x_1, \ldots, x_n) : x_1 > 0\}$ as follows;

\[
(2.1) \quad \left\{ \begin{array}{l}
X_1 = \{x_1^2 + l(x_2^2 + \cdots + x_n^2)\}^{\frac{1}{2}} \quad (X_1 > 0), \\
X_2 = \tan^{-1}\{(x_2^2 + \cdots + x_n^2)^{\frac{1}{2}} / x_1\} \quad (0 \leq X_2 < \frac{\pi}{2}),
\end{array} \right.
\]

and $X_3, \ldots, X_n$ are the parameters which are suitably chosen on the sphere $S^{n-2} = \{(x_2, \ldots, x_n) : x_2^2 + \cdots + x_n^2 = 1\}$. (For example, we may put $x_2 =$
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\[ (d + t)u = Q \]

\[ \sum_{i,j=1}^{n} g_{ij} \partial_{x_i} \partial_{x_j} f = (1/\sqrt{G}) \sum_{i,j=1}^{n} (g^{ij} \sqrt{G} f_{x_{i}}) x_{i} \]

\[ \sum_{i,j=1}^{n} |\partial f/\partial x_{i}|^2 = \sum_{i,j=1}^{n} g^{ij} f_{x_{i}} f_{x_{j}} \]

for a smooth function \(f\), where \(f_{x_{i}}=\partial f/\partial x_{i}\).

Now we give some lemmas specifying the asymptotic properties of \(g^{ij}\) and \(G\), which will play important roles in the following sections.

**Lemma 2.1.** \(g^{11} \to 1, X_{1}g_{11}^{11} / g^{11} \to 0, X_{1}g_{1}^{1} / g^{ij} \to 2l (i = j = 2 \text{ or } i,j \geq 3) \) and \(X_{1}G_{X_{1}} / G \to 2(n-1)l \) when \(X_{1} \to \infty\). These convergences are uniform in \(X_{2} \in [0, \theta]\) for any \(\theta < \pi/2\).

**Remark.** Because of the condition (1.3), there is a number \(\theta < \pi/2\), such that \(x_{2} < \theta\) for any point in \(Q_{1}(N)\).

**Proof of Lemma 2.1.** In the case of \(l=1\), the proof is easy. If \(0 < l < 1\), then \(r/X_{1} \to 0 \) (\(X_{1} \to \infty\)) uniformly when \(X_{2}\) varies in \([0, \theta]\), because, \(r=x_{1} \tan X_{2}\) and \(x_{1} \leq X_{1}\). From this \(g^{11} \to 1\) is obvious since \(g^{11} = \{X_{1}^{2} + (l^{2} - l)r^{2}\} / X_{1}^{2}\). The other convergences can be proved also by straightforward calculations if we use the facts that \(x_{1} X_{1} / \{X_{1}^{2} + (l^{2} - l)r^{2}\}\), \(r_{X_{1}} = l r X_{1} / \{X_{1}^{2} + (l^{2} - l)r^{2}\}\), \(G = g_{11} g_{22} r^{2(n-2)} \det (\tilde{g}_{ij})\), and \(\tilde{g}_{ij}\) are independent of \(X_{1}\) and \(X_{2}\).

**Lemma 2.2.** For any real \(\delta\), we have \(X_{1}^{-\delta}(\sqrt{G} X_{1}^{\delta}) / \sqrt{G} \to \delta + (n-1)l\), \(X_{1}^{-\delta} (g^{11} \sqrt{G} X_{1}^{\delta}) / \sqrt{G} \to \delta + (n-1)l\), and \(X_{1}^{-\delta} (g^{ij} \sqrt{G} X_{1}^{\delta}) / g^{ij} \sqrt{G} \to \delta + (n-3)l\) for \(i = j = 2\) or \(i,j \geq 3\), when \(X_{1} \to \infty\). These convergences are
uniform in $X_2 \in [0, \theta]$ for any $\theta < \frac{\pi}{2}$.

Proof. Lemma 2.1. and direct calculations lead us to this lemma. Q.E.D.

Let $\Omega_{AB}$, $\Omega_A$ and $S_A$ be the subsets of $\Omega_1(N)$ characterized by $A < X_1 < B$, $A < X_1 < \infty$ and $X_1 = A$ respectively, and put $\Gamma_{AB} = \partial \Omega_{AB} - (S_A \cup S_B)$ (the 'side' of $\Omega_{AB}$). If $u$ is a solution of (1.1) and (1.2), then $v = X^m u$ $(m \geq 0)$ satisfies

\[
\Delta V - \frac{2m}{X_1} g^{11} V X_1 + (M + \lambda) V = 0 \quad \text{(in $\Omega_A$)}
\]

and

\[
V |_{\Gamma_A} = 0
\]

for $A > \inf_{x_1 \in N} X_1 = N_0$, where $M = (m^2 + m) g^{11} / X_1^2 - m (g^{11} G) X_1 / X_1 \sqrt{G}$.

