Covariance Operators of Skew Distributions

By

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In this paper we extend the concept of a skew distribution on a real Hilbert space $H$ defined in [4] and [6] to that on a complex Hilbert space $K$ with an antiunitary involution $\Gamma$, and show the following result which stems from Theorem 3 in [4]. Whenever a skew distribution $m$ is given on even or infinite dimensional $(K, \Gamma)$, any two of the following conditions imply the other one:

(i) $m$ is a factor distribution;
(ii) $m$ is $\mathcal{U}$-invariant; and
(iii) any pair of $\Gamma$-invariant orthogonal subspaces are independent with respect to $m$.

In the appendix we give a correspondence of a pair of Fock and anti-Fock representations to a pair of orthogonal transformations $\{A, A^*\}$ with $A^2 = -1$ on $H$.

1. Notations and Definitions

In this section we prepare some notations and definitions from papers [2], [4] and [6] with slight modifications.

Let $\mathcal{H}$ be a separable complex Hilbert space, $\mathcal{A}$ a von Neumann algebra on $\mathcal{H}$ and $E$ a faithful normal trace on $\mathcal{A}$ with $E(1) = 1$. By $(K, \Gamma)$ we mean a complex Hilbert space $K$ with an antiunitary involution $\Gamma$, namely, $(\Gamma \xi | \Gamma \eta) = (\eta | \xi)$ for $\xi, \eta \in K$ and $\Gamma^2 = 1$. We denote by $F = (F, \mathcal{A})$ a strongly continuous ($\|F(\xi)\| \leq \lambda \|\xi\|$ for some $\lambda > 0$) and faithful (if $\xi$...
$F(\xi) \neq 0$) linear mapping of $(K, \Gamma)$ to $\mathbb{C}$. with $F(\Gamma \xi) = F(\xi)^*$. Since $E$ is faithful, $F \neq 0$. For any subset $K_0$ of $K$ the von Neumann algebra generated by $F(\xi)$, $\xi \in K_0$ is denoted by $\mathcal{A}(K_0)$. $\mathcal{A}(F)$ is the union of $\mathcal{A}(K_0)$ where $K_0$ runs over all finite subsets of $K$. Introduce the following equivalence relation into the set $F$ of such linear mappings. $(F_1, \mathcal{A}_1)$ and $(F_2, \mathcal{A}_2) \in F$ are equivalent if for any finite subset $K_0 = \{\xi_1, \ldots, \xi_n\}$ of $K$ there exists an isomorphism $\tau: F_1(\xi) \rightarrow F_2(\xi)$, $\xi \in K_0$ of $\mathcal{A}_1(K_0)$ onto $\mathcal{A}_2(K_0)$ such that

$$E_1(A) = E_2(\tau(A))$$

for $A \in \mathcal{A}_1(K_0)$, where $E_i$ is a faithful normal trace on $\mathcal{A}_i$ with $E_i(1) = 1$.

An equivalence class $m$ of $F$ classified by the above relation is called a distribution on $(K, \Gamma)$. Since algebraic structures of all $(F, \mathcal{A}) \in m$ is preserved by an isomorphism as above, the terminologies for $m$ is utilized as similarly as that for each $(F, \mathcal{A})$. For $(F, \mathcal{A}) \in m$, let $\mathcal{H}_F$, $\pi_F$ and $\Omega_F$ be the Hilbert space, the representation and the cyclic unit vector respectively such that

$$E(A) = \langle \pi_F(A) \Omega_F | \Omega_F \rangle$$

for $A \in \mathcal{A}(F)$. Denote

$$\pi_F(\mathcal{A}) = \{\pi_F(A) : A \in \mathcal{A}(F)\}''.$$  

Then it is easily seen that $(F, \mathcal{A})$ and $(\pi_F \circ F, \pi_F(\mathcal{A}))$ are equivalent. The latter is called the standard representative of $m$. From this we may assume that for every $(F, \mathcal{A}) \in m$,

$$\mathcal{A} = \{F(\xi) : \xi \in K\}''$$

in the following. If $\mathcal{A}$ is a factor for $(F, \mathcal{A}) \in m$, $m$ is called a factor distribution.

Let $K_1$ be a $\Gamma$-invariant subspace of $K$ such that $\Gamma_1 = \Gamma|K_1$. For distributions $m$ on $(K, \Gamma)$ and $m_1$ on $(K_1, \Gamma_1)$ choose suitable representatives $(F, \mathcal{A}) \in m$ and $(F_1, \mathcal{A}_1) \in m_1$. If $\mathcal{A}_1 \subset \mathcal{A}$ and $F(\xi) = F_1(\xi)$ for $\xi \in K_1$, then $m$ is called an extension of $m_1$ and denoted by $m_1 = m|K_1$.

Since for any distribution $m$ on $(K, \Gamma)$ there is $\lambda > 0$ such that
for all \( \xi \in K \), there exists a positive operator \( t \) on \( K \) such that
\[
E(F(\xi)^*F(\xi)) \leq \lambda \| \xi \|^2
\]
which is called the covariance operator of \( m \). It is easily seen that \( t = \Gamma t \Gamma \).

