On Quasifree States of the Canonical Commutation Relations (II)

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Abstract

A necessary and sufficient condition for the quasi-equivalence of two quasifree primary representations of the canonical commutation relations is derived.

§ 1. Introduction

A quasifree state of the self-dual CCR algebra $\mathfrak{H}(K, \gamma, I)$, which is a slight generalization of conventional canonical commutation relations, has been discussed in the preceding work [1]. In the present paper, we derive a necessary and sufficient condition for the quasi-equivalence of representations associated with quasifree states, when the representation is primary (i.e. the associated von Neumann algebra is a factor).

We believe that the following features of the present analysis is worth mentioning.

(1) Despite of many marked differences on mathematical structure between the present case of CCR and the case of CAR [2], such as unbounded B($f$) for CCR and bounded B($f$) for CAR, the indefinite metric $\gamma$ for the test function space $K$ of CCR and the definite metric for $K$ of CAR, and the difference in the details of the final statement, the two cases can be treated by essentially the same technique, yielding quite a similar results.

(2) For CAR, there is a unique $C^*$ norm for the * algebra generated by B($f$). In the present case, there is no intrinsic topology in the * algebra generated by B($f$). As a result, the topology induced by the representation plays an important role and serves as an invariant in the

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quasi-equivalence classification of quasifree states.

(3) On a Hilbert space of a definite metric, natural vector topologies are induced by the metric. On a Hilbert space of an indefinite metric, the natural topology is generally too weak. Indeed, it is the weak topology by its algebraic dual in the present case. It seems that a Hilbert space with an indefinite metric equipped with a certain class of vector topology is more canonical object to study. Problems concerning the structure \((K, \gamma, I')\) and Hilbert Schmidt class operators on \(K\), which we have treated with a help of ordinary tools on a Hilbert space with a definite metric, might serve as a testing ground for any general theory of a Hilbert space with an indefinite metric.

The study of quasifree states is probably of no direct physical interest. However, we know an example of the mathematical structure of free Bose gas analysed in \([3]\), which turned out to be common to a large class of systems \([4\sim 7]\). Our hope is that a complete analysis of CCR and CAR in the present paper and in \([2]\) presents similar useful examples.

In section 2, we obtain simpler properties of the von Neumann algebra associated with quasi-free states. In section 3, a quasifree state is viewed as a KMS state relative to a Bogoliubov automorphism. In section 4, a bilinear Hamiltonian is introduced which is used in section 5 to discuss the unitary implementability of a Bogoliubov transformation on a Fock type representation. In section 6, a necessary and sufficient condition for the quasi-equivalence of two quasifree primary states of CCR is obtained as the main Theorem.

We shall freely use the notation in \([1]\).

\section{Simple Properties of von Neumann Algebras Associated with Quasifree States}

\textbf{Lemma 2.1.} Let \(\omega'\) be a mapping of \(K'_S\) onto itself given by

\begin{equation}
(2.1) \quad \omega'(f \oplus g) = \Gamma_S g \oplus \Gamma_S f.
\end{equation}

\(\omega'\) leaves \(N'_S\) invariant and induces a mapping \(\omega\) of \(K'_S/N'_S\) onto itself. Its closure, denoted again by \(\omega\), as a mapping of \(\hat{K}_S\) onto itself is an
antilinear involution satisfying.

\[ [\omega, \hat{F}_s] = 0, \]
\[ (\omega h_1, \omega h_2) = (h_2, h_1), \]

and

\[ \hat{F}_S(\omega h_1, \omega h_2) = \hat{F}_S(h_2, h_1). \]

Let \( \tau(\omega) \) be a mapping of \( \hat{\mathfrak{G}}_S, \hat{\mathfrak{F}}_S, \hat{\mathfrak{F}}_S \) onto itself given by

\[ \tau(\omega) \sum_i c_i B(h_i^1) \cdots B(h_i^n) = \sum_i c_i^* B(\omega h_i^1) \cdots B(\omega h_i^n). \]

Then it is a conjugate * automorphism. There exists an antiunitary in-


volution \( T_{\Pi S}(\omega) \) on \( \xi_{\Pi S} \) uniquely determined by

\[ T_{\Pi S}(\omega) \pi_{\Pi S}(A) \Omega_{\Pi S} = \pi_{\Pi S}(\tau(\omega) A) \Omega_{\Pi S}. \]

**Proof.** From the definition (2.1), if follows that \( \omega' \) is an antilinear

involution leaving \( N_s \) invariant and hence the same is true for \( \omega \). (2.2)

follows immediately from \( [\omega', F_S'] = 0 \). (2.3) follows from the antiuni-
tarity of \( F_s \) relative to \( (f, g)_S \) and (6.4) of \( [1] \). (2.4) follows from

(6.2) of \( [1] \) and \( F_S \Gamma_S F_S = -\Gamma_S \).

By the antilinearity of \( \omega \), (2.2) and (2.4), \( \tau(\omega) \) defined by (2.5)

preserves the three relations in the definition of a selfdual CCR algebra

(Section 2 in \( [1] \)) and hence \( \tau(\omega) \) is a conjugate * automorphism.

Due to (2.3) and (2.4), we have \( \varphi_{\Pi S}(\tau(\omega) A) = \varphi_{\Pi S}(A)* \), from which

the existence of \( T_{\Pi S}(\omega) \) follows.

Q. E. D.

**Lemma 2.2.** The set of \( h \in \hat{\mathfrak{K}}_S \) satisfying \( \hat{F}_S([f], h) = 0 \) for all

\( f \in \text{Re } K \) is \( \{E_0 K_S \oplus K_S\}^\wedge \).

**Proof.** If \( f \in \text{Re } K \), \( f_1 \in E_0 K_S \) and \( g_1 \in K_S \), then \( \tau_S(f \oplus 0, f_1 \oplus g_1) = 0 \). Hence \( \hat{F}_S([f], h) = 0 \) for \( h \in \{E_0 K_S \oplus K_S\}^\wedge \).

Let \( f, g \) be elements of \( (1-E_0)K_S \) in the domain of \( (2S-1)^{-1} \). Let

\[ h(f \oplus g) = (2S-1)^{-1}(f + 2S^{1/2}(1-S)^{1/2} g) \]

\[ \oplus -(2S-1)^{-1}(g + 2S^{1/2}(1-S)^{1/2} f). \]

Then we have
(2.8) \[ \tau'_S(f_1 \oplus g_1, h(f_2 \oplus g_2)) = (f_1 \oplus g_1, f_2 \oplus g_2)'_S. \]

Since \( \hat{\tau}'_S[\bar{E}_0 K \oplus E_0 K] = 0 \), since the restriction of \( \hat{\tau}'_S \) to the orthogonal complement of \( \bar{E}_0 K \oplus E_0 K \) is 1 and since \( (1 - E_0)K \oplus (1 - E_0)K \) is orthogonal to \( E_0 K \oplus E_0 K \), we have from (2.8)

(2.9) \[ \{h(f_2 \oplus g_2)\}^\wedge = \hat{\tau}'_S \{(f_2 \oplus g_2)^\wedge\}. \]

Therefore \( \hat{\tau}'_S[\bar{K}] \) contains vectors \( \{h(f \oplus 0)\}^\wedge \), where \( f \) is any element of \( (1 - E_0)K \) in the domain of \( (2S - 1)^{-1} \). Therefore \( \hat{\tau}'_S[\bar{K}] + \{E_0 K \oplus K_3\}^\wedge \) is dense in \( \{K_3 \oplus K_3\}^\wedge \) and hence in \( \hat{K}_S \).

Since \( (\|f \oplus g\|^2)_S = \|f + g\|^2 \) for \( f \in E_0 K \) and \( g \in K_S \), \( \{E_0 K_S \oplus K_3\}^\wedge \) is closed. Therefore it is the orthogonal complement of \( \hat{\tau}'_S[\bar{K}] \). Q.E.D.

**Lemma 2.3.** Let \( R_S \) be the von Neumann algebra generated by spectral projections of all \( \pi_{\bar{H}}(\alpha_3 B(f)), f \in \Re K \) on the representation space of \( \mathcal{H}(\hat{K}_S, \hat{\tau}, \hat{F}_S) \) associated with \( \varphi_{\bar{H}} \). Then the following conditions are equivalent:

1. \( \mathcal{Q}_{\bar{H}} \) is cyclic for \( R_S \).
2. \( \mathcal{Q}_{\bar{H}} \) is separating for \( R_S \).
3. \( S \) does not have an eigenvalue 0.
4. \( S \) does not have an eigenvalue 1.

**Proof.** Using the notation of Lemma 5.5 of [1], we have

\[ R_S = R\bar{H}_S(H_1) \]

with \( H = \Re(\hat{K}_S \oplus \hat{E}_0 \hat{K}_S), \ H_1 = \Re K \oplus 0, \ \hat{H}_1 = \{\Re K_S \oplus 0\}^\wedge \oplus 0 \) where \( \hat{E}_0 \) is the eigenprojection of \( \Pi_S \) for an eigenvalue 1/2. (Since \( \|f \oplus 0\|^2 = \|f\|^2 \), \( \{K_3 \oplus 0\}^\wedge \) is closed.) Since \( \hat{E}_0 \hat{K}_S \subset \{K_3 \oplus 0\}^\wedge, \ H_1 \subset \Re \hat{K}_S \) and hence we obtain from Lemma 2.2 \( H_1^\wedge = \Re \{(E_0 K_S \oplus K_3)\}^\wedge \).

(1)\( \rightarrow \)(3): Assume that (3) does not hold and \( Sf = 0 \) for \( f \in K_S \), \( f \neq 0 \). Then \( h(0 \oplus f) = 0 \oplus f \) and hence \( \Pi_S \{(0 \oplus f)^\wedge\} = (0 \oplus f)^\wedge \). Further \( (0 \oplus f, g \oplus 0)_S^{} = 0 \) for any \( g \in K_S \). This implies \( (0 \oplus f)^\wedge \oplus 0, [g \oplus 0]_{\bar{H}_S} = 0 \) due to \( \hat{E}_0 \{(0 \oplus f)^\wedge\} = 0 \). Therefore \( \bar{H}_S(H_1 + iH_1) \) is not dense in \( \bar{H}_S \hat{K}_S \) and \( \mathcal{Q}_{\bar{H}_S} \), which can be identified with \( \mathcal{Q}_{\bar{H}_S} \), is not cyclic due to Lemma 5.5 (v) of [1].

(3)\( \rightarrow \)(1): Let
If \( f \in (1 - E_0)K_S \) is in the domain of \((2S - 1)^{-1}\), we have

\[
\text{(2.12) } h(k_\pm(f)) = \pm k_\pm(f).
\]

Hence \( \{k_+(f)\} \in \hat{E}_+K_S, \ \{k_-(f)\} \in \hat{E}_-K_S \), where \( \hat{E}_+ \) and \( \hat{E}_- \) are eigenprojections of \( \Pi_S \) for eigenvalues 1 and 0. By the continuity, this holds for any \( f \in K_S \). Further,

\[
\begin{align*}
\text{(2.13) } & k_+(S^{1/2}f) + k_-((1 - S)^{1/2}f) = (2S - 1)f \oplus 0, \\
\text{(2.14) } & k_-(S^{1/2}f) + k_+((1 - S)^{1/2}f) = 0 \oplus (2S - 1)f.
\end{align*}
\]

Therefore the set of \( k_+(f) \oplus k_-(f) \) is dense in \((1 - E_0)K_S\) and hence \( k_+(f) \) and \( k_-(f) \) are dense in \( \hat{E}_+K_S \) and \( \hat{E}_-K_S \), respectively.

Since

\[
\Pi_S\{(f \oplus 0)^\wedge\} = (1/2)(1 + \gamma_S)\{(f \oplus 0)^\wedge\} = k_+((2S - 1)^{-1}S^{1/2}f)^\wedge
\]

for \( f \in (1 - E_0)K_S \) and since \( f \in K_S \rightarrow k_+(f) \in \hat{K}_S \) is continuous, \( \Pi_S\{(K_S \oplus 0)\oplus 0\} = \hat{\Pi}_S\{(1 - E_0)K_S \oplus 0\} \oplus 0\) is dense in \( \hat{\Pi}_S\hat{K}_S \) if (3) holds.