Lemma 2.3. $X_1^2 M - g^{11} m^2 \rightarrow m(1 - (n-1)\lambda)$ uniformly when $X_1 \rightarrow \infty$ in $\Omega_1(N)$, and there exist positive constants $C_1$ and $N_1$ which are independent of $m \geq 0$ such that the inequalities $M \geq 0$ and $X M X_1 + 2(m/X_1)^2 \leq m C_1 / X_1^2$ hold in $\Omega_A$ for $A > N_1$.

Proof. It is easy to prove that $g_{X_1} = o(1/X_1^2)$ and $G_{X_1} = O(1/X_1^2)$ as $X_1 \rightarrow \infty$ in $\Omega_1(N)$. Using these facts and Lemmas 2.1–2, we have the lemma. Q.E.D.

The next two lemmas are concerned with the solutions of (2.3) and (2.4).

Lemma 2.4. Let $v$ be a real valued solution of (2.3) and (2.4). Then

\[
\int_{\partial_{\Omega}} \partial \phi |dv|^2 d\Omega = \left\{ \int_{S_B} - \int_{S_A} \right\} g^{11} \psi v_0 + \int_{\partial_{\Omega}} \psi (M + \lambda) v^2 d\Omega,
\]

where $\psi = \partial \phi$. Q.E.D.
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where \(B > A > N_0\), and \(\psi\) is smooth and depends only on \(X_1\).

**Proof.** Multiply (2.3) by \(\psi v\) and integrate over \(\Omega_{AB}\). Using (2.4) we have (2.5).

If we put \(m = 0\) in (2.5),

\[
\int_{\partial AB} \psi |F u|^2 d\Omega - \int_{\partial AB} \psi u u_{X_1} d\Omega + \lambda \int_{\partial AB} \psi u^2 d\Omega.
\]

**Lemma 2.5.** Let \(v\) be a real valued solution of (2.3) and (2.4). For any \(\delta > 0, \gamma > 0\) and \(m \geq 0\), we can find a real \(N_2 = N_2(\delta, \gamma)\) which is independent of \(m\) such that the inequality

\[
\int_{\partial AB} \psi |F u|^2 d\Omega - \int_{\partial AB} \psi u u_{X_1} d\Omega + \lambda \int_{\partial AB} \psi u^2 d\Omega \\
- (2m + \delta - \gamma) \int_{\partial AB} X_1^{\delta-1} g^{11} v_{X_1}^2 d\Omega \\
+ \frac{1}{2} \int_{\partial AB} X_1^{\delta-1} \left[ (\delta + (n-3)(\lambda + \gamma)) |F v|^2 \right. \\
- \left. \{(\delta + (n-1)(\lambda - \gamma)(M + \lambda) + X_1 M X_1) v^2 \} d\Omega \right] \\
\geq 0
\]

holds for \(B > A > N_2\).

**Proof.** We multiply (2.3) by \(X_1^\delta v_{X_1}\) and integrate over \(\Omega_{AB}(A > N_0)\). Integrating by parts, we have

\[
\int_{\partial AB} \psi |F u|^2 d\Omega - \int_{\partial AB} \psi u u_{X_1} d\Omega + \lambda \int_{\partial AB} \psi u^2 d\Omega \\
- 2m \int_{\partial AB} X_1^{\delta-1} g^{11} v_{X_1} d\Omega - \int_{\partial AB} \left\{ \delta X_1^{\delta-1} g^{11} \sqrt{G} \right. \\
- \left. \frac{1}{2} \left( g^{11} \sqrt{G} X_1^3 \right)_{X_1} v_{X_1}^2 d\Omega \right. \\
+ \frac{1}{2} \int_{\partial AB} \left[ \sum_{i,j \geq 2} \left( g^{ij} \sqrt{G} X_1^3 \right) X_i v_{X_i} v_{X_j} \right. \\
- \left. \int_{\partial AB} \psi u u_{X_1} d\Omega + \lambda \int_{\partial AB} \psi u^2 d\Omega \right] \\
\geq 0
\]
\[
- \{(M + \lambda) \sqrt{G} X_1^i X_i v^2 \}_{|a|} = - \frac{1}{2} \int_{\Gamma_{AB}} X_1^i |Fv|^2 \sqrt{g_{11}(\mathbf{n} \cdot \mathbf{X}_1)} dS,
\]
where \( \mathbf{X}_1 \) is the vector \( \mathbf{X}_1 = \frac{a}{|a|} = (x_1, l x_2, \ldots, l x_n)/(x_1^2 + l^2 x_2^2 + \cdots + l^2 x_n^2)^{1/2} \).