A family \( \{ K_t : t \in I \} \) of subspaces of \( K \) is said to be independent with respect to \( m \), if
\[
E(A_{t_1} \ldots A_{t_n}) = E(A_{t_1}) \ldots E(A_{t_n})
\]
for any \( A_t \in \mathcal{M}(F) \) with \( F_t \in m \rvert K_t \) and for any \( t_1, \ldots, t_n \in I \).

A distribution is skew, if for any \((F, \mathcal{H}) \in m \) and if \( F \in m \) implies \(-F \in m \). It follows from the last condition that
\[
E(F(\xi_1) \ldots F(\xi_{2n+1})) = 0.
\]

A unitary operator on \((K, \Gamma)\) which commutes with \( \Gamma \) is called a
Bogoliubov transformation. \( \mathcal{H} \) is a set of Bogoliubov transformations on \((K, \Gamma)\) whose commutant is the algebra of scalar operators. A skew distribution \( m \) is called to be \( \mathcal{H} \)-invariant if \((F, \mathcal{H}) \in m \) implies \((F \circ U, \mathcal{H}) \in m \) and if
\[
[F(U\xi), F(U\eta)]_\mathcal{H} = [F(\xi), F(\eta)]_\mathcal{H},
\]
here \((F \circ U)(\xi) = F(U\xi)\).

A self dual CAR algebra \( \mathcal{A}_{\text{SDC}}(K, \Gamma) \) over \((K, \Gamma)\) is a \(*\)-algebra generated by \( B(\xi), \xi \in K \), its adjoint \( B(\xi)^* \), \( \xi \in K \) and the identity which satisfy the following three relation: \( B(\xi) \) is linear in \( \xi \), \([B(\xi), B(\eta)]_\mathcal{H} = (\xi \rvert \Gamma \eta)1 \) and \( B(\xi)^* = B(\Gamma^* \xi) \). If \( K \) has a finite dimension, \( \mathcal{A}_{\text{SDC}}(K, \Gamma) \) has a finite dimension. Irrespective of the dimension of \( K \), \( \mathcal{A}_{\text{SDC}}(K, \Gamma) \) has a unique \( C^* \)-norm and \( \overline{\mathcal{A}}_{\text{SDC}}(K, \Gamma) \) denotes its \( C^* \)-completion.

A state \( \varphi \) on \( \overline{\mathcal{A}}_{\text{SDC}}(K, \Gamma) \) satisfying the following relation is called a quasifree state:
\[
\varphi(B(\xi_1) \ldots B(\xi_{2n+1})) = 0,
\]
\[ \varphi(B(\xi_1)\cdots B(\xi_{2n})) = \sum_{s} \text{sgn}(s) \prod_{j=1}^{n} \varphi(B(\xi_{s(2j-1)})B(\xi_{s(2j)})), \]

where \( n = 1, 2, \ldots \), the sum is over all permutations \( s \) satisfying
\[ s(1) < s(3) < \cdots < s(2n-1) \text{ and } s(2j-1) < s(2j) \]
for \( j = 1, \ldots, n \) and \( \text{sgn}(s) \) is the signature of \( s \).

### 2. Results

**Lemma 1.** Let \( t \) be a bounded positive operator. If \((t\xi | \eta) = 0\) for any pair of \( \xi \) and \( \eta \) in \( K \) with \((\xi | \eta) = (\xi | \Gamma \eta) = 0\), then \( t \) is a scalar operator.

**Proof.** Choose a complete orthonormal system \( \{\xi_\iota : \iota \in I\} \) of \( K \) with \( \xi_\iota = \Gamma \xi_\iota \). It follows from the hypothesis that \((t\xi_\iota | \xi_\iota) = 0\) for \( \iota, \kappa \in I \) with \( \iota \neq \kappa \), and hence \( t\xi_\iota = \lambda_\iota \xi_\iota \) for some \( \lambda_\iota \geq 0 \), \( \iota \in I \). For any \( \iota_0 \) and \( \iota_1 \) in \( I \) with \( \iota_0 \neq \iota_1 \), put
\[ \eta_{\iota_0} = \xi_{\iota_0} - \xi_{\iota_1}, \quad \eta_{\iota_1} = \xi_{\iota_0} + \xi_{\iota_1}, \]
and \( \eta_\iota = \xi_\iota \) for \( \iota \neq \iota_0 \) and \( \iota \neq \iota_1 \). Then \( \Gamma \eta_\iota = \eta_\iota \) for \( \iota \in I \) and \( \{\eta_\iota : \iota \in I\} \) is a complete orthogonal system. It follows from the hypothesis that \( t\eta_{\iota_0} = \mu \eta_{\iota_0} \) for some \( \mu \geq 0 \). On the other hand
\[ t\eta_{\iota_0} = t(\xi_{\iota_0} - \xi_{\iota_1}) = \lambda_{\iota_0} \xi_{\iota_0} - \lambda_{\iota_1} \xi_{\iota_1}. \]
Thus \( \lambda_{\iota_0} = \mu = \lambda_{\iota_1} \). Repeating the similar argument for each pair of elements in \( I \), we get \( \lambda_\iota = \mu \) for \( \iota \in I \). Consequently \( t = \mu 1 \).