(3) \( \implies \) (4). This follows from \( \Gamma_S\hat{\Gamma}_S = 1 - S \).

(4) \( \implies \) (2). By Lemma 5.5 (ii) of \([1]\), Lemma 2.1 and Lemma 2.2

\[
R_S' = T_{\Pi_S}(\omega)R_S T_{\Pi_S}(\omega).
\]

Furthermore, we have \( \varphi_{\Pi_S}(\tau(\omega)A^\circ) = \varphi_{1^-}(A) \) for \( A \in \alpha_\theta(\Pi_S, \gamma, \Gamma) \). Therefore by (3) \( \implies \) (1), \( \varphi_{\Pi_S} \) is cyclic for \( R_S' \) if and only if (4) holds. Since \( \varphi_{\Pi_S} \) is separating for \( R_S \) if and only if it is cyclic for \( R_S' \), we have (4) \( \implies \) (2).

\[
\text{Q. E. D.}
\]

\textbf{Lemma 2.4.} \textit{The center of } \( R_S \text{ is generated by } \exp i\pi_{\Pi_S}(B(h)), \ h \in \text{Re}\ (E_0K_S \oplus 0)^\wedge. \ \text{In particular, } R_S \text{ is a factor if and only if } K_0 = 0.}

\textit{Proof.} From the beginning part of the proof of Lemma 2.3, we have \( \hat{H}_1 \cap H_1^\perp = (E_0K_S \oplus 0)^\wedge \oplus 0 \) and hence this Lemma follows from Lemma 5.5 of \([1]\).
Lemma 2.5. \( \Omega_{\mathcal{H}_S} \) is separating for the center of \( R_S \).

Proof. Since \( \{ (\text{Re} E_0 K_S \oplus 0)^{\gamma} \oplus 0 \}^{\perp} = \text{Re} \tilde{K}_S \oplus 0 \) and since \( \tilde{\Pi}_S \tilde{K}_S \oplus 0 \) is dense in \( \tilde{\Pi}_S \tilde{K}_S \), we obtain the Lemma by Lemma 5.5 (vi) of [1].

Q. E. D.

§ 3. KMS Conditions

Lemma 3.1. Suppose that \( S \) does not have an eigenvalue 0. Let

\[
H_S = \log \{ S(1-S)^{-1} \}.
\]

Then \( \exp iH_S \) is a Bogoliubov transformation on \( K_S \) and \( \varphi_S \) is \( \tau(\exp iH_S) \) invariant.

Proof. Since \( 0 < S < 1 \), we have \( H_S^* = H_S \). Since \( S \) commute with \( \gamma_S \), we have \( \gamma(e^{iH_S f}, e^{iH_S g}) = \gamma(f, g) \). Since \( \Gamma_S \Gamma_S = 1 - S \), we have \( \Gamma_S H_S \Gamma_S = -H_S \). Therefore \( \exp iH_S \) is a Bogoliubov transformation.

Since \( S(f, g) \) is invariant under this transformation due to \( [S, H_S] = 0 \), \( \varphi_S \) is \( \tau(\exp iH_S) \) invariant.

Q. E. D.

Definition 3.2. Suppose that \( S \) does not have an eigenvalue 0. Let \( \Theta_S \) be an infinitesimal generator defined by

\[
\exp it\Theta_S = T_S(\exp itH_S),
\]

where \( T_S(\cdot) \) is defined on \( \mathcal{H}_S \) by Definition 3.6 of [1].

Lemma 3.3. Suppose that \( S \) does not have an eigenvalue 0 and identify \( \mathcal{H}_S \), \( \pi_S(A) \) and \( \Omega_S \) with \( \mathcal{H}_{\pi_S} \), \( \pi_{\pi_S}(\alpha_S A) \) and \( \Omega_{\pi_S} \). If \( A \in R_S \) then

\[
T_{\pi_S}(\omega)A\Omega_{\pi_S} = e^{-\theta_{\pi_S}^2/2}A^*\Omega_{\pi_S}.
\]

Proof. Let \( f \in D(S^{-1/2}) \). Then

\[
T_{\pi_S}(\omega)\pi_{\pi_S}B((f \oplus 0)^{\gamma})\Omega_{\pi_S} = \pi_{\pi_S}B((\Gamma_S f)^{\gamma})\Omega_{\pi_S}
\]

\[
= \pi_{\pi_S}B(((1 - S)^{1/2}S^{-1/2}f \oplus 0)^{\gamma})\Omega_{\pi_S}
\]

\[
= e^{-\theta_{\pi_S}^2/2}\pi_{\pi_S}B((f \oplus 0)^{\gamma})^*\Omega_{\pi_S},
\]

where we have used
not only for \( f \in (1 - E_0)K_S \cap D(S^{-1/2}) \) but also for \( f \in E_0K_S \). By using this result repeatedly, and by using the commutativity of \( T_{\Pi_S}(\omega)\pi_{\Pi_S}\[B((f\oplus0)^{\wedge})]\Pi_S(\omega) \) with \( e^{-\theta s^{1/2}}\pi_{\Pi_S}\[B((g\oplus0)^{\wedge})]\e^{\theta s^{1/2}} \), we obtain

\[
(3.4) \quad T_{\Pi_S}(\omega)\pi_{\Pi_S}\[B((f_1\oplus0)^{\wedge})]\cdots\pi_{\Pi_S}\[B((f_n\oplus0)^{\wedge})]\Pi_S = e^{-\theta s^{1/2}}\pi_{\Pi_S}\[B((f_1\oplus0)^{\wedge})]\cdots\pi_{\Pi_S}\[B((f_n\oplus0)^{\wedge})]\Pi_S.
\]

If the support of the Fourier transform of \( e^{iHsf} \) is in \([-l, l]\), then \( e^{-\theta s^{1/2}} \) is bounded by \( e^{-n^{1/2}} \) on \( \pi_{\Pi_S}\[B((f\oplus0)^{\wedge})]\Pi_S \). By the estimate (5.5) of \([1]\), we have the convergence of

\[
(3.5) \quad e^{-\theta s^{1/2}}W_S(f)\Pi_S = \sum_{n=0}^{\infty} n^{1-1}i^n e^{-\theta s^{1/2}}\pi_{\Pi_S}\[B((f\oplus0)^{\wedge})]\Pi_S
\]

where \( f \) is assumed to be in \( \text{Re} K_S \) and \( W_S(f) \) denotes \( W_{\phi_s}(f) = W_{\Pi_S}(\{(f\oplus0)^{\wedge}\oplus0\}) \). Therefore

\[
(3.6) \quad T_{\Pi_S}(\omega)W_S(f)\Pi_S = e^{-\theta s^{1/2}}W_S(-f)\Pi_S.
\]

A linear combination of \( W_S(f) \) such that \( e^{iHsf} \) has a Fourier transform with a compact support, is dense in \( R_S \). Therefore

\[
(e^{-\theta s^{1/2}}\Psi, A^*\Pi_S) = (\Psi, T_{\Pi_S}(\omega)A\Pi_S)
\]

holds for any \( A \in R_S \) and \( \Psi \) in the domain of \( e^{-\theta s^{1/2}} \). This implies that \( A^*\Pi_S \in D(e^{-\theta s^{1/2}}) \) and (3.3) holds.

\[ Q. \ E. \ D. \]

**Corollary 3.4.** \( \phi_s \) is a KMS state of \( (K_S, \gamma_S, \Gamma_S) \) for the automorphism \( \tau(\exp itH_S) \).

**Proof.** This follows from the antiunitarity of \( T_{\Pi_S}(\omega) \).

**Corollary 3.5.** Let \( j(A) = \Pi_S(\omega)AT_{\Pi_S}(\omega) \) for \( A \in R_S \). Then \( \langle \Omega_{\Pi_S}, A\pi_s(\Omega_{\Pi_S}) \rangle \geq 0 \).

**§ 4. Bilinear Hamiltonian**

**Lemma 4.1.** Assume that \( \gamma \) is non-degenerate. Let \( K_1 \) be finite dimensional subspace of \( K \). Then there exists a \( \Gamma \)-invariant finite dimensional
subspace $K_1'$ of $K$ such that $K_1' \supset K_1$, and the restriction of the hermitian form $\gamma$ to $K_1'$ is nondegenerate. Further, there exists a basis projection $P$ for $(K_1', \gamma, \Gamma)$.

Proof. Let $f_1 \cdots f_n$ be a linearly independent basis of $K_1$. Let $K_2 = K_1 + \Gamma K_1$ and $f_j'$ be a maximal linearly independent subset of $f_k + \Gamma f_k$, $i(f_k - \Gamma f_k)$, $k = 1, \ldots, n$. $f_j'$ is a complete set of linearly independent $\Gamma$ invariant vectors of $K_2$.

Since $f_j'$ is invariant, $\gamma(f_{j_1}', f_{j_2}') = \gamma(f_{j_1}, f_{j_2}) = \gamma(\Gamma f_{j_1}', \Gamma f_{j_2}') = -\gamma(f_{j_1}', f_{j_2}')$ and hence $\gamma(f_{j_1}', f_{j_2}')$ is purely imaginary. In particular, $\gamma(f_{j_1}', f_{j_2}') = 0$.

If $\gamma$ restricted to $K_2$ is not identically 0, let $f_{j_1}'$ and $f_{j_2}'$ be a pair such that $\gamma(f_{j_1}', f_{j_2}') = 0$ and set $e_1 = f_{j_1}'$, $e_2 = i[\gamma(f_{j_1}', f_{j_2}')]^{-1} f_{j_2}' - i\gamma(e_2, f_{j_2}') e_1 + i\gamma(e_1, f_{j_2}') e_2$. Then $e_1$ and $e_2$ are $\Gamma$ invariant, $\gamma(e_1, e_2) = i$ and $\gamma(e_1, f_{j_2}') = \gamma(e_2, f_{j_2}') = 0$. Apply the same procedure to $f_{j_2}'$. Repeat this process until we obtain $e_{2k-1} e_{2k}$, $k = 1, \ldots, l$ and $f_{j_1}^{(l+1)}$, $j > 2l$, such that $\gamma(f_{j_1}^{(l+1)}, f_{j_1}^{(l+1)}) = \gamma(e_j, f_{j_1}^{(l+1)}) = 0$, $\gamma(e_j, e_{j'}) = 0$ unless $(j, j') = (2k-1, 2k)$ or $(2k, 2k-1)$ and $\gamma(e_{2k-1}, e_{2k}) = i$, $k = 1, \ldots, l$.

Next, let $g_k$, $k = 1, \ldots, s$ be a maximal linearly independent subset of $f_{j_1}^{(l+1)}$. Let $h_1 \in K$ such that $\gamma(g_1, h_1) \neq 0$. Let $h_1'$ be either one of $\Gamma h_1 + h_1$ and $i(h_1 - \Gamma h_1)$ such that $\gamma(g_1, h_1') \neq 0$. Let $e_{2l+1} = h_1' - i \sum_{k=1}^{s} \gamma(e_{2k}, h_1') e_{2k-2} - \gamma(e_{2k-1}, h_1') e_{2k}$. We have $\gamma(e_{2l+1}, g_k) = \gamma(h_1', g_k) = 0$. Let $e_{2l+2} = i \gamma(e_{2l+1}, g_k) e_{2l} - i \gamma(e_{2l+1}, g_k) e_{2l+2}$. Next apply this procedure to $g_k$, $k = 2, \ldots, s$. After repeating this process $s$ times, we obtain $e_j$, $j = 1, \ldots, 2l + 2s$ such that $\Gamma e_j = e_j$, $\gamma(e_j, e_{j'}) = 0$ unless $(j, j') = (2k-1, 2k)$ or $(2k, 2k-1)$ and $\gamma(e_{2k-1}, e_{2k}) = i$, $k = 1, \ldots, l + s$. Further, the subspace $K_1'$ of $K$ generated by $e_1 \cdots e_{2l+2s}$ and the projection $P$ defined by $P f = i \sum_k \{ \gamma(e_{2k}, f) e_{2k-1} - \gamma(e_{2k-1}, f) e_{2k} \}$ have the desired properties.