Here we have used the fact that \( v x_i (F v \cdot n) = \sqrt{g_{11}} |Fv|^2 (X_1 \cdot n) \) on \( \Gamma_{AB} \), which follows from the boundary condition (2.4). In view of the condition (1.4), the right side of (2.8) is non-negative. In consequence of Lemma 2.2, for any \( \eta > 0 \), we can take \( N_{\eta}^*(\delta, \eta) \) such that the inequalities

\[
\delta g^{11} X_1^{i-1} - \frac{1}{2} (g^{11} \sqrt{G} X_1^i)_{X_i} \sqrt{G} \geq \frac{1}{2} (\delta - (n - 1) l - \gamma) g^{11} X_1^{i-1},
\]

\[
(g^{ij} \sqrt{G} X_1^i)_{X_j} \sqrt{G} \leq (\delta + (n - 3) l + \gamma) g^{ij} X_1^{i-1} (i, j \geq 2),
\]

\[
(\sqrt{G} X_1^i)_{X_i} \geq (\delta + (n - 1) l - \gamma) X_1^{i-1}
\]

hold if \( X_1 > N_{\eta}^*(\delta, \eta) \). Thus we have the inequality (2.7) for \( B > A > N_2(\delta, \eta) = \max(N_0, N_1, N_{\eta}^*(\delta, \eta)) \) by Lemma 2.3. Q.E.D.

### 3. On a Solution Which Does Not Satisfy (1.5)

In this and following sections, we use the abbreviations \( X, f_X \) and \( \gamma \) which stand for \( X_1, f_{X_1} = \partial f/\partial X_1 \) and \( g^{11} \) respectively.

**Lemma 3.1.** Let \( u \) be a solution of (1.1) and (1.2). If

\[
\liminf_{t \to \infty} t^3 \int_{S_t} (|u|^2 + |F u|^2) dS = 0
\]

for some \( \delta > 0 \), then

\[
\int_{\partial(S_1(N))} X_m (|u|^2 + |F u|^2) d\Omega < \infty
\]

for any \( m \geq 0 \).

**Proof.** We may assume that \( u \) is real valued. If we put \( m = 0 \) in Lemma 2.5, we have
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\begin{align*}
(\delta - l) \int_{\Sigma_{AB}} X^{\delta - 1} u_x^2 d\Omega - \frac{1}{2} \int_{\Sigma_{AB}} X^{\delta - 1} \{(\delta + (n - 3)l + \eta) |P u|^2 - \lambda(\delta + (n - 1)l - \eta)u^2\} d\Omega,
\end{align*}

for $B > A > N_2(\delta, \eta)$. On the other hand, taking $X^{\delta - 1}$ as $\psi$ in (2.6), we see

\begin{align*}
\int_{\Sigma_{AB}} X^{\delta - 1} u u_x \sqrt{\gamma} dS = \int_{\Sigma_{AB}} X^{\delta - 1} (|P u|^2 - \lambda u^2) d\Omega. 
\end{align*}

From (3.3) and (3.4), we have, for $A > N_2$,

\begin{align*}
\int_{\Sigma_{AB}} X^{\delta - 1} u u_x \sqrt{\gamma} dS = & \int_{\Sigma_{AB}} \left\{ X^{\delta} \left\{ \gamma u_x^2 - \frac{|P u|^2}{2} + \lambda u^2 \right\} \sqrt{\gamma} dS ight. \\
+ & \frac{(n - 1)}{2} \left\{ \int_{\Sigma_{AB}} X^{\delta - 1} u u_x \sqrt{\gamma} dS \right\} (\delta - l) \int_{\Sigma_{AB}} X^{\delta - 1} \gamma u_x^2 d\Omega \\
- & \frac{1}{2} \int_{\Sigma_{AB}} X^{\delta - 1} \{(\delta - 2l + \eta) |P u|^2 - \lambda(\delta - \eta)u^2\} d\Omega \\
+ & \frac{(n - 1)}{2} (\delta - 1) \int_{\Sigma_{AB}} X^{\delta - 2} u u_x d\Omega \\
= & \frac{1}{2} \int_{\Sigma_{AB}} X^{\delta - 1} \{(\delta - 2\eta) |P u|^2 + \lambda(\delta - 2\eta)u^2\} d\Omega \\
+ & \left( l - \delta + \frac{\eta}{2} \right) \int_{\Sigma_{AB}} X^{\delta - 1} |P u|^2 d\Omega + (\delta - l) \int_{\Sigma_{AB}} X^{\delta - 1} \gamma u_x^2 d\Omega \\
+ & \frac{\eta \lambda}{2} \int_{\Sigma_{AB}} X^{\delta - 2} u^2 d\Omega + \frac{(n - 1)(\delta - 1)}{2} \int_{\Sigma_{AB}} X^{\delta - 2} u u_x d\Omega.
\end{align*}