**Corollary.** Let \( t \) be the covariance operator of a skew distribution \( m \) on \( (K, \Gamma) \). If \( [m(\xi), m(\eta)] = 0 \) for any pair of \( \xi \) and \( \eta \) in \( K \) with \((\xi | \eta) = (\xi | \Gamma \eta) = 0\), then \( t \) is a scalar operator.

**Proof.** It is clear from
\[ 2(t\xi | \eta) = E(\square F(\xi), F(\eta)^* \square) \]
for \((F, \mathcal{H}) \in m \).

It should be noted that, since the underlying Hilbert space \( \mathcal{H} \) of \( \mathcal{H} \) for \((F, \mathcal{H}) \in m \) is assumed to be separable, \( \mathcal{H}(F)' \) is generated by a countable family of elements. Furthermore, since \( F \) is assumed to be faithful, it follows that \( K \) is separable.
Lemma 2. A skew distribution \( m \) with the covariance operator \( t \) has a representative \((F, \xi) \in m\) such that there exist a locally compact space \( Z \), a positive measure \( \nu \) whose carrier is \( Z \), a \( \nu \)-measurable field \( \zeta \to \mathcal{S}(\zeta) \) of Hilbert spaces, a \( \nu \)-measurable field \( \zeta \to \mathcal{V}(\zeta) \) of von Neumann algebras, a \( \nu \)-measurable field \( \zeta \to E_\zeta \) of finite normal traces and a \( \nu \)-measurable operator valued function \( \zeta \to t(\zeta) \) with \( t(\zeta) \in B(K) \) and that

\[
\begin{align*}
\mathcal{S} &= \int \mathcal{S}(\zeta) d\nu(\zeta), \quad \mathcal{V} = \int \mathcal{V}(\zeta) d\nu(\zeta), \quad E = \int E_\zeta d\nu(\zeta), \\
\{F(\xi), F(\eta)^*\} = 2\int (t(\zeta)\xi|\eta)1(\zeta) d\nu(\zeta)
\end{align*}
\]

and

\[
t = \int t(\zeta)E_\zeta(1(\zeta)) d\nu(\zeta),
\]

where \( B(K) \) is a full operator algebra on \( K \) and \( 1(\zeta) \) is the identity in \( \mathfrak{N}(\zeta) \).

Proof. According to the reduction theory and the separability of \( \mathcal{N} \), we can conclude that there exist a locally compact space \( Z \) which satisfies the second axiom of countability, a positive measure \( \nu \) whose carrier is \( Z \), a \( \nu \)-measurable field \( \zeta \to \mathcal{S}(\zeta) \) of non zero Hilbert spaces on \( Z \) and a \( \nu \)-measurable field \( \zeta \to \mathfrak{N}(\zeta) \) of factor von Neumann algebras over \( \mathfrak{S}(\zeta) \) on \( Z \) such that \( \mathfrak{N} \) is spatially isomorphic to

\[
\int \mathfrak{N}(\zeta) d\nu(\zeta) \quad \text{over} \quad \int \mathcal{S}(\zeta) d\nu(\zeta).
\]

Since \( K \) is separable, it contains a \( \Gamma \)-invariant countable dense \( \mathbb{Q} \) linear subset \( K_0 \), where \( \mathbb{Q} \) denotes the field of rational complex numbers. Since \( F \) is strongly continuous, a \( * \)-algebra \( \mathfrak{N}_0 \) generated by \( \{F(\xi) : \xi \in K\} \) and the identity has a countable base with respect to the uniform topology. Hence we may associate with \( T \in \mathfrak{N}_0 \) a \( \nu \)-measurable field \( \zeta \to T(\zeta) \) of operators with \( T(\zeta) \in \mathfrak{N}(\zeta) \) such that

\[
T = \int T(\zeta) d\nu(\zeta), \quad \|T(\zeta)\| \leq \|T\|
\]

and a mapping \( \Phi_T : T \to T(\zeta) \) is a \( * \)-homomorphism. Put \( F_\xi(\xi) = \Phi_T(F(\xi)) \) for \( \xi \in K \). Then
\[ F(\xi) = \int F_\xi(\xi) d\nu(\zeta) \]

and \( F_\xi \) belongs to some skew distribution \( m_\xi \) on \((K, \Gamma)\). It follows that

\[ [F(\xi), F(\eta)^*]_+ = \int \lambda_\xi(\xi, \eta) 1(\zeta) d\nu(\zeta) \]

for some complex number \( \lambda_\xi(\xi, \eta) \in \mathbb{C} \). Since \( K \) is separable, it contains a countable dense linear subset \( K_0 \) on \( \mathbb{Q} \). Then there is a \( \nu \)-null set \( N \subset Z \) such that

\[ [F_\xi(\xi), F_\xi(\eta)^*]_+ = \lambda_\xi(\xi, \eta) 1(\zeta) \]

and \( \|F_\xi(\xi)\| \leq \|F(\xi)\| \) for \( \zeta \in N \) and \( \xi, \eta \in K_0 \). Since