Q. E. D.

Remark. We see from the above proof that if $K$ has a finite dimension, then dim $K$ is even and there exists a basis projection $P$ for $(K, \gamma, \Gamma)$.

Lemma 4.2. If $\gamma$ is non-degenerate, $\mathcal{U}(K, \gamma, \Gamma)$ is simple (as a
* algebra) and has a trivial center.

Proof. Let \( \mathfrak{X} \) be a non-zero two sided \(*\) ideal of \( \mathfrak{A}(K, \gamma, \Gamma) \) and \( A \) be a non-zero element of \( \mathfrak{X} \). \( A \) is a polynomial of \( B(f_1), \ldots, B(f_n) \) for a finite number of \( f_1, \ldots, f_n \in K \). Let \( K_1 \) be the subspace of \( K \) generated by \( f_1, \ldots, f_n \) and apply the preceding Lemma. \( A \) can be written as

\[
A = \sum_{n=0}^{N} B(e_1)^n \mathcal{P}_n,
\]

where \( \mathcal{P}_n \) is a polynomial of \( B(e_k), k=2, 3, \ldots \). We may assume that \( \mathcal{P}_N \neq 0 \) if \( A \neq 0 \). Let \( A_1 = \mathcal{P}_0 \) if \( N=0 \) and \( A_1 = [B(e_2), \ldots [B(e_2), A] \ldots ] \) (the \( N \) fold commutator with \( B(e_2) \)) if \( N>0 \). Then \( A_1 \in \mathfrak{X} \), \( A_1 = (-i)^N \mathcal{P}_N \neq 0 \), and \( A_1 \) no longer contains \( B(e_1) \). Repeating this process, we see that \( 1 \in \mathfrak{X} \) and hence \( \mathfrak{X} = \mathfrak{A}(K, \gamma, \Gamma) \). Namely \( \mathfrak{A}(K, \gamma, \Gamma) \) is simple as a \(*\) algebra.

Next, let \( A \) be a central element of \( \mathfrak{A}(K, \gamma, \Gamma) \) and (4.1) holds. Assume that \( \mathcal{P}_N \neq 0 \). If \( N \neq 0 \), then we have \( P_N = (N!)^{-1} i^N [B(e_2), \ldots [B(e_2), A] \ldots ] = 0 \), which is a contradiction. Hence \( N=0 \). Repeating this reasoning, we see that \( A \) must be a multiple of the identity operator.

Q. E. D.

Definition 4.3. Assume that \( \gamma \) is non-degenerate. Let \( H \) be a finite rank operator on \( K \) satisfying

\[
Hf = \sum_{j=1}^{N} \gamma(g_j, f) f_j
\]

for any \( f \in K \). Then \( (B, HB) \in \mathfrak{A}(K, \gamma, \Gamma) \) is defined by

\[
(B, HB) = \sum_{j=1}^{N} B(f_j) B(\Gamma g_j).
\]

Remark. An operator \( H \) on \( K \) is called a finite rank operator if its domain is \( K \) and its range has a finite dimension. Any finite rank operator \( H \) can be written as \( Hf = \sum_{j=1}^{N} e_j(f) f_j \) for all \( f \in K \), where \( f_j \in K \) and \( e_j \) is in the algebraic dual of \( K \). The trace of \( H \) is then defined by

\[
\text{tr} H = \sum_{j=1}^{N} e_j(f_j)
\]

and is independent of the choice of \( f_j \) and \( e_j \) for a given \( H \). If \( H \) is of
finite rank, then \( AH \) is of finite rank for any linear operator \( A \) defined on the whole \( K \), and if \( H \) is given by (4.2), then

\[
\text{tr } AH = \sum_{j=1}^{N} \gamma(g_j, Af_j).
\]

**Lemma 4.4.** Assume that \( \gamma \) is non-degenerate. \( (B, HB) \) is independent of the choice of \( f_j \) and \( g_j \), for a given \( H \), is linear in \( H \) and satisfies

\[
\begin{align*}
\varphi_s((B, HB)) &= \sum_j S(f_j, g_j), \\
(B, HB) &= (B, H^\dagger B),
\end{align*}
\]

where \( H^\dagger \) is defined by

\[
H^\dagger f = \sum_{j=1}^{N} \gamma(f_j, f) g_j
\]

if \( H \) satisfies (4.2), and

\[
\alpha(H) = \frac{1}{2}(H - \Gamma H^\dagger \Gamma),
\]

which satisfies

\[
\Gamma \alpha(H)^\dagger \Gamma = -\alpha(H),
\]

\[
(B, HB) = (B, \alpha(H)B) - \frac{1}{2} \text{tr } H.
\]

Conversely, if \( H \) satisfies

\[
\Gamma H^\dagger \Gamma = -H,
\]

then

\[
\alpha(H) = H.
\]

**Proof.** (4.11) obviously satisfies (4.10) and (4.10) uniquely specifies \( H^\dagger \) by the non-degeneracy of \( \gamma \). (4.9) follows immediately from (4.11). We have
The text is a continuation of the previous paragraph, discussing commutation relations and their implications. It involves the use of operators \( B \) and \( H \) in a Hilbert space context, with a focus on the demonstration of certain properties and equalities. The notation and symbols are consistent with the previous discussion, involving sums, products, and transformations.

The key points include:

1. **Commutation Relations**: The relation \( [(B, H B), B(f)] = \sum_{j=1}^{N} B(f_j) r(g_j, f) + r(\Gamma f_j, f) B(\Gamma g_j) \)
   
2. **Equality**: Use of the equality \( (4.18) \) to simplify expressions.

3. **Further Properties**:
   - Suppose the same \( H \) can be written in two different ways in terms of \( f_j, g_j \) and \( f'_j, g'_j \).
   - Let \( K_1 \) be the subspace spanned by \( f_j, g_j, f'_j, g'_j \).
   - Let \( K_{\gamma_1} \) be the subspace spanned by \( f_j, g_j \).
   - Then \( (B, H B) \) given by \( (4.3) \) can be considered as an element of \( \gamma_1 \).

4. **Independence**:
   - Since the center of \( \gamma(K, \gamma, \Gamma) \) is trivial, \( (4.6) \) implies that \( (B, H B) \) is independent of the choice of \( f_j, g_j \) for a given \( H \).

5. **Final Result**: The expression \( (4.19) \) is shown to be independent of the choice of \( f_j, g_j \) for a given \( H \).

The text concludes by proving certain properties and formulas, maintaining the coherency and logical progression of the mathematical arguments.
By (4.13), we have $\alpha(H')g_j = -\Gamma\alpha(H')^*g_j$ and hence
$\begin{bmatrix} (B, H'B), (B, HB) \end{bmatrix} = 2(B, \{\alpha(H')H - H\alpha(H')\}B).
By (4.14), this is the same as $2(B, [\alpha(H'), \alpha(H)]B)$. Q. E. D.

**Remark 4.5.** $(B, HB)$ defined by (4.3) and $H^\dagger$ defined by (4.11) are not uniquely determined by $H$ for a general $\gamma$.

**Lemma 4.6.** For any choice of $f_j$ and $g_j$ satisfying (4.2), the formulae (4.6), (4.7) and (4.8) hold, $H^\dagger$ defined by (4.11) satisfies (4.8) and (4.10), and $\alpha(H)$ defined by (4.12) satisfies (4.14). If $\gamma(f, Hg) = \gamma(Hf, g)$ for all $f$ and $g$ in $K$, then there exists a choice of $f_j$ and $g_j$ such that $H^\dagger$ defined by (4.11) coincides with $H$.

**Proof.** First half follows from the computation in the proof of Lemma 4.4. For the second half, assume that $\gamma(f, Hg) = \gamma(Hf, g)$ for all $f$ and $g$ in $K$ and $H$ is expressed as $Hf = \sum_{j=1}^{N'} \gamma(g_j^*, f)f_j^*$. Let $H_1 = H - H^\dagger$, where $H^\dagger$ is defined by (4.11) using $f_j$ and $g_j$. From (4.10) and the assumption, $\gamma(H_1f, g) = 0$ for all $f$ and $g$. Namely, the range of $H_1$ is in the Null space of $\gamma$ (the set of $f$ such that $\gamma(f, g) = 0$ for all $g \in \mathcal{K}$). Then $H_1$ has a representation $H_1f = \sum_{j=1}^{N'} \gamma(g_j^*, f)f_j^*$ where $f_j^*$ is in the Null space of $\gamma$. Then $H_1^\dagger$ defined by (4.11) using $f_j^*$ and $g_j^*$ is 0 as operator due to $\gamma(f_j^*, f) = 0$ for all $f$. Hence we have $H_1f = \sum_{j=1}^{N'} \gamma(g_j^*, f)f_j^* + \sum_{j=1}^{N^*} \gamma(g_j^*, f)f_j^*$. $H^\dagger$ using this representation is $H$. Q. E. D.

**Lemma 4.7.** Let $\varphi_\eta$ be a Fock type state. Let $E_0$ be the eigenprojection of $\Pi$ for an eigenvalue $1/2$. Let $H$ be a finite rank operator on $\mathcal{K}_\eta$ such that $E_0H = HE_0 = 0$. Then $H$ can be represented by (4.2) with $f_j$ and $g_j$ in $(1 - E_0)\mathcal{K}_\eta$ and $(B, HB)$ defined by (4.3) and $H^\dagger$ defined by (4.11) do not depend on the choice of such $f_j$ and $g_j$. If $\gamma(f, Hg) = \gamma_n(H_\eta f, g)$ for all $f$ and $g$, then $H^\dagger = H$.

**Proof.** Since $H = (1 - E_0)H(1 - E_0)$, we may restrict our attention to $(1 - E_0)\mathcal{K}$, where $\gamma$ is non-degenerate. Hence the present Lemma follows from Lemma 4.4. Q. E. D.
§ 5. Unitarily Implementable Bogoliubov Automorphisms

**Lemma 5.1.** Let $H$ be a finite rank operator such that $\gamma(f, Hg) = \gamma(Hf, g)$ and $\Gamma H \Gamma^* = -H$. Fix $f_i$ and $g_i$ in (4.2) such that $H^r = H$ and define the corresponding $(B, HB)$. Let $\varphi_\Omega$ be a Fock type state of $(K, \gamma, \Gamma)$. Then $D_0 = \pi_\Omega[\mathcal{B}(K, \gamma, \Gamma)]\Omega_\Omega$ is a dense set of analytic vectors for $\pi_\Omega[\mathcal{B}(B, HB)]$. The unitary operator

\[ Q_n(H) = \exp(i/2)\pi_\Omega[\mathcal{B}(B, HB)] \]

satisfies

\[ Q_n(H)W_n(f)Q_n(H)^* = W_n(e^{iH}f) \]

for $f \in \Re K$ and

\[ Q_n(H)\pi_\Omega(A)Q_n(H)^* = \pi_\Omega(\tau(e^{iH})A) \]

for $A \in \mathcal{B}(K, \gamma, \Gamma)$, where $\pi_\Omega(A)$ denotes the closure of $\pi_\Omega(A)$.

**Proof.** We shall use Lemma 5.8 of [1] and identify $\pi_\Omega(f \oplus 0)$ and $\Omega_\Omega$ with $\pi_\Omega(f)$ and $\Omega_\Omega$. By Lemma 5.5 (v) of [1], it is easily seen that $\Omega_\Omega$ is cyclic for $\pi_\Omega(\mathcal{B}(K, \gamma, \Gamma))$ and hence we may also identify the whole space $\mathfrak{D}_\Omega$ with $\mathfrak{D}_\Omega$. It follows that $D_0$ is dense.