Without loss of generality, we may assume $\delta < l$. So we can consider that

\begin{align*}
(|\delta - l| \int_{\Sigma_{AB}} X^{\delta - 1} \gamma u_x^2 d\Omega) \leq (l - \delta) \int_{\Sigma_{AB}} X^{\delta - 1} |P u|^2 d\Omega.
\end{align*}

Moreover, $|(\delta - 1)(n - 1)u u_x / X| \leq \eta(\gamma u_x^2 + \lambda u^2)$ for sufficiently large $X$,.
say $X > N_3(\eta, \lambda)$. Thus, passing to the limit for $B \to \infty$, it follows from (3.5) that

$$
(3.6) \quad \int_{S_A} X^8(|\nabla u|^2 - \lambda u^2)^{dS/\sqrt{\gamma}} - (n-1)I \int_{S_A} X^{8-1} u u_x \sqrt{\gamma} dS \\
= \frac{\delta}{2} \int X^{8-1}(|\nabla u|^2 + \lambda u^2) d\Omega
$$

for $A > N_4(\delta, \lambda) = \max \left\{ N_2\left(\frac{\delta}{4}\right), N_3\left(\frac{\delta}{4}, \frac{\lambda}{4}\right) \right\}$.

We integrate (3.6) with respect to $A$ from $\xi_0$ to $\xi_1$ ($\xi_1 > \xi_0 > N_4$). Using $|uu_x| < (u^2 + |\nabla u|^2)/2$ and (2.6) in which we replace $\phi$ by $X^8$, we have

$$
(3.7) \quad \frac{\partial}{2} \int_{\xi_0}^{\xi_1} \int_{S_t} X^{8-1} \{|\nabla u|^2 + \lambda u^2\} d\Omega \\
\leq C_2 \int_{S_{t_1}} X^{8-1}(|\nabla u|^2 + u^2) d\Omega \\
+ \left\{ \int_{S_{t_1}} - \int_{S_{t_0}} \right\} X^{8} u u_x \sqrt{\gamma} dS,
$$

where $C_2 = C_3(\delta)$ is some positive constant which is independent of $\xi_0$ and $\xi_1$. By (3.1) and

$$
\int_{\xi_0}^{\xi_1} \int_{S_t} X^{8-1}(|\nabla u|^2 + \lambda u^2) d\Omega \\
= \int_{S_{t_1}} (X - \xi_0) X^{8-1}(|\nabla u|^2 + \lambda u^2) d\Omega + (\xi_1 - \xi_0) \int_{S_{t_1}} X^{3-1}(|\nabla u|^2 + \lambda u^2) d\Omega
$$

(3.7) implies

$$
\frac{\partial}{2} \int_{S_{t_0}} (X - \xi_0) X^{8-1}(|\nabla u|^2 + \lambda u^2) d\Omega \\
\leq C_2 \int_{S_{t_0}} X^{8-1}(|\nabla u|^2 + u^2) d\Omega \\
- \int_{S_{t_0}} X^{8} u u_x \sqrt{\gamma} dS
$$

($\xi_0 > N_4$). Integrating this inequality with respect to $\xi_0$ from $\xi_1$ to $\infty$ ($\xi_1 > N_4$), we find

$$
\int_{S_{t_1}} (X - \xi_1)^2 X^{3-1}(|\nabla u|^2 + \lambda u^2) d\Omega \leq C_3 \int_{S_{t_1}} X^{3}(|\nabla u|^2 + u^2) d\Omega,
$$
where \( C_3 \) does not depend on \( \xi_1 \). Repeating this process, we arrive at (3.2).

Q.E.D.

**Lemma 3.2.** Under the assumption of Lemma 3.1,

\[
\lim_{t \to \infty} \int_{S_t} |u|^2 dS = 0
\]

for \( \alpha < \sqrt{\lambda l/(1-l)} \). If \( l=1 \), \( \alpha \) may be taken arbitrarily.