\[
\begin{align*}
\int \lambda_\xi(\lambda \xi_1 + \xi_2, \eta) 1(\zeta) d\nu(\zeta) &= [F(\lambda \xi_1 + \xi_2), F(\eta)^*]_+ \\
&= \lambda [F(\xi_1), F(\eta)^*]_+ + [F(\xi_2), F(\eta)^*]_+ \\
&= \int \{ \lambda \lambda_\xi(\xi_1, \eta) + \lambda_\xi(\xi_2, \eta) \} 1(\zeta) d\nu(\zeta),
\end{align*}
\]

and

\[ |\lambda_\xi(\xi, \eta)| \leq \|F(\xi), F(\xi)^*\|_+ \]

for \( \xi, \xi_1, \xi_2, \eta \in K_0 \) and \( \lambda \in \mathbb{Q} \), it follows that \( \lambda_\xi \) is a positive definite bounded bilinear functional on \( K_0 \) as well as on \( K \). Hence a bounded positive operator \( \iota(\zeta) \) exists on \( K \) for \( \zeta \in Z - N \) satisfying

\[ [F_\xi(\xi), F_\xi(\eta)^*]_+ = \lambda_\xi(\xi, \eta) 1(\zeta) = 2(\iota(\zeta) \xi | \eta) 1(\zeta) \]

for \( \xi, \eta \in K \). Define \( \iota(\zeta) = 0 \) for \( \zeta \in N \). Then the mapping \( \zeta \mapsto \iota(\zeta) \) is a \( \nu \)-measurable function on \( Z \) with values in \( B(K) \) and

\[ [F(\xi), F(\eta)^*]_+ = 2 \int (\iota(\zeta) \xi | \eta) 1(\zeta) d\nu(\zeta) \]

for any \( \xi, \eta \in K \). Since \( E \) is a faithful normal trace with \( \|E\| = 1 \) over \( \mathfrak{H} \), there exists a \( \nu \)-measurable field \( \zeta \mapsto E_\xi \) of finite faithful normal traces on \( Z \) such that

\[ E = \int E_\xi d\nu(\zeta). \]
It follows that

\[ 2\langle t\xi | \eta \rangle = E(\xi | F(\xi) , F(\eta) ) = \int_\mathcal{H} (\ell(\zeta) \xi | \eta ) d\nu(\zeta) \]

and hence

\[ t = \int E_\zeta(1(\zeta)) \ell(\zeta) d\nu(\zeta). \]

**Definition.** A distribution \( m \) is a canonical skew distribution, if for \((F, \mathcal{M}) \in m \)

\[ \mathcal{M}(F(\xi), F(\eta)^*) = \langle \xi | \eta \rangle. \]

**Remark 1.** If \( m \) is a factor skew distribution with the covariance operator \( t \), then

\[ \mathcal{M}(F(\xi), F(\eta)^*) = \langle \xi | \eta \rangle 1. \]

Since \( t = \Gamma t \Gamma \), \( K \) is a pre-Hilbert space with respect to the inner product \( \langle \cdot | \cdot \rangle_\Gamma \) defined by

\[ \langle \xi | \eta \rangle_\Gamma = \langle 2t\xi | \eta \rangle \]

for \( \xi, \eta \in K \). Denote by \( K_t \) the completion of \( K \) with respect to \( \langle \cdot | \cdot \rangle_\Gamma \). Since \( \Gamma \) can be extended to \( K_t \), we shall denote it by the same letter \( \Gamma \). Then

\[ \mathcal{M}(F(\xi), F(\eta)^*) = \langle \xi | \eta \rangle_\Gamma 1, \]

that is, \( F \) is a canonical skew distribution on \((K_t, \Gamma)\), which generates a self dual CAR algebra \( \mathcal{M}_{SC}(K_t, \Gamma) \). Choosing a standard representative \((F, \mathcal{M})\), we know that \( \mathcal{M} \) is a hyperfinite \( II_1 \) factor, if \( K \) is of infinite dimension.

If the underlying Hilbert space \( \mathcal{H} \) of a von Neumann algebra \( \mathcal{A} \) is separable, then \( \mathcal{M} \) has a countable generator \( \mathcal{M} \). Let \( \mathcal{B}_0 \) and \( \mathcal{B} \) denote \( * \)-algebras algebraically generated by \( \mathcal{M} \) on \( \mathcal{Q} \) and \( \mathbb{C} \) respectively. Then \( \mathcal{B}_0 \) is countable and the unit ball of \( \mathcal{B}_0 \) is uniformly dense in the unit ball of \( \mathcal{B} \). Since \( \mathcal{B} \) is strongly dense in \( \mathcal{A} \), the unit ball of \( \mathcal{B} \) is strongly dense in the unit ball \( \mathcal{A}_1 \) of \( \mathcal{A} \) by the Kaplansky’s density theorem and
hence $\mathfrak{N}_1$ is separable. Since the unit ball of a countably decomposable von Neumann algebra is metrizable by the strong topology, $\mathfrak{N}_1$ satisfies the second axiom of countability. Thus any subset $\mathfrak{N}$ of $\mathfrak{N}_1$ contains a countable subset $\mathfrak{N}_0$ which is strongly dense in $\mathfrak{N}$.