From (5.5) of [1] we obtain

\[ \|\pi_\Omega[\mathcal{B}(B, HB)]\Psi\| \leq [(N+2)(N+1)]^{1/2}G\|\Psi\| \]

for a constant $G$ independent of $N$, where $\Psi \in \sum_{n=0}^{N} (\mathfrak{D}_\Omega)_n$. Since $\pi_\Omega[\mathcal{B}(B, HB)]$ increases $N$ at most by 2, we have

\[ \|\pi_\Omega[\mathcal{B}(B, HB)]^n\Psi\| \leq [(N+2n)!N!^{-1}]^{1/2}G^n\|\Psi\|. \]

Therefore, $\sum n!^{-1}t^n||\pi_\Omega[\mathcal{B}(B, HB)]^n\Psi|| < \infty$ for small $t > 0$ and such $\Psi$ is analytic for $\pi_\Omega[\mathcal{B}(B, HB)]$. Since $(B, HB)^* = (B, HB)$ by (4.9), the closure $\pi_\Omega[\mathcal{B}(B, HB)]$ is selfadjoint and

\[ Q_n(iH)\Psi = \sum_{n=0}^{\infty} n!^{-1}((it/2)\pi_\Omega[\mathcal{B}(B, HB)])^n\Psi \]

for sufficiently small $t$.

From (5.5) of [1] and (5.4), we also have the convergence of

\[ \sum_{n=0}^{\infty} n!^{-1}\pi_\Omega(B(f))\{(it/2)\pi_\Omega[\mathcal{B}(B, HB)]\}^n\Psi \]
for small $t$. Therefore $Q_{\mathcal{H}}(tH)\Psi$, $\Psi \in D_0$ is in the domain of $\pi_{\mathcal{H}}(B(f))$ for small $t$ and (5.7) gives $\pi_{\mathcal{H}}(B(f))Q_{\mathcal{H}}(tH)$.

From the absolute convergence of (5.6) and (5.7) for small $t$, we have absolute convergence of

$$
(5.8) \quad (Q_{\mathcal{H}}(tH)\Psi, \pi_{\mathcal{H}}(B(f))Q_{\mathcal{H}}(tH)\Phi) = \sum_n (n!m!)^{-1}i^{n-m}(t/2)^{n+m}(\Psi, \pi_{\mathcal{H}}(B, HB)^mB(f)(B, HB)^n\Phi)
$$

for $\Phi, \Psi \in D_0$ and small $t$. By re-ordering the summation, we obtain

$$
(5.9) \quad \sum_n n!^{-1}(-it/2)^n(\Psi, \pi_{\mathcal{H}}[(B, HB)[\ldots[B, HB], B(f)\ldots]])\Phi)
= \sum_n n!^{-1}(-it)^n(\Psi, \pi_{\mathcal{H}}[B(H^*f)]\Phi).
$$

Since $(\Psi, \pi_{\mathcal{H}}(g)\Phi)$ is continuous in $g$ when $g$ is restricted to a finite dimensional subspace, (5.9) becomes

$$(\Psi, \pi_{\mathcal{H}}B(\sum_n n!^{-1}(-itH)^n f)\Phi) = (\Psi, \pi_{\mathcal{H}}B(e^{-itH}f)\Phi).$$

Therefore we have

$$
(5.10) \quad Q_{\mathcal{H}}(tH)^*\pi_{\mathcal{H}}(B(f))Q_{\mathcal{H}}(tH)\Phi = \pi_{\mathcal{H}}[B(e^{-itH}f)]\Phi
$$

for small $t$ and all $\Phi \in D_0$. Since a unitary transform of a selfadjoint operator is selfadjoint and $\pi_{\mathcal{H}}[B(e^{-itH}f)]$ is essentially selfadjoint on $D_0$, we have

$$
(5.11) \quad Q_{\mathcal{H}}(tH)^*\pi_{\mathcal{H}}(B(f))Q_{\mathcal{H}}(tH) = \pi_{\mathcal{H}}[B(e^{-itH}f)],
$$

for sufficiently small $t$. By using (5.11) repeatedly, we obtain (5.3) for a general $t$ and for a general $A$.

From (5.2) of [1] and (5.3), we obtain (5.2). Q. E. D.

**Lemma 5.2.** Assume that $\gamma$ is non-degenerate. Let $P$ be a basis projection and $K=K_P$. Then

$$
(5.12) \quad (Q_P, Q_P(tH)Q_P) = \det_P(Pe^{-itH}P)^{-1/2},
$$

where $\det_P$ is the determinant of $PK$ and the branch of the square root is determined by the continuity in $t$.

**Proof.** Let
(5.13) $f(t) = (\mathcal{Q}_P, Q_P(tH)\mathcal{Q}_P).$

Then

(5.14) $\frac{2}{i}f'(t) = (\mathcal{Q}_P, Q_P(tH)\pi_P((B, HB))\mathcal{Q}_P).$

We decompose $f_j$ as

(5.15) $f_j = f_j + f_j' = 0$

(5.16) $f_j' = e^{-itH}(Pe^{-itH})^{-1}Pf_j,$

(5.17) $f_j'' = f_j - f_j' \in (1 - P)K.$

The operator $C_t = Pe^{-itH}P$ is a bounded operator on $PK_P$, because a finite rank operator $H$ is bounded on $PK_P$. Since $\langle f, g \rangle = \langle f, g \rangle_P$ for $f, g \in PK$, we have

(5.18) $C_t^*C_t = C_t^*C_t = Pe^{itH}Pe^{-itH}P = PP'P$

where

(5.19) $P' = e^{itH}Pe^{-itH}$

is another basis projection. Since

(5.20) $\langle f, P(1-P')Pf \rangle_P = \gamma(Pf, (1-P')Pf) \leq 0,$

we have

(5.21) $PP'P = P - P(1-P')P \geq P.$

Therefore $(Pe^{-itH}P)^{-1}$ is also bounded on $PK_P$ and

(5.22) $\infty > |\det_P(Pe^{-itH}P)^{-1}| = \det_P(PP'P)^{-1/2} \neq 0.$

Therefore $f_j'$ and $f_j''$ are well defined and we have

(5.23) $\pi_P(B(f_j'))^*Q_P(tH)^*\mathcal{Q}_P = Q_P(tH)^*\pi_P(e^{itH}f_j')^*\mathcal{Q}_P = 0,$

(5.24) $\pi_P(B(f_j'))\mathcal{Q}_P = 0.$

Therefore

(5.25) $f'(t) = (i/2)k(t)f(t),$

(5.26) $k(t)1 = \sum_{j=1}^{N} \left[ B(f_j'), B(C_j) \right]$. 


\[= - \sum_{j=1}^{N} \tau(g_j, f_j^r)\]
\[= - \text{tr} \, H(1 - P) \{1 - e^{-itH}(Pe^{-itH})^{-1}P \}.\]

From \( \Gamma H^\dagger \Gamma = -H \), we have
\[
(5.27) \quad \text{tr} \, H(1 - P) = \text{tr} \, H \Gamma P \Gamma = \sum \tau(\Gamma f_j, \Gamma P \Gamma g_j) = - \sum \tau(\Gamma f_j, \Gamma P \Gamma g_j) = \text{tr} \, \{(\Gamma H^\dagger \Gamma)P\} = - \text{tr} \, HP.
\]

Hence
\[
(5.28) \quad k(t) = \text{tr} \, HP + \text{tr} \, H(1 - P)e^{-itH}(Pe^{-itH})^{-1}P = \text{tr} \, He^{-itH}(Pe^{-itH})^{-1}P = \frac{id}{dt} \text{tr} \, P \log(Pe^{-itH}P)
\]
where the trace is taken on \( PK \). Substituting (5.28) into (5.25), we obtain
\[
(5.29) \quad f(t) = \exp\left(\frac{-1}{2}\text{tr} \, P \log(Pe^{-itH}P)\right) = \det_P(Pe^{-itH}P)^{-1/2}. \quad \text{Q. E. D.}
\]

**Corollary 5.3.** Let \( \varphi_H \) be a Fock type state such that \( K = K_H \). Let \( H \) be a finite rank operator satisfying \( \tau(Hf, g) = \tau(f, Hg) \), \( \Gamma H \Gamma = -H \) and \( E_0 H = HE_0 = 0 \). Then
\[
(5.30) \quad (Q_H, Q_H(tH)Q_H) = \det_E(E_+ e^{-itH}E_+)^{-1/2}
\]
where \( E_+ \) is the eigenprojection of \( \Pi \) for an eigenvalue 1.

**Proof.** Immediate from Lemmas 4.7 and 5.2.

**Lemma 5.4.** Let \( P_1 \) and \( P_2 \) be basis projections for \( (K, \gamma, \Gamma) \). \( \gamma \) is necessarily non-degenerate. Assume that \( K \) is complete with respect to \( \|f\|_{F_j} = \gamma(f, (2P_j - 1)f)^{1/2} \), \( j = 1, 2 \), and that the topologies by these two norms are the same.

1. Let \( \theta(P_1, P_2) \) be a non-negative selfadjoint operator on \( K \) satisfying
\[
(5.31) \quad [\sinh \theta(P_1, P_2)]^2 f = -(P_1 - P_2)^2 f.
\]
Such \( \theta(P_1, P_2) \) exists, is bounded and commutes with \( P_1, P_2, \gamma_{P_1} = 2P_1 - 1, \gamma_{P_2} = 2P_2 - 1 \) and \( \Gamma \). \( \theta(P_1, P_2) \) and \( \theta(P_2, P_1) \) are the same operator.
Let $v_2 = \sinh \theta(P_1, P_2)$, $v_1 = \cosh \theta(P_1, P_2)$ and $u_{ij}(P_1/P_2)$ be bounded operators on $K$ satisfying

\begin{equation}
  u_{11}(P_1/P_2) = P_1 F_{10}, \quad u_{22}(P_1/P_2) = (1 - P_1) F_{10},
\end{equation}

\begin{equation}
  u_{12}(P_1/P_2) = (v_1 v_2)^{-1} P_1 P_2 (1 - P_1),
\end{equation}

\begin{equation}
  u_{21}(P_1/P_2) = -(v_1 v_2)^{-1} (1 - P_1) P_2 P_1.
\end{equation}

where $1 - F_{10}$ is the eigenprojection of $\theta(P_1, P_2)$ for an eigenvalue 0.

Such $u_{ij}(P_1/P_2)$ is unique and satisfies

\begin{equation}
  u_{ij}(P_1/P_2) u_{kl}(P_1/P_2) = \delta_{jk} \delta_{il} u_{ij}(P_1/P_2),
\end{equation}

\begin{equation}
  u_{ij}(P_1/P_2)^* = u_{ij}(P_1/P_2),
\end{equation}

\begin{equation}
  \Gamma u_{12}(P_1/P_2) \Gamma = u_{21}(P_1/P_2),
\end{equation}

\begin{equation}
  P_2 = \sum_{k, l = 1}^{2} v_k v_l (1 -)^{k-1} u_{kl}(P_1/P_2) + (1 - F_{10}) P_1,
\end{equation}

\begin{equation}
  1 - P_2 = \sum_{k, l = 1}^{2} v_{(3-k)} v_{(3-l)} (1 -)^{l-1} u_{kl}(P_1/P_2) + (1 - F_{10}) (1 - P_1),
\end{equation}

\begin{equation}
  \gamma P_1 u_{ij}(P_1/P_2) = (-1)^{i+j} u_{ij}(P_1/P_2) \gamma P_1.
\end{equation}

(3) Let

\begin{equation}
  H(P_1/P_2) = -i\theta(P_1, P_2) (u_{12}(P_1/P_2) + u_{21}(P_1/P_2)).
\end{equation}

Then $H(P_1/P_2)^*= - H(P_1/P_2)$ ($*$ is relative to $(f, g)_{P_1}$), $\Gamma H(P_1/P_2) H = - H(P_1/P_2)$,

\begin{equation}
  U(P_1/P_2) \exp i H(P_1/P_2) = v_1 + v_1^{-1} [P_1, P_2]
\end{equation}

is a Bogoliubov transformation for $(K, \gamma, \Gamma)$, and

\begin{equation}
  U(P_1/P_2)^T P_1 U(P_1/P_2) = P_2.
\end{equation}

Proof. (1) Since $P_2$ is bounded with respect to the norm $|| \cdot ||_{P_2}$, it
is also bounded with respect to the norm $|| \cdot ||_{P_1}$. Therefore $(P_1 - P_2)$ is bounded. The operator on the right hand side of (5.31) can be rewritten as

\begin{equation}
  -(P_1 - P_2)^2 = -P_1(1 - P_2) P_1 - (1 - P_1) P_2 (1 - P_1).
\end{equation}

Since $(f, P_1(1 - P_2) P_1 f)_{P_1} = \gamma (P_1 f, (1 - P_2) P_1 f)_{P_1} \leq 0$ for $f \in K$ and $(1 - P_1) P_2 (1 - P_1) = \Gamma P_1 (1 - P_2) P_1 \Gamma$, we have $-(P_1 - P_2)^2 \geq 0$. Hence there exists
a nonnegative selfadjoint operator $\theta(P_1, P_2)$ satisfying (5.31). $(P_1 - P_2)^2$ obviously commutes with $\Gamma$. It commutes with $P_1$ due to (5.44). By symmetry, it commutes with $P_2$.