**Proof.** We may assume that \( u \) is real valued. Put \( v=X^m u \). In Lemma 2.5, we replace \( \delta \) by \( l \) and let \( B \to \infty \). Then, by Lemma 3.1, we have

\[
\begin{align*}
\int_{S_A} X^l \left( \gamma v^2 \right) - \frac{|Fv|^2}{2} + \frac{M+\lambda}{2} v^2 dS & \to -2m \int_{S_A} X^{l-1} \gamma v^2 d\Omega \\
& \quad + \frac{1}{2} \int_{S_A} X^{l-1} \left\{ (n-2)l + \eta \right\} |Fv|^2 - \left\{ (nl - \eta)(M+\lambda) + XM_X v^2 \right\} d\Omega \\
& \geq 0
\end{align*}
\]

for \( A > N_2(l, \eta) \). On the other hand, taking \( X^{l-1} \) as \( \phi \) in (2.5) we see

\[
\begin{align*}
\int_{S_A} X^{l-1} |Fv|^2 d\Omega = - \int_{S_A} X^{l-1} \gamma vv_X dS \\
& \quad - \int_{S_A} X^{l-2} \left\{ (l-1) + 2m \right\} \gamma vv_X d\Omega + \int_{S_A} X^{l-1} (M+\lambda)v^2 d\Omega.
\end{align*}
\]

From (3.9) and (3.10), we have

\[
\begin{align*}
\int_{S_A} X^l \left\{ \gamma v^2 - \frac{|Fv|^2}{2} + \frac{1}{2} (M+\lambda) v^2 \right\} dS & \to \frac{(n-2)l + \eta}{2} \int_{S_A} X^{l-1} \gamma vv_X dS + 2m \int_{S_A} X^{l-1} \gamma v^2 d\Omega \\
& \quad + \frac{1}{2} (l-1 + 2m) \left\{ (n-2)l + \eta \right\} \int_{S_A} X^{l-2} \gamma vv_X d\Omega \\
& \quad + \frac{1}{2} \left\{ (2(l-\eta)(M+\lambda) + XM_X v^2 \right\} d\Omega \leq 0
\end{align*}
\]
for $A > N_2$. Note that the fourth term of (3.11) is estimated as follows;

$$
\left| \frac{1}{2} (l-1+2m) \{ (n-2) l + \eta \} \int_{\partial A} d\Omega \right| \leq 2m \int_{\partial A} X^{l-1} v^2 d\Omega \\
+ C_4 \frac{m}{2} \int_{\partial A} X^{l-3} v^2 d\Omega
$$

where $C_4$ is a positive constant independent of $A > N_0$ and $m \geq 1$. Thus we have the inequality

$$
(3.12) \quad \int_{S_A} X^{l} \left\{ \gamma v_X^2 - \frac{|Fv|^2}{2} + \frac{(M+\lambda)v^2}{2} \right\} dS \\
- \frac{(n-2)l+\gamma}{2} \int_{S_A} X^{l-1} \sqrt{\gamma} |vv_X| dS + \frac{1}{2} \int_{\partial A} X^{l-1} \{ 2(l-\varepsilon)(M+\lambda) \\
+ XM_X - mC_4/X^2 \} v^2 d\Omega \leq 0
$$

for $A > N_2$. Using the equality

$$
|Fv|^2 = X^{2m} |Vu|^2 + 2mX^{2m-1}u u_X + m^2 X^{2m-2} u^2,
$$

the first term of (3.12) can be written in the form

$$
\int_{S_A} X^{l} \left\{ \gamma v_X^2 + \frac{X^{2m}}{2} (M-\gamma m^2/X^2) u^2 - mX^{2m-1} u u_X \right\} dS \\
+ \frac{1}{2} \int_{S_A} X^{2m-l} (- |Fv|^2 - \lambda u^2) dS.
$$

Multiplying this by $A^{2-2m-l}$ and integrating with respect to $A$ from $\xi$ to $\infty (\xi > N_2)$, we have, by Lemma 3.1,

$$
\int_{\partial t} X^{2-2m} \gamma v_X^2 d\Omega + \frac{1}{2} \int_{\partial t} (X^{2m} - \gamma m^2) u^2 d\Omega + (1-m) \int_{\partial t} X \gamma u u_X d\Omega \\
+ \frac{1}{2} \int_{S_t} X^2 \sqrt{\gamma} u u_X dS \\
= \int_{\partial t} X^{2-2m} \gamma v_X^2 d\Omega + \frac{1}{2} \int_{\partial t} \{ X^{2m} - \gamma m^2 + (m-1)(X\gamma G)_X/\sqrt{G} \} u^2 d\Omega \\
+ m-1 \frac{1}{2} \int_{S_t} X^2 \sqrt{\gamma} u u_X dS.
Here we have used (2.6) with \( \psi = X^2 \). Thus we have from (3.12)