Utilizing the same notations as in the last lemma, we have the following

**Lemma 3.** A necessary and sufficient condition that a skew distribution $m$ be $\mathfrak{U}$-invariant is that there be $(F, \mathfrak{N}) \in m$ and $t(\xi)$ as in Lemma 2 such that

(i) $E(F(U\xi_1) \ldots F(U\xi_n)) = E(F(\xi_1) \ldots F(\xi_n))$ for every $U \in \mathfrak{U}$; and
(ii) $t(\xi)$ is a scalar operator $\nu$-almost everywhere.

**Proof.** Necessity: By virtue of Lemma 2, there exists a representative $F \in m$ such that

$$2\int (t(\xi)U\xi|U\eta)1(\xi)d\nu(\xi) = [F(U\xi), F(U\eta)^*]_+$$

$$= [F(\xi), F(\eta)^*]_+ = 2\int (t(\xi)^*|\eta)1(\xi)d\nu(\xi).$$

Since $\mathfrak{N}$ is separable, $\mathfrak{U}$ contains a countable family $\mathfrak{U}_0$ which is strongly dense in $\mathfrak{U}$ by the preceding discussion. Thus there is a $\nu$-null set $N \subset Z$ such that

$$(t(\xi)U\xi|U\eta) = (t(\xi)\xi|\eta)$$

for any $\xi \in Z - N$, $U \in \mathfrak{U}_0$ and $\xi, \eta \in K_0$. Since $(\mathfrak{U}_0)' = \mathfrak{U}'$ is the algebra of scalar operators, it follows that $t(\xi)$ is a scalar operator for $\xi \in Z - N$.

Sufficiency: If $t(\xi)$ is a scalar operator $\nu$-almost everywhere, then

$$[F(U\xi), F(U\eta)^*]_+ = 2\int (t(\xi)U\xi|U\eta)1(\xi)d\nu(\xi)$$

$$= 2\int (t(\xi)^*|\eta)1(\xi)d\nu(\xi) = [F(\xi), F(\eta)^*]_-.$$ 

Further, since $F(U\xi)$, $U \in \mathfrak{U}$ is strongly continuous, faithful linear mapping in $\xi$ with $F(U^*\xi) = F(U\xi)^*$, $U$ induces an automorphism $\tau(U)$ of $\mathfrak{U}(K_0)$
for any finite $K_0$ in $K$ such that $\tau(U)F(\xi) = F(U\xi)$, $\xi \in K_0$ and $E(\tau(U)A) = E(A)$, $A \in \mathcal{A}(K_0)$ by (i).

**Lemma 4.** Let $B$ be a canonical skew distribution. If $[\xi_i, \Gamma \xi_i; i=1, \ldots, n]$ and $[\eta_j, \Gamma \eta_j; j=1, \ldots, m]$ are orthogonal, then the quasifree state $E$ with $E(B(\eta) * B(\xi)) = 2^{-1} (\xi | \eta)$ on a C*-algebra of a self dual CAR algebra $\mathcal{A}_{SDC}(K, \Gamma)$ satisfies

$$E(B(\xi_1) \ldots B(\xi_n) B(\eta_1) \ldots B(\eta_m)) = E(B(\xi_1) \ldots B(\xi_n)) E(B(\eta_1) \ldots B(\eta_m)),$$

where $\omega_k; k=1, \ldots, l]$ denotes the subspace spanned by $\omega_1, \ldots, \omega_l$.

**Proof.** If $n$ or $m$ are odd, then the left side and at least one of the factors in the right side are 0 due to $E(B(\xi)B(\eta)) = E(B(\eta)B(\xi)) = 0$. If $n$ and $m$ are even, $E(B(\xi_1) \ldots B(\xi_n) B(\eta_1) \ldots B(\eta_m)) = \sum_{\sigma} \text{sgn}(\sigma) \prod_{j=1}^{k} E(B(\xi_{\sigma(2j-1)}) B(\xi_{\sigma(2j)}))$

where $2k = n + m$, $\xi_{n+i} = \eta_i$ for $i = 1, \ldots, m$, $\text{sgn}$ is the signature of the permutation $\sigma$ satisfying

$$\sigma(1) < \sigma(3) < \cdots < \sigma(2k-1) \text{ and } \sigma(2j-1) < \sigma(2j)$$

for $j = 1, \ldots, k$. If there is $j$ with $1 \leq j \leq k$ such that $1 \leq \sigma(2j-1) \leq n$ and $n+1 \leq \sigma(2j) \leq n+m$, then

$$\prod_{j=1}^{k} E(B(\xi_{\sigma(2j-1)}) B(\xi_{\sigma(2j)}) = 0.$$ 

Therefore we have only to consider the sum over all permutations $\sigma$ satisfying

$$1 \leq \sigma(2j-1) < \sigma(2j) \leq n$$

for $j = 1, \ldots, 2^{-1}n$ and

$$n+1 \leq \sigma(2i-1+n) < \sigma(2i+n) \leq n+m$$

for $i = 1, \ldots, 2^{-1}m$. Let $s$ and $t$ denote the permutations

$$s = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix} \text{ and } t = \begin{pmatrix} n+1 & n+2 & \cdots & n+m \\ \sigma(n+1) & \sigma(n+2) & \cdots & \sigma(n+m) \end{pmatrix},$$
Then \( \text{sgn}(\sigma) = \text{sgn}(s) \text{sgn}(t) \) for \( \sigma \) considered. Consequently