Let $K = \max \{\|(P_1 - P_2)^2\|_{\mathcal{F}}, \|(P_1 - P_2)^2\|_{\mathcal{F}}\}$. Let $f_n(x)$ be a sequence of real polynomials of $x$ such that

$$\lim_{n \to \infty} \sup_{x \in [0, K]} |f_n(x) - \sinh^{-1}\sqrt{x}| = 0.$$ 

Such $f_n$ exists due to the Weierstrass approximation theorem. Then $
\theta(P_1, P_2) = \lim_{n \to \infty} f_n(- (P_1 - P_2)^2) = \theta(P_2, P_1)$.

(2) We see that (5.33) is partially isometric by the following computation:

$$\{P_1 P_2 (1 - P_1)\}^* P_1 P_2 (1 - P_1) = (1 - P_1) \gamma_{\Gamma, P_2} \Gamma P_1 P_2 (1 - P_1)$$

$$= - (1 - P_1) P_2 P_1 P_2 (1 - P_1) = \{(P_1 - P_2)^2 - (P_1 - P_2)^3\} (1 - P_1)$$

$$= u_1^2 u_2^2 (1 - P_1).$$

We also see that $u_{21}(P_1/P_2) = \Gamma u_{12}(P_1/P_2) \Gamma$ is also partially isometric and (5.35) holds. From $P_1^* = P_1$, $P_2^* = \gamma_{\Gamma, P_2} \gamma_{\Gamma, P_1}$, $P_1 \gamma_{\Gamma, P_1} = P_1$ and $(1 - P_1) \gamma_{\Gamma, P_1} = -(1 - P_1)$, we have (5.36).

Since $P_1 P_2 P_1 + (1 - P_1) P_2 (1 - P_1) + P_1 P_2 (1 - P_1) + (1 - P_1) P_2 P_1 = P_2$ and $P_1 (1 - F_{1o}) = P_2 (1 - F_{1o})$, we have (5.38) and (5.39). (5.40) follows from $\gamma_{\Gamma, P_1} P_1 = P_1$ and $(1 - P_1) \gamma_{\Gamma, P_1} = -(1 - P_1)$.

(3) From (5.36) and $\theta(P_1, P_2)^* = \theta(P_1, P_2)$, we have $H(P_1/P_2)^* = - H(P_1/P_2)$. From (5.40) and $[\gamma_{\Gamma, P_1}, \theta(P_1, P_2)] = 0$, we have $H(P_1/P_2)^* = \gamma_{\Gamma, P_1} H(P_1/P_2)^* \gamma_{\Gamma, P_1} = H(P_1/P_2)$. From (5.37) and $[\theta(P_1/P_2), \Gamma] = 0$, we have $\Gamma H(P_1/P_2) \Gamma = - H(P_1/P_2)$. Since $(u_{12}(P_1/P_2) + u_{21}(P_1/P_2))^2 = F_{1o}$, $v_1 (1 - F_{1o}) = (1 - F_{1o}), [P_1, P_2] (1 - F_{1o}) = 0$, and $u_{12}(P_1/P_2) + u_{21}(P_1/P_2) = (v_1 v_2)^{-1}$ $[P_1, P_2]$, we obtain (5.42).

Finally we have

$$U(P_1/P_2)^* = \gamma_{\Gamma, P_1} U(P_1/P_2)^* \gamma_{\Gamma, P_1} = v_1 - v_1^{-1} \gamma_{\Gamma, P_1} P_1, P_2],$$

$$v_1^2 + P_1 [P_1, P_2] - [P_1, P_2] P_1 - v_1^{-2} [P_1, P_2] P_1 [P_1, P_2] = P_2.$$ 

Hence we have (5.43). Q. E. D.
Lemma 5.5. Let $P_1$ and $P_2$ be basis projections for $(K, \gamma, \Gamma)$. Assume that $\| \cdot \|_j$, $j=1,2$, give the same topology, with respect to which $K$ is complete and that $\theta(P_1, P_2)$ is in the Hilbert Schmidt class. Then there exists a unique unitary operator $T(P_1, P_2)$ on $\mathcal{S}_{P_1}$ such that
\begin{align}
T(P_1, P_2)^* \pi_{P_j}(A) T(P_1, P_2) = \pi_{P_j} \left( \tau \left[ U(P_1/P_2) \right] A \right),
\end{align}
\begin{align}
(\mathcal{Q}_{P_j}, T(P_1, P_2) \mathcal{Q}_{P_j}) = \det P_j \left[ \text{sech} \theta(P_1, P_2) \right]^{1/2}.
\end{align}

Proof. Let $F_{1r}$ be the spectral projection of $\theta(P_1, P_2)$ for the infinite interval $(r, \infty)$ relative to the inner product $(f, g)_{P_j}$. Then $H(P_1/P_2)F_{1r}$ is of finite rank for $r>0$. Therefore it is of the form (4.2). Since $\theta(P_1, P_2)$ commutes with $\Gamma$, $P_1$ and $\gamma P_1$, $F_{1r}$ commutes with them and $(H(P_1/P_2) F_{1r})^\dagger = H(P_1/P_2) F_{1r}$, $\Gamma P_1 H(P_1/P_2) F_{1r}$, $\Gamma P_1 = -H(P_1/P_2) F_{1r}$. Let
\begin{align}
U_r = \exp i H(P_1/P_2) F_{1r},
\end{align}
\begin{align}
P_{(r)} = U_r^\dagger P_1 U_r,
\end{align}
\begin{align}
T_r' = Q(H(P_1/P_2) F_{1r}).
\end{align}
\begin{align}
\alpha_r = (\mathcal{Q}_{P_j}, T_r' \mathcal{Q}_{P_j}) \left| (\mathcal{Q}_{P_j}, T_r' \mathcal{Q}_{P_j}) \right|^{-1}
\end{align}
\begin{align}
T_r = \alpha_r^* T_r'.
\end{align}

From (5.12) and (5.22), we have
\begin{align}
(\mathcal{Q}_{P_j}, T_r'^* \mathcal{Q}_{P_j}) = \left| (\mathcal{Q}_{P_j}, (T_r')^* \mathcal{Q}_{P_j}) \right|
= \det P_j \left( P_1 \{ P_2 F_{1r} + (1 - F_{1r}) \} P_1 \right)^{-1/4}
= \det P_j \left[ \text{sech} \{ \theta(P_1, P_2) F_{1r} \} \right]^{1/2}.
\end{align}
In particular, it does not vanish and hence $\alpha_r$ is well defined.

If $\theta(P_1, P_2)$ is in the Hilbert Schmidt class, then $1 - \text{sech} \theta(P_1, P_2)$ is in the trace class and
\begin{align}
\lim_{r \to 0} (\mathcal{Q}_{P_j}, T_r'^* \mathcal{Q}_{P_j}) = \det P_j \left[ \text{sech} \theta(P_1, P_2) \right]^{1/2} > 0.
\end{align}
Further
\begin{align}
\left| (T_r'^* \mathcal{Q}_{P_j}, T_r'^* \mathcal{Q}_{P_j}) \right| = \left| (\mathcal{Q}_{P_j}, Q(H(P_1/P_2) \left[ F_{1r} - F_{1r}' \right]) \mathcal{Q}_{P_j}) \right|
= \det P_j \left[ \text{sech} \{ \theta(P_1, P_2) (F_{1r} - F_{1r}') \} \right]^{1/2} \to 1.
as \( r, r' \to 0 \).

Setting \( \alpha_{r, r'} = (T_r^* \Omega_{P_2}, T_r^* \Omega_{P_1}) (T_{r'}^* \Omega_{P_2}, T_{r'}^* \Omega_{P_1})^{-1} \), we see from (5.54) that \( T_r^* \Omega_{P_2} - \alpha_{r, r'} T_r^* \Omega_{P_1} \to 0 \) as \( r, r' \to 0 \). Substituting this to (5.53), we see that \( \alpha_{r, r'} \to 1 \) as \( r, r' \to 0 \) and hence \( T_r^* \Omega_{P_1} \) is a Cauchy sequence as \( r \to 0 \).

Let

\[
\Omega'_{P_1} = \lim_{r, r' \to 0} T_r^* \Omega_{P_1}
\]

We now have

\[
(\Omega'_{P_2}, W_{P_1}(f) \Omega'_{P_2}) = \lim_{r, r' \to 0} (\Omega_{P_2}, W_{P_1}(\tau(U_r)f) \Omega_{P_2})
= (\Omega_{P_2}, W_{P_1}(f) \Omega_{P_2}).
\]

Since \( \{W_{P_1}(f)\} \) is irreducible, \( \Omega'_{P_2} \) is cyclic and the vector state of \( \pi_{P_2} \)
\( (\pi(K, r, \Gamma)) \) by \( \Omega'_{P_2} \) is \( \varphi_{P'} \).

Let \( T(P_1, P_2) \) be a closed linear operator satisfying

\[
T(P_1, P_2) W_{P_1}(f) \Omega'_{P_2} = W_{P_1}(\tau(U_r)f) \Omega'_{P_2}.
\]

Since the mapping defined by (5.57) is isometric and the range is total, such \( T(P_1, P_2) \) exists as a unitary operator.

We have (5.45) from (5.57) and (5.46) from (5.53).

Corollary 5.6. Let \( U \) be a Bogoliubov transformation on \( K \) and \( P \) be a basis projection. Then \( \tau(U) \) is unitarily implementable on \( \hat{\pi}_P \) if \( PU(1-P)U^\dagger P \) is in the trace class on \( K_P \).

Proof. Let \( UPU^\dagger = P' \), \( U_1 = U(P/P') \), \( U_2 = U(P/P')U \). Then \( U = U_1U_2 \). Since \( U_2 \) commutes with \( P \), \( \varphi_P \) is invariant under \( U_2 \) and \( \tau(U_2) \) is implementable by a unitary operator \( T_P(U_2) \) on \( \hat{\pi}_P \):

\[
T_P(U_2) \pi_P(A) T_P(U_2)^* = \pi_P(\tau(U_2)A).
\]

Because \( PU(1-P)U^\dagger P = -(\sinh \theta(P, P'))^2 P \) is in the trace class, \( \Gamma PU(1-P)U^\dagger P' = -(\sinh \theta(P, P'))^2 (1-P) \) is also in the trace class and hence \( \theta(P, P') \) is in the H. S. class. Then \( T(P, P') \) implements \( \tau(U^\dagger) \) and \( T(P, P')T_P(U_2) \) implements \( \tau(U) \).

Lemma 5.7. Let \( (f, g)_1 \) and \( (f, g)_2 \) be positive definite inner products
on $K$ such that $K$ is complete with respect to both $\|f\|_1 = (f, f)^{1/2}$ and $\|f\|_2 = (f, f)^{1/2}$ and the two norms give the same topology on $K$.

(1) There exists a bounded linear operator $\alpha$ on $K$ with a bounded inverse $\alpha^{-1}$ such that

$$(5.59) \quad (f, g)_2 = (\alpha f, \alpha g)_1.\quad \text{(1)}$$

(2) A bounded linear operator $Q$ on $K$ is in the Hilbert Schmidt (resp. trace) class relative to $(f, g)_2$ if and only if $Q$ is in the H.S. (resp. trace) class relative to $(f, g)_1$.

(3) If $Q$ is in the trace class, the trace of $Q$ relative to $(f, g)_1$ and $(f, g)_2$ are the same.