\[
(3.13) \quad \int_{\partial \Sigma} \{X^2 M - \gamma m^2 + (m-1)(X\sqrt{G})_X/\sqrt{G} - (nl - 2l + \eta)^2/4\} u^2 d\Omega \\
+ \int_{S_{\xi}} X^2 \sqrt{\gamma} uu_x dS + (m-1) \int_{S_{\xi}} X \sqrt{\gamma} u^2 dS \\
+ \int_{\xi}^{\infty} A^{-2m-1} dA \int_{\partial \Delta} X^{l-1} \{2(l-\eta)(M+\lambda) + XM_X - mC_4/X^2\} v^2 d\Omega \\
\leq 0
\]

for \( \xi > N_2 \). (Note that \( \{(n-2)l + \eta\} X^{l-2m} |vv_x| \leq X^{l-2m} v_x^2 + \frac{1}{4} \{(n-2)l + \eta\}^2 X^{-2m} v^2 \).

Put

\[
\Phi(\xi) = \int_{S_{\xi}} X^2 \sqrt{\gamma} u^2 dS.
\]

Then

\[
(3.14) \quad \frac{1}{2} \frac{d\Phi}{d\xi} = \int_{S_{\xi}} \{X^2 \sqrt{\gamma} uu_x + (X^2 \sqrt{G})_X u^2 \sqrt{\gamma} \} dS.
\]

By Lemma 2.2, we can choose \( C_5 > 0 \) such that \( (\gamma X^2 \sqrt{G}/\sqrt{G}) < (C_5 - 1)\gamma X \) for \( X > N_0 \). (3.13) and (3.14) give

\[
(3.15) \quad \int_{\partial \Sigma} \{X^2 M - \gamma m^2 + (m-1)(X\sqrt{G})_X/\sqrt{G} - (nl - 2l + \eta)^2/4\} u^2 d\Omega \\
+ \frac{1}{2} \frac{d\Phi}{d\xi} + (m-C_5) \frac{1}{\xi} \Phi \\
+ \int_{\xi}^{\infty} A^{-2m-1} dA \int_{\partial \Delta} X^{l-1} \{2(l-\eta)(M+\lambda) + XM_X - mC_4/X^2\} v^2 d\Omega \\
\leq 0.
\]

The coefficient of \( u^2 \) in the first integral of (3.15) tends to \( 2m-1-(n-1)l-(nl-2l+\eta)^2/4 \) as \( X \to \infty \). See Lemmas 2.1, 2.2 and 2.3. So it is positive if \( m > C_6 \{1+(n-1)l/2+(nl-2l+\eta)^2/8 \) and \( X \) is sufficiently large, say \( X > N_5 \). We can take \( N_5 \) independently of \( m \), at least, when \( m > C_6 \). If we put
then, by Lemma 2.3, we can take a positive constant $C_7$ such that $|h| < C_7/X^2$ for $X> N_6$. Now if we put in (3.15) and (3.16) $m = m(\xi, \eta)$, then there exists positive $N_6(\eta)$ such that $m > C_6$, $m > C_7/\eta$ for $\xi > N_6$. Note that

$$ (3.16) = 2(l-\eta-1)(l-\eta)\frac{\xi^2}{X^2}/(1-l + \frac{3}{2}\eta)X^2 + 2(l-\eta)\lambda - mh > 0 $$

if $X > \xi > N_6$. Taking $\eta$ sufficiently small we may assume $m(\xi, \eta)/\xi > \alpha$. Moreover, for such $\eta$, we can take $N_7(\eta) (> N_6(\eta))$ so that $(m(\xi, \eta) - C_5)/\xi > \alpha$ for $\xi > N_7$. Thus we have from (3.15) the differential inequality

$$ \frac{d\Phi}{d\xi} + 2\alpha\Phi \leq 0 $$

for large $\xi$. This proves the lemma. Q.E.D.

**Lemma 3.3.** Under the assumption of Lemma 3.1,

$$ (3.17) \int_{\Omega(N)} e^{2\alpha X}(u^2 + |\nabla u|^2) d\Omega < \infty $$

for $\alpha < \sqrt{\lambda l}/(1-l)$. If $l = 1$, $\alpha$ may be taken arbitrarily.