\[
E(B(\xi) \ldots B(\xi_n)B(\eta_1) \ldots B(\eta_m)) \\
= \sum_{\sigma} \text{sgn}(\sigma) \left\{ \prod_{j=1}^{m/2} E(B(\xi_{\sigma(2j-1)})B(\xi_{\sigma(2j)})) \right\} \left\{ \prod_{i=1}^{m/2} E(B(\xi_{\sigma(2i-1+n)})B(\xi_{\sigma(2i+n)})) \right\}
= \left\{ \sum_{\sigma} \text{sgn}(s) \prod_{j=1}^{m/2} E(B(\xi_{\sigma(2j-1)})B(\xi_{\sigma(2j)})) \right\}
\]

\[
\sum_{t} \text{sgn}(t) \prod_{i=1}^{m/2} E(B(\eta_{i(2i-1)})B(\eta_{i(2i+1)}))
\]

\[
= E(B(\xi) \ldots B(\xi_n))E(B(\eta_1) \ldots B(\eta_m)),
\]

where \( s \) and \( t \) satisfy the condition in the definition of a quasifree state.

Since \( E \) is faithful and \( \nu \) has the carrier \( Z \), it follows that a function \( \zeta \mapsto E_\zeta(1(\xi)) \) is \( \nu \)-measurable and \( 0 < E_\zeta(1(\xi)) < +\infty \) \( \nu \)-almost everywhere. Define a probability measure \( \mu \) on \( Z \) by \( d\mu(\zeta) = E_\zeta(1(\xi))d\nu(\zeta) \). Then \( \mu \) and \( \nu \) are equivalent and hence the \( \mu \)-measurability and the \( \nu \)-measurability coincide.

**Theorem.** Let \( m \) be a skew distribution on \( (K, \Gamma) \) whose dimension is even or infinite. Any two of the following conditions imply the remaining one:

(i) \( m \) is a factor distribution;

(ii) \( m \) is \( \mathcal{U} \)-invariant;

(iii) any pair of \( \Gamma \)-invariant orthogonal subspaces are independent with respect to \( m \).

In this case \( m \) is the canonical skew distribution up to a scalar constant.

**Proof.** (i) and (ii) imply (iii): According to (i), (ii) and Lemma 3, the covariance operator \( t \) of \( m \) is a scalar operator, say \( 2t = \lambda 1 \) for \( \lambda > 0 \). Put \( B(\xi) = \lambda^{-1/2} F(\xi) \). Again by (i)

\[
[ B(\xi), B(\eta)^* ]_+ = E( [ B(\xi), B(\eta)^* ]_+ )_1 = \lambda E( [ F(\xi), F(\eta)^* ]_+ )_1 = (\xi \mid \eta)_1,
\]

hence \( B \) is a canonical skew distribution. Then \( B(\xi), \xi \in K \) generate a self dual CAR algebra \( \mathcal{A}_{SC}(K, \Gamma) \), we have
for any orthogonal subspaces \([\xi_i, \Gamma \xi_i; i = 1, \ldots, n]\) and \([\eta_j, \Gamma \eta_j; j = 1, \ldots, m]\) by Lemma 4.

(ii) and (iii) imply (i): Employing the same notations as in Lemma 2, we find a \(\mu\)-measurable field \(\zeta \mapsto F_\zeta(\xi)\) of operators on \(Z\) and a \(\mu\)-measurable operator valued function \(\zeta \mapsto t(\zeta)\) with the remaining properties in Lemma 2 such that

\[
F(\xi) = \int_0^\infty F_\zeta(\xi) d\nu(\zeta)
\]

and

\[
t = \int t(\zeta) d\nu(\zeta).
\]

It follows from (ii) and Lemma 3 that \(t(\zeta)\) is a scalar operator, say \(2t(\zeta) = \lambda(\zeta)1\) and \(\lambda(\zeta) > 0\) for \(\zeta \in Z - N\) and \(\mu(N) = 0\). Putting \(B_\zeta(\xi) = \lambda(\zeta)^{-1/2} F_\zeta(\xi)\) for \(\zeta \in Z - N\) and \(B_\zeta(\xi) = 0\) for \(\zeta \in N\), we have a \(\mu\)-measurable field \(\zeta \mapsto B_\zeta(\xi)\) of canonical skew distributions. According to Lemma 4

\[
E(F(\xi_1) \ldots F(\xi_{2n}) F(\eta_1) \ldots F(\eta_{2m}))
= \lambda^{n+m} E(B(\xi_1) \ldots B(\xi_{2n}) B(\eta_1) \ldots B(\eta_{2m})) d\nu(\zeta)
= \lambda^{n+m} E(B(\xi_1) \ldots B(\xi_{2n}) E(\eta_1) \ldots E(\eta_{2m})) d\nu(\zeta).
\]