Proof. (1) If there is no positive $a$ such that $\|f\|_2 \leq a \|f\|_1$ for all $f \in K$, then there exists a sequence $f_n$ such that $\|f_n\|_1 = 1$ and $\|f_n\|_2 \geq n$. Then $\|f_n/n\|_1 \to 0$ while $\|f_n/n\|_2 = 1$ does not tend to 0. Hence, under our assumption that the two norms induce the same topology on $K$, there exists $a > 0$ such that $\|f\|_2 \leq a \|f\|_1$ for all $f \in K$. Similarly there exists $a' > 0$ such that $\|f\|_1 \leq a' \|f\|_2$ for all $f \in K$.

Since $|(f, g)_2| \leq \|f\|_2 \|g\|_2 \leq a^2 \|f\|_1 \|g\|_1$, there exists by the Riesz theorem a bounded linear operator $\alpha_0$ such that

$$(f, g)_2 = (f, \alpha_0 g)_1.\quad \text{(5.59)}$$

Since $(f, g)_2$ is hermitian and positive, $\alpha_0$ is hermitian and positive relative to $(f, g)_1$. Setting $\alpha = (\alpha_0)^{1/2}$, where the positive square root is taken relative to $(f, g)_1$, we obtain (5.59) and $\alpha = \alpha^* > 0$ relative to $(f, g)_1$.

From $(f, \alpha^2 f)_1 = \|f\|_2 \geq (a')^{-2} \|f\|_2^2$, we obtain $\alpha^2 \geq (a')^{-2}$ and $\alpha \geq (a')^{-1}$ relative to $(f, g)_1$ and hence $\|\alpha^{-1}\|_1 \leq a'$. Namely $\alpha$ has a bounded inverse.

(2) Assume that $\{e_\beta\}$ is an orthonormal basis of $K$ relative to $(f, g)_1$. Let $e'_\beta = \alpha^{-1} e_\beta$. Since $\alpha^{-1}$ has a bounded inverse and $\{e_\beta\}$ is total, $\{e'_\beta\}$ is also total in $K$. Hence $\{e'_\beta\}$ is an orthonormal basis of $K$ relative to $(f, g)_2$.

Let $Q^*$ denote the adjoint of $Q$ relative to $(f, g)_1$ and $|Q|$ denote
the absolute value of $Q$ relative to $(f, g)$. We have $|Q|^2 = Q^*Q$. From

$$(f, Qg)_2 = (f, \alpha^2 Qg)_1 = (Q^*\alpha^2 f, g)_1 = (\alpha^{-2} Q^*\alpha^2 f, g)_2,$$

we obtain $|Q|_2^2 = \alpha^{-2} Q^*\alpha^2 Q$.

Assume that $Q$ is in the H.S. class relative to $(f, g)_1$. Then $\text{tr}_1 |Q|_1^2 = \sum_{\beta}^{} \|Qe_\beta\|_2^2 < \infty$. It is then known that $\alpha Q\alpha^{-1}$ is also in the H.S. class relative to $(f, g)_1$. Hence

\begin{equation}
(5.60) \quad \text{tr}_2 |Q|_2^2 = \sum_{\beta}^{} \|Qe_\beta\|_2^2 = \sum_{\beta}^{} (e_\beta^*, \alpha^{-2} Q^*\alpha^2 Q e_\beta)_1 = \text{tr}_1 (\alpha Q\alpha^{-1})^* (\alpha Q\alpha^{-1}) < \infty.
\end{equation}

and $Q$ is in the H.S. class relative to $(f, g)_2$.

Next assume that $Q$ is in the trace class relative to $(f, g)_1$. Let $Q = W_j |Q|_j$ be the polar decomposition of $Q$ relative to $(f, g)_j$. Let $\overline{W}_j$ be the adjoint of $W_j$ relative to $(f, g)_j$. Then $\overline{W}_j Q = |Q|_j$, $W_j$ and $\overline{W}_j$ are bounded. Hence $|Q|_2 = \overline{W}_2 W_1 |Q|_1$ is in the trace class relative to $(f, g)_1$ and so is $\alpha |Q|_2 \alpha^{-1}$. We now have

\begin{equation}
(5.61) \quad \text{tr}_2 |Q|_2^2 = \sum_{\beta}^{} (e_\beta^*, |Q|_2 e_\beta)_2 = \sum_{\beta}^{} (e_\beta^*, \alpha |Q|_2 \alpha^{-1} e_\beta)_1 = \text{tr}_1 \alpha |Q|_2 \alpha^{-1} < \infty.
\end{equation}

Therefore $Q$ is in the trace class relative to $(f, g)_2$.

(3) The computation of (5.61) shows $\text{tr}_2 Q = \text{tr}_1 Q$ for any $Q$.

**Lemma 5.8.** In Lemma 5.5, $\theta(P_1, P_2)$ is in the H.S. class if and only if $P_1 - P_2$ is in the H.S. class.

**Proof.** If $P_1 - P_2$ is in the H.S. class, $-(P_1 - P_2)^2 = \sinh^2 \theta (P_1, P_2)$ is in the trace class and hence $\theta(P_1, P_2)$ is in the H.S. class. If $\theta(P_1, P_2)$ is in the H.S. class, then $P_1 - P_2$ is in the H.S. class, as is obvious from (5.38).

Q. E. D.

§ 6. Quasi-equivalence for Non-degenerate Case

In this section, we shall be concerned with $S$ and $S'$ which do not
have an eigenvalue 1/2. Since the corresponding $\Pi_S$ and $\Pi_{S'}$ give Fock states, we shall denote $P_S$ and $P_{S'}$ instead of $\Pi_S$ and $\Pi_{S'}$.

**Definition 6.1.** Two quasifree representations $\pi_S$ and $\pi_{S'}$ are quasi-equivalent if there exists a (homeomorphic) $*$ isomorphism $p$ from $R_S := \{W_S(f); f \in \text{Re } K\}$ onto $R_{S'} := \{W_{S'}(f); f \in \text{Re } K\}$ such that $pW_{S}(f) = W_{S'}(f)$ for all $f \in \text{Re } K$.

**Lemma 6.2.** Let $\varphi_S$ and $\varphi_{S'}$ be quasifree states of $\mathfrak{H}(K, \gamma, \Gamma)$ such that $S$ and $S'$ do not have an eigenvalue 1/2. Assume that the following 3 conditions hold:

1. $N_S = N_{S'}$.
2. The topologies induced by $\|f\|_S$ and $\|f\|_{S'}$ on $K$ are equivalent.

Identify $K_S, \gamma_S, \Gamma_S$ with $K_{S'}, \gamma_{S'}, \Gamma_{S'}$ by the closure of the identification map $\hat{f} \in K_S \rightarrow \hat{f} \in K_{S'}$ for $f \in K$. Identify $K_S, \gamma_S, \Gamma_S$ with $K_{S'}, \gamma_{S'}, \Gamma_{S'}$ through definitions (6.1)~(6.3) of [1]. Identify $\hat{K}_S, \tilde{\gamma}_S, \tilde{\Gamma}_S$ with $\hat{K}_{S'}, \tilde{\gamma}_{S'}, \tilde{\Gamma}_{S'}$ due to Lemma 6.1 (6) of [1].

3. $P_S - P_{S'}$ is in the Hilbert-Schmidt class on $\hat{K}_S = \hat{K}_{S'}$.

Then $\pi_S$ and $\pi_{S'}$ are quasi-equivalent.

If $S$ and $S'$ do not have an eigenvalue 0 in addition, $\varphi_S$ and $\varphi_{S'}$ are unitarily equivalent.

**Proof.** (1)~(3) imply the unitary equivalence of $(\mathcal{S}_{P_S}, \pi_{P_S}, \varrho_{P_S})$ and $(\mathcal{S}_{P_{S'}}, \pi_{P_{S'}}, \varrho_{P_{S'}})$ due to Lemma 6.1 (6) of [1], Lemmas 5.5 and 5.8. The vector $\varrho_{P_S}$ and $\varrho_{P_{S'}}$ are separating for the center of $R_S$ and $R_{S'}$ by Lemma 2.5. Therefore $\pi_S$ and $\pi_{S'}$, which are restrictions of $\pi_{P_S}$ and $\pi_{P_{S'}}$ to a subalgebra, are quasi-equivalent.

If $S$ and $S'$ do not have an eigenvalue 0, then $\varrho_{P_S}$ and $\varrho_{P_{S'}}$ are cyclic for $R_S$ and $R_{S'}$. Hence $\pi_S$ and $\pi_{S'}$ are unitarily equivalent.

Q. E. D.

**Lemma 6.3.** Let $\mathcal{S}_0$ be the set of positive semidefinite hermitian forms $S$ on $K$ satisfying (3.4) of [1] and such that the associated operator $S$ does not have an eigenvalue 1/2. Define the quasi-equivalence $S \sim S'$ by
the requirements (1)～(3) of Lemma 6.2. Then this is an equivalence relation.

This is obvious from the form of requirements (1)～(3).

If $N_S=N_{S'}$, $S$ and $S'$ do not have an eigenvalue 1/2 and $\dim (K/N_S)<\infty$, then we have a representation of canonical commutation relations for a finite degree of freedom and hence all representations are quasi-equivalent, a well known result of von Neumann. Hence we have a common $W^*$ algebra $R_S=R_{S'}$ generated by $W_S(f), f \in \text{Re } K$, and $\varphi_S$ can be viewed as the unique state of $R_S=R_{S'}$ satisfying $\varphi_S(W_S(f))=\Omega_S, W_S(f)\Omega_S)$. This makes it meaningful to speak of the norm $||\varphi_S-\varphi_{S'}||$ for such $S$ and $S'$.

**Lemma 6.4.** Let $\varphi_S$ and $\varphi_{S'}$ be quasifree states of \( \mathfrak{H}(K, \gamma, \Gamma) \) such that $N_S=N_{S'}$, and $S$ and $S'$ do not have an eigenvalue 1/2 nor 1. Assume that $\dim (K/N_S)<\infty$. Then

$$||\varphi_S-\varphi_{S'}|| \geq 2\{1-\det P_S(P_{S'}P_S)^{-1/4}\}. \quad (6.1)$$

**Proof.** Let $\omega$ be as in Lemma 2.1. (2.3) and (2.4) imply $[\omega, \gamma_S]=0$ and hence $[\omega, P_S]=0$. Since $\omega$ is the same for $S$ and $S'$ in the present case, we also have $[\omega, P_S]=0$. This implies $\omega H(P_S/P_{S'}) = -H(P_S/P_{S'})\omega$ and $\tau(\omega)(B, H(P_S/P_{S'})B) = -(B, H(P_S/P_{S'})B)$. Hence we obtain

$$[T_{P_S}(\omega), Q_{P_S}\{H(P_S/P_{S'})\}]=0. \quad (6.2)$$

Therefore, $Q_{P_S}$ and $Q'=Q_{P_S}\{H(P_S/P_{S'})\}Q_{P_S}$ are invariant under $T_{P_S}(\omega)$ and vector states of the representation $\pi(B(f))=\pi_{P_S}[B(\bar{f}^\perp0), f \in K$ of $\mathfrak{H}(K, \gamma, \Gamma)$ by $Q_{P_S}$ and $Q'$ are $\varphi_S$ and $\varphi_{S'}$.

By Lemma 6.5 of [2], Lemma 3.3 and Corollary 3.5, we obtain

$$||\varphi_S-\varphi_{S'}|| \geq 2(1-|\Omega_{P_S}, \Omega'|). \quad (6.3)$$

By (5.46), we obtain $|\Omega_{P_S}, \Omega'|=\det P_S(P_{S'}P_S)^{-1/4}$ and hence (6.1).

Q. E. D.

**Lemma 6.5.** Let $S$ and $S'$ be hermitian forms belonging to $\mathfrak{E}_0$ such that $N_S=N_{S'}$, and $\tau_S=\tau_{S'}$. Let
(6.4) \( \chi(S) = \text{Tanh}^{-1}2S^{1/2}(1-S)^{1/2} \), \( \sigma(S) = |2S-1|^{-1}(2S-1) \),
(6.5) \( \chi(S') = \text{Tanh}^{-1}2(S')^{1/2}(1-S')^{1/2} \), \( \sigma(S') = |2S'-1|^{-1}(2S'-1) \).