**Proof.** We assume that $u$ is real valued. Replace $\psi$ in (2.6) by $e^{2\alpha X}$, then we have

$$ (3.18) \int_{\Omega(N)} e^{2\alpha X}|\nabla u|^2 d\Omega = \int_{\Omega(N)} \left\{ \int_{S^A} - \int_{S^B} \right\} \sqrt{\gamma} e^{2\alpha X} uu_X dS $$

$$ -2\alpha \int_{\Omega(N)} \gamma e^{2\alpha X} uu_X d\Omega + \lambda \int_{\Omega(N)} e^{2\alpha X} u^2 d\Omega. $$
Next note that

\[\int_{a_B} \tau e^{2a_X} u u_X d\Omega = -\frac{1}{2} \int_{a_B} (\sqrt{\gamma} e^{2a_X})_X \sqrt{\gamma} u^2 d\Omega \]

\[+ \frac{1}{2} \left\{ \int_{S_B} - \int_{S_A} \right\} \sqrt{\gamma} e^{2a_X} u^2 dS.\]

In view of Lemma 3.2, the limit of (3.19) exists when \(B \to \infty\). Hence

\[\lim_{B \to \infty} \inf \left| \int_{S_B} \sqrt{\gamma} e^{2a_X} u u_X dS \right| = 0.\]

Thus the limit of (3.18) exists when \(B \to \infty\). Q.E.D.

4. Proof of Theorem 1.1

If the assertion of Theorem 1.1 is not true, there exists some \(\delta > 0\), and

\[\lim_{t \to \infty} \inf \int_{P_t} (|u|^2 + |\nabla u|^2) dS = 0.\]

This is nothing but (3.1) of Lemma 3.1. Thus, for the proof of Theorem 1.1, it suffices to show the following assertion.

Let \(u\) be a solution of (1.1) and (1.2). If \(u\) satisfies (3.1), then \(u \equiv 0\) on the whole of \(\Omega\).

First note that

\[\int_{\partial_i(N)} X^\delta e^{mX^\delta} (|u|^2 + |\nabla u|^2) d\Omega < \infty\]

for any \(m > 0\), \(\kappa > 0\) and \(\beta < 1\). This is a direct consequence of Lemma 3.3.

Put \(v = e^{mX^\delta} u\), then

\[\Delta v - 2m\beta X^{\beta-1} \gamma v_X + (L + \lambda)v = 0,\]

where
We multiply (4.2) by $X^k v_X$ and integrate over $\Omega_A$. From (4.1) we have

\begin{equation}
-\int_{\Omega_A} X^k \left\{ \gamma v_X^2 - \frac{|\nabla v|^2}{2} + \frac{(L + \lambda)}{2} v^2 \right\} dS
- \int_{\Omega_A} (2m\beta X^{2+k-1} + (k-l)X^{\beta-1}) \gamma v_X^2 d\Omega +
\frac{1}{2} \int_{\Omega_A} X^{k-1} [(k+(n-3)l+\gamma)|\nabla v|^2 - ((k+(n-1)l-\gamma)(L+\lambda))
+ XL_X]v^2 d\Omega \geq 0
\end{equation}

for $\gamma > 0$ and sufficiently large $A$, say $A > N_9(\gamma)$. (See the proof of Lemma 2.5.) If we put $k = (3-n)l - \gamma$ in (4.4),

\begin{equation}
\int_{\Omega_A} X^{(3-n)l-\gamma} \left\{ \gamma v_X^2 - \frac{|\nabla v|^2}{2} + \frac{(L + \lambda)}{2} v^2 \right\} dS
\leq - \int_{\Omega_A} \left\{ 2m\beta X^{2+(3-n)l-\gamma-1} + ((2-n)l-\gamma)X^{(3-n)l-\gamma-1} \right\} \gamma v_X^2 d\Omega
- \int_{\Omega_A} X^{(3-n)l-\gamma-1} [(l-\gamma)(L+\lambda) + XL_X]v^2 d\Omega = I_1 + I_2.
\end{equation}

There is $N_9(\gamma) (> N_9(\gamma))$ such that $I_1 \leq 0$ for $A > N_9$. Assuming $m \geq 1$, and $\frac{1}{2} < \beta < 1$, we can take $N_9$ independently of $m$ and $\beta$.

Next note that

\begin{equation}
L \geq m^2 \beta^2 X^{2\beta-2} - mC_9(\beta)X^{\beta-2},
XL_X \leq m^2 \beta^2 (2\beta - 2) X^{2\beta-2} + mC_9(\beta)X^{\beta-2},
\end{equation}

where $C_9(\beta)$ and $C_9(\beta)$ are constants which are independent of $m$.

Now let us assume $\gamma$ is small so that $l-\gamma > 0$. If we take $\beta(<1)$ near to 1, then $l-\gamma > 2(1-\beta)$, and hence

\begin{equation}
(l-\gamma)(L+\lambda) + XL_X
\end{equation}
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\[ \geq m^2 (l - \eta + 2\beta - 2) X^{2\beta - 2} - m \{ (l - \eta) C_6(\beta) - C_3(\beta) \} X^{\beta - 2} \]
\[ \geq 0 \]

for large \(m\) and \(X\), say \(m \geq C_{10}\) and \(X \geq N_{10}(\eta) (> N_5)\), where \(N_{10}\) is independent of \(m\) (>\(C_{10}\)). Thus we have from (4.5)

\[ (4.6) \quad \int_{S_A} X^{(3-n)^{1-\theta}} \left\{ \tau v^2 - \frac{|Pv|^2}{2} + \frac{(L+\lambda) v^2}{2} \right\} dS I_1 + I_2 \leq 0 \]

for \(A > N_{10}\) and \(m > C_{10}\).