On the other hand

\[
E(F(\xi_1) \ldots F(\xi_{2n})) E(F(\eta_1) \ldots F(\eta_{2m}))
= \lambda^{n+m} E(B(\xi_1) \ldots B(\xi_{2n}) d\nu(\zeta) \lambda^{n+m} E(B(\xi_1) \ldots B(\xi_{2m}) d\nu(\zeta).
\]

Selecting \(\xi_i\) and \(\eta_j\) being mutually orthogonal such that \(\xi_{n+i} = \Gamma \xi_i\) and \(\eta_{m+j} = \Gamma \eta_j\). Then by (iii)

\[
\int \{\lambda(\zeta)/2\}^{n+m} d\mu(\zeta) = \int \{\lambda(\zeta)/2\}^n d\mu(\zeta) \int \{\lambda(\zeta)/2\}^m d\mu(\zeta).
\]
for any integers $n \geq 0$ and $m \geq 0$. Therefore $\lambda(\zeta)$ is constant $\mu$-almost everywhere and hence $\lambda(\zeta) = \lambda$ for some $\lambda > 0$ $\mu$-almost everywhere. Let $B(\xi)$ be an operator in $\mathfrak{A}$ which corresponds to $\zeta \rightarrow B_\xi(\xi)$. Then

$$F(\zeta) = \int_\mathfrak{A} F_\xi(\zeta) d\nu(\zeta) = \int_\mathfrak{A} \lambda^{1/2} B_\xi(\zeta) d\nu(\zeta) = \lambda^{1/2} B(\xi)$$

and hence $B$ is a canonical skew distribution on $(K, \Gamma)$. Taking a standard representative $(\mathfrak{F}, \mathfrak{A}) \in m$, we can conclude that the von Neumann algebra $\mathfrak{A}$ is a factor.

(iii) and (i) imply (ii): By (iii), if $(\xi | \eta) = (\zeta | \Gamma \eta) = 0$, then

$$(\xi \xi | \eta) = E(m(\eta)^* m(\xi)) = E(m(\eta)^* E(m(\xi))) = 0.$$}

It follows from Lemma 1 that $2t = \lambda 1$ for some $\lambda > 0$. Put $B(\xi) = \lambda^{-1/2} F(\xi)$. Then $B$ is a canonical skew distribution by (i). Since

$$E(F(\xi_1) \cdots F(\xi_n)) = \lambda^n E(B(\xi_1) \cdots B(\xi_2n))$$

$$= \lambda^n \sum_\sigma \text{sgn}(\sigma) \prod_{j=1}^n E(B(\xi_{\sigma(2j-1)}) B(\xi_{\sigma(2j)}))$$

$$= \lambda^n \sum_\sigma \text{sgn}(\sigma) \prod_{j=1}^n E(B(U\xi_{\sigma(2j-1)}) B(U\xi_{\sigma(2j)}))$$

$$= \lambda^n E(B(U\xi_1) \cdots B(U\xi_{2n})) = E(F(U\xi_1) \cdots F(U\xi_{2n})),$$

it follows from Lemma 3 that $m$ is $\mathfrak{A}$-invariant.

**Remark 2.** In case where the last theorem is valid, the covariance operator of $m$ is a scalar operator. Making use of the results of Segal [5], we have that if $A \in \mathfrak{A}$ and $\tau(U) A = A$ for all $U \in \mathfrak{A}$, then $A$ is a scalar operator, where $\tau(U)$ is the $^*$-automorphism induced by

$$\tau(U) F(\xi) = F(U\xi).$$

### 3. Appendix

Let $K$ be a complex Hilbert space and $\Gamma$ be an antiunitary involution. A projection $e$ on $K$ is called a *basis* projection if $\Gamma e \Gamma = 1 - e$. Restricting the coefficient to the real number field and define the inner product $(\quad | \quad)$, by
we have a real Hilbert space $(K, (\cdot | \cdot)_r)$, which we denote by $K_r$. Here $\mathbb{R}$ denotes the real part. Then the operator $i$ of multiplying $i$ on $K_r$ is an orthogonal transformation on $K_r$ with $i^2 = -1$. Since the set $\{\xi \in K: i^*\xi = \xi\}$ forms a real Hilbert space for the inner product $(\cdot | \cdot)_r$, we designate it by $H$. Then $iH$ is a set $\{\xi \in K: i^*\xi = -\xi\}$ and $K_r = H \oplus iH$. The complexification $H + iH$ of $H$ with the inner product

$$(\xi_1 + i\xi_2 | \eta_1 + i\eta_2) = (\xi_1 | \eta_1) - (\xi_2 | \eta_2) + i((\xi_2 | \eta_1) - (\xi_1 | \eta_2))$$

is naturally isomorphic to $K$. For any orthogonal transformation $v$ on $H$ there corresponds a Bogoliubov transformation $u$ on $K$ such that

$$u(\xi_1 + i\xi_2) = v\xi_1 + iv\xi_2$$

for $\xi_1, \xi_2 \in H$. Conversely, since every Bogoliubov transformation is reduced by $H$, the restriction $v$ of $u$ onto $H$ is an orthogonal transformation on $H$. Since $v = 1$ iff $u = 1$, this correspondence is bijective. Let $A$ be an orthogonal transformation on $H$ with $A^2 = -1$ and $e$ be the projection onto the complex subspace of $K$ such that

$$eK = \{\xi - iA\xi: \xi \in H\}.$$

Then $e$ is a basis projection, $\Gamma'(\xi + iA\xi) = \xi - iA\xi$ and

$$(1 - e)K = \{\xi + iA\xi: \xi \in H\}.$$

**Proposition 1.** There is a bijection between the family of pairs $\{e, 1 - e\}$ of basis projections on $(K, \Gamma')$ and the family of pairs $\{A, iA\}$ of orthogonal transformations with $A^2 = -1$ on $H$.