Then \( e^{-\chi(S)}e^{\chi(S')} \) and \( e^{-\chi(S')}e^{\chi(S)} \) are bounded and the following conditions are equivalent:

1. \( P_S - P_{S'} \) is in the Hilbert Schmidt class.
2. \( 1 - \sigma(S)e^{-\chi(S)}e^{\chi(S')}\sigma(S') \) is in the Hilbert Schmidt class.
3. \( 1 - \sigma(S')e^{-\chi(S')}e^{\chi(S)}\sigma(S) \) is in the Hilbert Schmidt class.

**Proof.** For each \( f \in K_S \), let

\[(6.6) \|f\|_S^2 = \sup \| (f, \tau_S g) \|_S = \|(2S-1)f\|_S \]

and \( K_S^* \) be the completion of \( K_S \) relative to (6.6). Let \( (f, g)_S = (f, (2S-1)^2g)_S \). Consider \( \tilde{K}_S = K_S \oplus K_S^* \) equipped with an inner product \((f_1 \oplus g_1, f_2 \oplus g_2)_S = (f_1, f_2)_S + (g_1, g_2)_S^* \). If \( \tau_S = \tau_{S'} \), \( K_S \) is identified with \( K_{S'} \).

Since \( (f, \tau_S g)_S = (f, \tau_{S'} g)_{S'} \), \( K_S^* \) can be identified with \( K_{S'}^* \) and hence \( \tilde{K}_S = \tilde{K}_{S'} \).

The mapping from \( f \in K_S \) to \( (2S-1)f \in K_S \) is isometric as a mapping from \( K_S^* \) into \( K_S \). Since \( |2S-1| = \text{sech} \chi(S) \), \( e^{\chi(S)}|2S-1| = 2 \left[ 1 + e^{-2\chi(S)} \right]^{-1} \) is bounded above and below by 2 and 1 both on \( K_S \) and on \( K_S^* \).

Hence the closure of \( e^{-\chi(S)} \) is a bounded mapping from \( K_S^* \) onto \( K_S \) and its inverse is a bounded mapping from \( K_{S'} \) onto \( K_S^* \). We shall denote the closure and its inverse again by \( e^{-\chi(S)} \) and \( e^{\chi(S)} \). A similar statement holds for \( S' \). Hence \( e^{-\chi(S)}e^{\chi(S')} \) and \( e^{-\chi(S')}e^{\chi(S)} \) are bounded on \( K_S \).

Due to

\[(6.7) 2\|[f \oplus g]\|_S^2 = \|[S^{1/2} + (1-S)^{1/2}]f + g\|_S^2 + \|[S^{1/2} + (1-S)^{1/2}]^{-1}(f - g)\|_S^2 \]

and \( \sqrt{2} \geq S^{1/2} + (1-S)^{1/2} \geq 1 \), the mapping \( v_S \) from \( f \oplus g \in \tilde{K}_S \) to \( (f + g) \oplus (f - g) \in \tilde{K}_{S'} \) and its inverse are bounded, where \( f \) and \( g \) are in \( K_S \).

Let \( \bar{v}_S \) be the closure of \( v_S \), which is a bounded mapping from \( \tilde{K}_S \) onto \( \tilde{K}_{S'} \) with a bounded inverse. Obviously \( \bar{v}_S = \bar{v}_{S'} \).

By a direct computation, we have

\[(6.8) 2\bar{v}_S P_S \bar{v}_S^{-1} = \begin{bmatrix} 1 & e^{-\chi(S)}\sigma(S) \\ e^{\chi(S)}\sigma(S) & 1 \end{bmatrix} \]
Hence the condition (1) is equivalent to the following two conditions: 
\( (\alpha)\sigma(S)e^{-x(S)} - e^{-x(S')}\sigma(S') \) is in the H.S. class as a mapping from \( K_S^2 = K_{S'} \) into \( K_S = K_{S'} \). \( (\beta)\sigma(S)e^{x(S)} - e^{x(S')}\sigma(S') \) is in the H.S. class as a mapping from \( K_S \) into \( K_S^2 \). Here a mapping \( Q \) from a Hilbert space \( H_1 \) into another Hilbert space is in the H.S. class if \( \sum ||Qe^\lambda||^2 < \infty \) where \( e^\lambda \) is a complete orthonormal set on \( H_1 \) and the norm is in \( H_2 \).

Since \( ||Qf||_H^2/||e^{-x(S)}Qf||_S \) is bounded below and above uniformly, \( (\beta) \) is equivalent to the condition (2). The condition (2) implies that \( e^{-x(S')}\sigma(S') - e^{-x(S)}\sigma(S) = (1 - \sigma(S)e^{x(S')}\sigma(S'))e^{-x(S')}\sigma(S') \) is in the H.S. class as a mapping from \( K_S^2 \) into \( K_S \) (cf. Lemma 5.7), and hence \( (\alpha) \). Namely (1) and (2) are equivalent.

By symmetry, (1) and (3) are equivalent.

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**Remark 6.6.** From the above calculation, we have

\[
(6.9) \quad 4||P_S - P_{S'}||^2_{H.S.} = ||\beta[1 - \sigma(S)e^{-x(S)}e^{x(S')}\sigma(S')]\beta^{-1}||^2_{H.S.} + ||\beta[1 - \sigma(S')e^{-x(S')}e^{x(S)}\sigma(S)]\beta^{-1}||^2_{H.S.}
\]

where the H.S. norm is relative to \((f, g)\) on the left hand side, relative to \((f, g)_s\) on the right hand side and

\[
\beta = \sqrt{2[1 + \exp - 2x(S)]^{-1/2}} = (S^{1/2} + (1 - S)^{1/2})
\]

satisfies \( 1 \leq \beta \leq \sqrt{2} \). As a consequence

\[
(6.10) \quad ||P_S - P_{S'}||^2_{H.S.} \geq (1/8)||1 - \sigma(S)e^{-x(S)}e^{x(S')}\sigma(S')||^2_{H.S.}
\]

**Lemma 6.7.** Assume that \( K \) is separable. Let \( S \) and \( S' \) be in \( H_0 \). Assume that \( S \) and \( S' \) do not have an eigenvalue 1, \( N_S = N_{S'} \), \( \tau_S = \tau_{S'} \), and \( P_S - P_{S'} \) is not in the H.S. class. Then there exists a sequence of \( \Gamma \) invariant finite dimensional subspaces \( K_n \) of \( K_S \) such that hermitian forms \( S_n(f, g) = (f, Sg)_s \) and \( S'_n(f, g) = (f, S'g)_{S'} \), \( f, g \in K_n \), satisfy \( \lim ||\varphi_{S_n} - \varphi_{S'_n}|| = 2 \).

**Proof.** Let \( E_+ \) and \( E_n \) be the spectral projection of \( S \) for intervals \([1/2, 1]\) and \([1/2 + 1/n, 1]\), \( n = 3, 4, \ldots \), respectively. Let \( f_\alpha \) be a complete orthonormal basis of \( E_+ K_S \) relative to \((f, g)_s\) such that \( E_n f_\alpha = f_\alpha \).
or 0 for all $n$ and $\alpha$. For any subset $A$ of $\alpha$, let $K(A)$ be the subspace of $K_S = K_{S'}$ spanned by $f_a$ and $\Gamma f_a$, $\alpha \in A$ and $E(A)$ be the orthogonal projection on $K(A)$ relative to $(f, g)$. Let $I_n$ be the set of $\alpha$ such that $E_n f_a = f_a$ and $I$ denote any finite set of $\alpha$.

Let $\tilde{\tau}(f, g) = (f, \tilde{\tau} g) = (f, \tilde{\tau} g)$. Let $S(A)$ and $S'(A)$ be the hermitian forms on $K(A)$, defined by $S(A)(f, g) = (f, S g)A, S'(A)(f, g) = (f, S' g)_S$. The restriction of $\tilde{\tau}$ and $\Gamma_S = \Gamma_{S'}$ to $K(A)$ are denoted by the same letters.

By construction of $K_S$, $N_{S(A)} = N_{S'(A)} = 0$. Since $\tilde{\tau}(f, \Gamma f_a) = 0$, $\tilde{\tau}(\Sigma c_a f_a, \Sigma c_a f_a) > 0$ for $\Sigma c_a f_a \neq 0$ and $\tilde{\tau}(\Sigma c_a f_a, \Sigma c_a \Gamma f_a) < 0$ for $\Sigma c_a f_a \neq 0$, we see that $S(A)$ and $S'(A)$ belong to $\mathbb{C}_0$ for $(K(A), \tilde{\tau}, \Gamma_S)$. We have $S(A)E(A) = E(A)S(A)$ and $S'(A)E(A) = E(A)^*S'(A)E(A)$, where $E(A)^*$ is the adjoint of $E(A)$ relative to $(f, g)$. For $\tau_S = \tau_{S'}$, there exists an operator $\alpha$ with a bounded inverse such that $(f, g)_S = (f, \alpha g)_S$ and $\alpha$ is hermitian and positive relative to $(f, g)_S$. Then $E(A)^* = \alpha^{-1}E(A)\alpha$.

We have $\lim E(I) = E(I_\alpha)$ and $\lim E(I_\alpha) = 1$. Hence $\lim \lim S(I)E(I) = S$ and $\lim S'(I)E(I) = S'$ (as strong limits of operators). Since $[E(I_\alpha), S] = 0$ and $x(S)E(I_\alpha)$ is bounded, we have $\lim \{x(S)E(I_\alpha)\}E(I) = (x(S)E(I_\alpha))\exp -\alpha = (x(S)E(I_\alpha))\exp -\alpha E(I_\alpha)$. (If $f(x)$ is piecewise continuous and its jump points are not eigenvalues of $Q$, then $Q = \lim Q = Q$ implies $f(Q) = f(Q)$. $[S]$.) We also have $\lim \{x(S'(I_\alpha))\exp -\alpha = (x(S'(I_\alpha))\exp -\alpha E(I_\alpha)\}E(I_\alpha) = x(S'(I_\alpha))E(I_\alpha)$ and $\lim \{x(S'(I_\alpha))\exp -\alpha = (x(S'(I_\alpha))E(I_\alpha)\}E(I_\alpha) = x(S'(I_\alpha))E(I_\alpha)$, since $\alpha = \alpha E(A) \leq \alpha E(A) \leq ||\alpha||E(A)$, hence $x(S(A)) = [E(A)\alpha E(A)]^{-1}$ (on $K(A)$) and its inverse are uniformly bounded. Hence $e^{-x(S'(I))}e^{x(S'(I))}$ and its inverse $e^{-x(S'(I))}e^{x(S'(I))}$ are uniformly bounded due to $1/2 \leq e^{-x} \leq 1$ and hence

$$\lim_{n \to \infty} \lim_{m \to \infty} \sigma(S'(I))e^{-x(S'(I))}e^{x(S'(I))}\sigma(S'(I))$$

$$= \sigma(S')e^{-x(S')}e^{x(S')}\sigma(S').$$

Since $x^{-1}$ is continuous for $x \in [\|\alpha\|^{-1}, \|\alpha^{-1}\|]$, we also obtain an inverse of (6.11)

$$\lim_{n \to \infty} \lim_{m \to \infty} \sigma(S'(I))e^{-x(S'(I))}e^{x(S'(I))}\sigma(S'(I)) = \sigma(S)e^{-x(S')}e^{x(S')}\sigma(S').$$
We now have
\begin{equation}
\lim_{n \to \infty} \lim_{I \uparrow I_n} \| (1-\sigma(S(I))e^{-\chi(S(I))}e^{\chi(S'(I))}\sigma(S'(I))\|_{H.S.} \geq \|1-\sigma(S)e^{-\chi(S)}e^{\chi(S')}\sigma(S')\|_{H.S.} = \infty,
\end{equation}
where the H.S. norm is relative to \((f, g)_{S(I)} = (f, g)_S\) for \(f, g \in K(I)\).