On the other hand, if we put

\[ \int_{S_A} X^{(3-n)^{1-\theta}} \left\{ \tau v^2 - \frac{|Pv|^2}{2} + \frac{(L+\lambda) v^2}{2} \right\} dS = m^2 M_1(u, A) + m M_2(u, A) + M_3(u, A), \]

(where \(M_1, M_2\) and \(M_3\) are independent of \(m\),) then it is easy to see \(M_1(u, A) > 0\) when \(u \equiv 0\) on \(S_A\). Note that \(v = e^x v^* u\). If we assume \(M_1(u, A) > 0\) for some \(A > N_{10}\),

\[ \int_{S_A} X^{(3-n)^{1-\theta}} \left\{ \tau v^2 - \frac{|Pv|^2}{2} + \frac{(L+\lambda) v^2}{2} \right\} dS > 0 \]

for sufficiently large \(m\). This contradicts (4.6), hence we see \(u \equiv 0\) on \(\Omega_{N_{10}}\). The unique continuation theorem for the second order elliptic equations enables us to conclude \(u \equiv 0\) on the whole of \(\Omega\). The proof of Theorem 1.1 is now complete.

5. On the Spectrum of \(-\Delta\)

This final brief section concerns the spectrum of \(-\Delta\) in \(\Omega\) with the Dirichlet boundary condition.

Let \(L\) be the operator in \(L^2(\Omega)\) with the domain \(D(L) = \{ f : f \in D_{1,1}, \Delta f \in L^2(\Omega) \}\), and \(Lu = -\Delta u\), where \(D_{1,1}\) is the completion of \(C^\infty_0(\Omega)\) with regard to the norm

\[ \| f \|_1 = \left\{ \int_{\Omega} \left( |f|^2 + | Pf |^2 \right) d\Omega \right\}^{\frac{1}{2}} \]
Then $L$ is a non-negative self-adjoint operator in $L^2(\mathcal{Q})$.

**Theorem 5.1.** Under the assumption on $\mathcal{Q}$ in §1, $L$ has no point eigenvalues. Moreover, the continuous spectrum of $L$ fills up the non-negative half of the real axis.

**Proof.** $L$ is a non-negative operator, and so it has no negative eigenvalues. Let $u \in D(L)$, and $Lu = 0$. Integrate $uLu$ over $\mathcal{Q}$, we have

$$
\int_{\mathcal{Q}} |Lu|^2 \, d\mathcal{Q} = 0.
$$

Hence $u = \text{constant}$. By the Dirichlet condition, $u = 0$, and so $\lambda = 0$ cannot be an eigenvalue of $L$. If $u \in D(L)$, then

$$
\lim_{t \to 0} \inf \int_{P_t} (|u|^2 + |Lu|^2) \, dS = 0.
$$

This shows that the non-existence of positive eigenvalues is a consequence of Theorem 1.1.

Next let us prove the latter half of the theorem. That is, we must prove that any non-negative real number $\lambda$ belongs to the spectrum of $L$.

Let $\varphi = \varphi(X_1, \cdots, X_n)$ be a function which is in $C^\infty(\mathcal{Q}, N_0, k_0)$, and $\varphi_m = \varphi(X_1/m, X_2, \cdots, X_n)$. Put

$$
\Phi_m = e^{i \sqrt{\lambda} X_n} \varphi_m / \nu_m,
$$

where $\nu_m = ||\varphi_m||_{L^2(\mathcal{Q})}$. It is not difficult to show that

$$
||L\Phi_m - \lambda \Phi_m||_{L^2(\mathcal{Q})} \to 0 \quad (m \to \infty).
$$

Taking subsequence if necessary, we may assume $	ext{supp} \Phi_i \cap \text{supp} \Phi_j = \emptyset$ ($i \neq j$) ($\Phi_i$ and $\Phi_j$ $(i \neq j)$ are orthogonal), where $\text{supp} \Phi$ denotes the support of $\Phi$. This shows that $\lambda$ is in the spectrum of $L$, because, if not, $\{\Phi_m\}$ would tend to 0, which is impossible, however, on account of $||\Phi_m||_{L^2(\mathcal{Q})} = 1$.

Q.E.D.
Solutions of $(d + \lambda)u = 0$

References