**Proof.** With $A$ satisfying $\Gamma A = A' A = 1$ and $A^2 = -1$, we associate basis projections $e$ and $1 - e$ as above.

Suppose that $e$ is a basis projection. Let $H$ be the real Hilbert space $\{\eta + \Gamma\eta: \eta \in eK\}$. Choose $\xi \in H$. If $\xi = \eta + \Gamma\eta$ and $\xi = \eta' + \Gamma\eta'$ for $\eta$ and $\eta' \in eK$. Then $\eta - \eta' = \Gamma(\eta - \eta')$ and $\eta - \eta' \in eK$, which implies that $\eta = \eta'$. Thus we may define a transformation $A_0$ on $H$ by

$$A_0(\eta + \Gamma\eta) = i(\eta - \Gamma\eta) = i\eta + \Gamma(i\eta)$$
for \( \eta \in eK \). Then \( A_0 \) is an orthogonal transformation with \( A_0^2 = -1 \). Since for \( \xi = \eta + \Gamma \eta \) and \( \eta \in eK \) we have

\[
\xi + iA_0\xi = 2\Gamma\eta \in (1 - e)K \quad \text{and} \quad \xi - iA_0\xi = 2\eta \in eK,
\]

it follows that \( A_0 \) is associated with \( e \) and \( 1 - e \). Hence the mapping \( \{A, 'A\} \rightarrow \{e, 1 - e\} \) defined above is onto. Suppose now that \( e \) is a basis projection associated with a given orthogonal transformation \( A \) with \( A^2 = -1 \) such that

\[
eK = \{\xi - iA\xi : \xi \in H\},
\]

and that \( A_0 \) is associated with \( e \) as above. Since, then, for any \( \xi \in H \) we have \( \eta = \xi - iA\xi \in eK \) and \( \Gamma\eta = \xi + iA\xi \in (1 - e)K \), it follows that

\[
\eta + \Gamma\eta = 2\xi \quad \text{and} \quad i(\eta - \Gamma\eta) = 2A\xi.
\]

Therefore \( A_0\xi = A\xi \) for \( \xi \in H \), that is, \( A_0 = A \).

**Remark 3.** It is clear that a self dual CAR algebra \( \mathfrak{A}_{SDC}(K, \Gamma) \) on \((K, \Gamma)\) and a Clifford algebra \( \mathfrak{A}_{CL}(H) \) coincide, if \( K \) is of even or infinite dimension.

A family \( \{K_t : t \in I\} \) of \( \Gamma \)-invariant subspaces of \((K, \Gamma)\) is said to be independent with respect to a quasifree state \( \varphi \) on \( \mathfrak{A}_{SDC}(K, \Gamma) \) if

\[
\varphi(A_{t_1} \cdots A_{t_n}) = \varphi(A_{t_1}) \cdots \varphi(A_{t_n})
\]

for any \( A_t \in \mathfrak{A}_{SDC}(K, \Gamma) \) and for any \( t_1, \cdots, t_n \in I \), where \( \mathfrak{A}_{SDC}(K, \Gamma) \) is the C*-subalgebra of \( \mathfrak{A}_{SDC}(K, \Gamma) \) generated by \( B(\xi) \), \( \xi \in K \) and the identity. It is shown in [2] that there is a one to one correspondence between an operator \( s \) on \((K, \Gamma)\) with \( 0 \leq s \leq 1 \) and \( \Gamma s\Gamma = 1 - s \), and a quasifree state \( \varphi \) on \( \mathfrak{A}_{SDC}(K, \Gamma) \) as the following

\[
(s\xi | \eta) = \varphi(B(\xi)B(\xi))
\]

**Proposition 2.** Let \( \varphi \) be a quasifree state on \( \mathfrak{A}_{SDC}(K, \Gamma) \). Then \( \varphi \) is central if and only if any pair of \( \Gamma \)-invariant subspaces are independent with respect to \( \varphi \).

**Proof.** The only if part is clear from Lemma 4. It suffices to show the if part. If \( \xi \) and \( \eta \) are non zero vectors with \((\xi | \eta) = (\xi | \Gamma \eta)\), then
\[(s\xi|\eta) = \varphi(B(\eta)^*B(\xi)) = \varphi(B(\eta)^*)\varphi(B(\xi)) = 0.\]

It follows from Lemma 1 that \(s\) is a scalar operator, that is, \(s = 2^{-11}\).

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