By Remark 6.6, we have \(\lim_{n \to \infty} \lim_{I \uparrow I_n} \| P_{S(I)} - P_{S'(I)} \|_{H.S.} = \infty\) where the H.S. norm is relative to \((f, g)_{S(I)}\). Since \(\det Q \geq 1 + \text{tr}(Q - 1)\) for \(Q \geq 1\), we obtain
\begin{equation}
\lim_{n \to \infty} \lim_{I \uparrow I_n} \det_{\mathcal{P}_{S(I)}}(P_{S(I)}P_{S'(I)}P_{S(I)})^{-1/4} = 0.
\end{equation}
Hence we can find a finite set \(I(n)\) such that \(\|\varphi_{S(I(n))} - \varphi_{S'(I(n))}\| \geq 2\left[1 - \left(1/n\right)\right]\) due to Lemma 6.4.

Lemma 6.8. Assume that \(K\) is separable. Let \(S\) and \(S'\) be in \(\mathcal{B}_0\). Assume that \(N_S = N_{S'}\), \(\tau_S = \tau_{S'}\) and \(\text{tr}(P_S - P_{S'})^2 = -\infty\). Then \(\pi_S\) and \(\pi_{S'}\) are not quasi-equivalent.

Proof. First assume that \(S\) and \(S'\) do not have an eigenvalue 1. Then \(\varrho_{P_S}\) and \(\varrho_{P_{S'}}\) are cyclic and separating by Lemma 2.3. Hence, if \(\pi_S\) and \(\pi_{S'}\) are quasi-equivalent, they are unitarily equivalent and there exists a unitary mapping \(W_0\) from \(\varrho_{P_S}\) to \(\varrho_{P_{S'}}\) such that \(W_0W_S(f)W_0^{-1} = W_{S'}(f)\) for all \(f \in \text{Re } K\). By continuity, we have
\[W_0W_{P_S}(f \oplus 0)W_0^{-1} = W_{P_{S'}}(f \oplus 0)\]
for all \(f \in K_S = K_{S'}\).

Since the set of \(W'\varrho_{P_S}\) with isometric \(W'\) in \((R_S)'\) is total, there exists such \(W'\) satisfying \(b = (W'\varrho_{P_S}, W_0^{-1}\varrho_{P_S}) \neq 0\). The vector states of \(R_S\) by \(\varrho_{P_S}\) and \(\varrho_{P_{S'}}\) are denoted by \(\varphi_S\) and \(\varphi_{S'}\) (cf. Lemma 6.4). Then
\begin{equation}
\|\varphi_S - \varphi_{S'}\| \leq 2(1 - |b|^2)^{1/2} = 2 - \delta.
\end{equation}

We now have a contradiction because there exists a \(I\) invariant subspace \(K_0\) of \(K_S\) such that the restriction of \(\varphi_S - \varphi_{S'}\) to the subalgebra generated by \(W_{P_S}(f \oplus 0), f \in K_0\) has a norm larger than \(2 - \delta\), due to Lemma 6.7.

For the general case, let \(E_1\) be the eigenprojection of \(S\) for an
eigenvalue 1 (relative to \((f, g)_S\)) and \(z\) be a positive Hilbert Schmidt class operator on \(E_1K_S\). Let

\[
S_{10} = S + (-1 + \cosh^2 z)E_1 + \Gamma_S (\sinh^2 zE_1)\Gamma_S,
\]

(6.17) \(S_1(f, g) = (f, S_{10} g)_S, \quad f, g \in K\).

Then \(S_1\) is in \(\mathfrak{E}_0\) for \((K, \gamma, \Gamma)\). We have \(N_{S_1} = N_S\),

(6.18) \((f, g)_S = (f, \{1 + (\cosh 2x - 1)(E_1 + \Gamma_S E_1 \Gamma_S)\} g)_S, \quad f, g \in K, \tau_{S_1} = \tau_S\) and

(6.19) \(S_1 = S + (-1 + \cosh^2 z \text{sech}2x)E_1 + \Gamma_S (\sinh^2 z \text{sech}2x)E_1\Gamma_S,\)

(6.20) \(z(S_1) = z(S) + 2z(E_1 + \Gamma_S E_1 \Gamma_S)\).

Note that \(z(S) = 0\) on \(E_1K_S \oplus \Gamma_S E_1K_S\). Hence we have

(6.21) \(||1 - \sigma(S)e^{-z(S)}e^{z(S)}\sigma(S)||_{H,S}^2 = 2||1 - e^{2x}E_1||_{H,S} < \infty.\)

By Lemma 6.2, \(\pi_S\) and \(\pi_{S_1}\) are quasi-equivalent. By (6.19), \(S_1\) does not have an eigenvalue 1. By Lemma 6.5, \(S_1S_1\). Similarly, there exists \(S'_1 \in \mathfrak{E}_0\) such that \(S'_1\) does not have an eigenvalue 1, \(\pi_{S'_1}\) and \(\pi_{S'}\) are quasi-equivalent and \(S'_1 \neq S'_1\). Since \(P_{S_1} - P_{S'_1} = P_S - P_S' = (P_S - P_{S_1}) + (P_{S_1} - P_{S'_1})\) is not in the H.S. class, \(\pi_{S_1}\) and \(\pi_{S'_1}\) are not quasi-equivalent by preceding conclusion. Hence \(\pi_S\) and \(\pi_{S'}\) are not quasi-equivalent. Q.E.D.

**Lemma 6.9.** If \(S\) and \(S'\) are in \(\mathfrak{E}_0\) and quasifree representations \(\pi_S\) and \(\pi_{S'}\) are quasi-equivalent, there exists an operator \(E\) on \(K_S = K_{S'}\) such that \(E\) commutes with \(S\), \(S'\) and \(\Gamma\), is an orthogonal projection relative both \((f, g)_S\) and \((f, g)_{S'}\), \(Sf = S'f\) and \((g, f)_S = (g, f)_{S'}\) for \(f \in K\) to \((1 - E)\) \(K\) and \(EK_S\) is separable.

**Proof.** Since \(\tau_S = \tau_{S'}\) due to the quasi-equivalence of \(\pi_S\) and \(\pi_{S'}\), there exists an operator \(\alpha\) with a bounded inverse such that \((f, g)_{S'} = (\alpha f, \alpha g)_S\) for all \(f, g \in K_S = K_{S'}\) and \(\alpha\) is hermitian and positive relative to \((f, g)_S\).

We can construct inductively a separable subspace \(K_\mu\) of \(K_S\) for each ordinal \(\mu < \mu_S\) in such a way that \(K_\mu\) is mutually orthogonal relative to
both \((f, g)_S\) and \((f, g)_{S'}\) invariant under \(S, S', \Gamma\) and \(\alpha\) and \(K_S = \bigoplus K_\mu\).

The construction of such \(K_\mu\) proceeds as follows: Assume that \(K_\mu, \mu' < \mu,\) be given. If the orthogonal complement, relative to \((f, g)_S,\) of the union of \(K_\mu, \mu' < \mu,\) is 0, then set \(\mu = \mu_S.\) Otherwise take a vector \(f_\mu\) from there and let \(K_\mu\) be the subspace of \(K_S\) spanned by \(Qf_\mu\) where \(Q\) runs over arbitrary polynomials of \(\alpha, S, S', \Gamma\) and 1. The required properties are satisfied inductively.

Let \(E_\mu\) be an orthogonal projection onto \(K_\mu\) relative to \((f, g)_S.\) \((1 - E_\mu)K_S\) is spanned by the union of \(K_\mu, \mu' = \mu,\) and hence \(E_\mu\) is hermitian relative to \((f, g)_{S'}\) due to the \(\alpha\) invariance of each \(K_\mu.\)

Corresponding to the decomposition \(K_S = \bigoplus K_\mu,\) we have the decomposition \(K_S = \bigoplus K_\mu.\) Let \(\hat{E}_\mu\) be the orthogonal projection onto \(K_\mu\) relative to \((f, g)^S.\) By similar reason as before, \(\hat{E}_\mu\) is hermitian also relative to \((f, g)^{S'}\) and commutes with \(P_S, P_S', \) and \(\hat{F}_S.\) Let \(P_\mu, \gamma_\mu, \hat{F}_\mu\) be the restrictions of \(P_S, \gamma_S, \hat{F}_S\) to \(K_S.\) Let \(\Omega_\mu, \pi_\mu, \varphi^{\mu}\) be the Fock representation of \(\mathfrak{A}(\hat{K}_\mu, \gamma_\mu, \hat{F}_\mu)\) corresponding to the basis projection \(P_\mu.\) Then \(\Omega_{P_\mu, \pi_\mu, \varphi^{\mu}}\) is (unitarily equivalent to) the incomplete infinite tensor product \(\otimes(\varphi^{\mu}, \Omega_\mu)\) and \(W_{P_\mu}(f) = \otimes W_{P_\mu}(f_\mu)\) for \(f = \otimes f_\mu.\)

Any normal state \(\varphi\) of \(R_S = \{W_S(f); f \in K_S\}\) is a countable sum of vector states. Each vector in \(\Omega_S \subset \Omega_{P_\mu}\) is a countable linear combination of product vectors. Each product vector has a form \(\otimes \varphi^{\mu}\) where all \(\varphi^{\mu}\) except a countable number is \(\Omega_\mu.\) Therefore there exists a countable set \(A\) of \(\mu\) for each normal state \(\varphi\) such that \(\varphi(W_S(f)) = (\Omega_S, W_S((1 - E)f)\Omega_S)\varphi(W_S(Ef))\) for \(f \in K_S\) where \(E = \sum_{\mu \notin A} E_\mu.\)

If \(\pi_S\) and \(\pi_S'\) is quasi-equivalent, then \((\Omega_{S'}, W_{S'}(f)\Omega_{S'})\) has an extension to a normal state of \(R_S\) and hence

\[(\Omega_{S'}, W_{S'}(f)\Omega_{S'}) = (\Omega_S, W_S((1 - E)f)\Omega_S)(\Omega_{S'}, W_{S'}(E f)\Omega_{S'}).\]

This implies that

\[(6.22) \quad S'(f, g) = S([1 - E]f, [1 - E]g) + S'(Ef, Eg)\]

and hence \(E\) has the required properties. Q. E. D.

Theorem. Two primary quasifree representations \(\pi_S\) and \(\pi_{S'}\) are quasi-equivalent if and only if the following 3 conditions hold:
(1) Coincidence of the kernel: \( N_S = N'_S \). (\( N_S \) is the set of \( f \in K \) such that \( S(f, f) + S(\Gamma f, \Gamma f) = 0 \).)

(2) Coincidence of the induced topology: \( \tau_S = \tau'_S \). (\( \tau_S \) is the topology induced on \( K/N_S \) by \( \| f \|_S = \left[ S(f, f) + S(\Gamma f, \Gamma f) \right]^{1/2} \).)

(3) \( 1 - e^{-\chi(S)} e^{\chi(S')} \) is in the Hilbert Schmidt class, where \( \chi(S) = \tanh^{-1} 2 S^{1/2}(1 - S)^{1/2} \), \( S \) is defined by \( S(f, g) = S(f, Sg) + S(\Gamma f, \Gamma Sg) \) and the positive square root is relative to \( (f, g)_S = S(f, g) + S(\Gamma f, \Gamma g) \).

The condition (3) is equivalent to the condition that \( P_S - P'_S \) is in the H.S. class.

Proof. By Lemma 2.4, a quasifree state \( \varphi_S \) is primary (i.e. \( R_S \) is a factor) if and only if \( S \) does not have an eigenvalue \( 1/2 \) (i.e. \( S \in \mathbb{C}_0 \)). Since \( W_S(f) = 1 \) if and only if \( f \in N_S \), the condition (1) is obviously necessary. By Lemma 6.4 of [1] and due to the equivalence of topologies induced by quasi-equivalent representations, the condition (2) is necessary. The equivalence of (3) and an alternative condition is in Lemma 6.5. By Lemmas 6.9 and 6.8, the condition (3) is necessary. By Lemma 6.2, the three conditions are sufficient. Q. E. D.

Corollary. A Bogoliubov transformation \( U \) is unitarily implementable on a Fock representation \( \pi_P \) if and only if \( P - UPU^\dagger \) is in the H.S. class.

Remark. In general, the condition (3) is not equivalent to the condition that \( S^{1/2} - (S')^{1/2} \) is in the Hilbert Schmidt class. They become equivalent if \( \chi(S) \) is bounded.

References

[5] Hugenholtz, N. M., On the factor type of equilibrium states in quantum statis-

