On a Mixed Problem for Hyperbolic Equations with Discontinuous Boundary Conditions

By

Kazunari Hayashida*

1. Mixed problems for hyperbolic equations have been studied by many authors. For variable coefficients the powerful tool is the theorem of Hille-Yosida ([3], [4], [5], [7]). In this case the mixed problems have been treated completely by Mizohata [5] and Ikawa [3].

On the other hand Čehlov [2] has shown the existence of a weak solution for mixed problems under discontinuous boundary conditions. He has imposed the assumption that the space domain is a half space and the equation is the wave equation. His method is the Fourier-Laplace transformation.

In this note we consider a mixed problem under discontinuous boundary conditions of Dirichlet or Neumann type. We proceed mainly along the lines of [3] and [5].

2. Let $\Omega$ be a bounded domain in the $n$-dimensional Euclidean space $\mathbb{R}^n$ with boundary $\partial \Omega$ of class $C^\alpha$. We assume that $\partial \Omega$ consists of two measurable sets $\partial_1 \Omega$ and $\partial_2 \Omega$ having no common points. Further let us assume that

\begin{equation}
\partial_2 \Omega \cap \overline{\partial_1 \Omega} = \emptyset.
\end{equation}

We set

\[ (u, v)_k = \int_D \sum_{|a| \leq k} D^a u \, \overline{D^a v} \, dx, \]

\[ ||u||_k^2 = (u, u)_k \]

and

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* Department of Mathematics, Nagoya University, Chikusa-ku, Nagoya, Japan.
\[(u, v) = (u, v)_0, \quad ||u|| = ||u||_0.\]

Let us denote by \(H^k(\Omega)\) the Sobolev space with norm \(|| \cdot ||_k\) and by \(K(\Omega)\) the completion of all \(u\) each of which belongs to \(C^\infty(\Omega)\) and vanishes in a neighborhood of \(\partial_1 \Omega\) with \(H^1(\Omega)\) norm.

Consider the elliptic operator \(L\) of second order on \(\bar{\Omega} \times [0, T]\):

\[(2.2) \quad L = - \sum_{i,j} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_i b_i(x, t) \frac{\partial}{\partial x_i} + c(x, t),\]

where the coefficients are all in \(C^\infty(\bar{\Omega} \times [0, T])\). We assume that \(a_{ij}(x) (= a_{ji}(x))\) are real valued and positive definite on \(\bar{\Omega}\) and we consider the following equation for real valued \(h_i(x), h(x) \in C^\infty(\bar{\Omega})\):

\[(2.3) \quad \frac{\partial^2}{\partial t^2} u + Lu + \left( 2 \sum_i h_i(x) \frac{\partial}{\partial x_i} + h(x) \right) \frac{\partial}{\partial t} u = f.\]

Let us impose the following boundary condition

\[(2.4) \quad B_1 u(x, t) = u(x, t) = 0 \quad \text{on} \quad \partial_1 \Omega \times [0, T],\]

\[(2.5) \quad B_2 u(x, t) = \left\{ \frac{d}{dn} - <h, \gamma> \frac{\partial}{\partial t} + \sigma(x, t) \right\} u(x, t) = 0 \quad \text{on} \quad \partial_2 \Omega \times [0, T],\]

where

\[\frac{d}{dn} = \sum_{i,j} a_{ij}(x) \cos(\nu, x_i) \frac{\partial}{\partial x_i} \quad (\nu \text{ is the exterior normal vector}),\]

\[<h, \gamma> = \sum_i h_i(x) \cos(\nu, x_i)\]

and \(\sigma(x)\) is \(C^\infty\) on \(\partial_2 \Omega\). The equation (2.3) has been considered in \([3]\) and \([5]\) under the boundary condition \(B_1 u = 0\) or \(B_2 u = 0\) on \(\partial \Omega \times [0, T]\).

Now we define the boundary condition (2.5) in the weak sense as follows:

**Definition 2.1.** Let \(u(\cdot, t)\) be in \(H^1(\Omega)\) and \((Lu)(\cdot, t)\) be in \(L^2(\Omega)\) for \(0 \leq t \leq T\). Further we assume that \(u\) is in \(\mathcal{H}^1_1(\Omega, [0, T])\). Then \(u\) is said to satisfy the boundary condition (2.5) weakly on \(\partial_2 \Omega \times [0, T]\).

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1) For the Banach space \(E\) the letter \(\mathcal{H}^k_1(E)[0, T]\) means the set of \(E\)-valued functions which are \(k\)-times continuously differentiable in \(0 \leq t \leq T\).
if the following equality holds on \([0, T]\):

\[
\left\{ -\frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial}{\partial x_j} \right) \right\} u, \varphi = \sum_{i,j} \left( a_{ij} \frac{\partial u}{\partial x_i}, \frac{\partial \varphi}{\partial x_j} \right) + \int_{\partial_2 \Omega} (\sigma u - \langle h, \gamma \rangle \frac{\partial u}{\partial t}) \varphi \, dS
\]

for any \(\varphi \in K(\Omega, \Omega, T)\).

In addition we define the boundary condition for vector functions as follows.

**Definition 2.2.** Let \(U = \{u, v\} \in H^1(\Omega) \times H^1(\Omega)\) and \(Lu\) be in \(L^2(\Omega)\). Then \(U\) is said to satisfy the boundary condition \((B_2)\) on \(\partial_2 \Omega\), if the following equality holds for any \(\varphi \in K(\Omega)\):

\[
\left\{ -\frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial}{\partial x_j} \right) u, \varphi \right\} = \sum_{i,j} \left( a_{ij} \frac{\partial u}{\partial x_i}, \frac{\partial \varphi}{\partial x_j} \right) + \int_{\partial_2 \Omega} (\sigma u - \langle h, \gamma \rangle v) \varphi \, dS.
\]

In this note we shall prove the following theorems, where we assume that \(f(x, t) \in \mathcal{S}(K(\Omega))\) \([0, T]\) and \(f(x, 0)\) has compact support in \(\Omega\).

**Theorem 1.** Suppose that \(u_0(x), v_0(x) \in K(\Omega)\), \(Lu_0 \in L^2(\Omega)\) and \(\{u_0, v_0\}\) satisfies the boundary condition \((B_2)\) on \(\partial_2 \Omega\). Then there is a unique solution \(u(x, t) \in \mathcal{S}(K(\Omega))\) \([0, T]\) \(\cap \mathcal{S}(L^2(\Omega))\) \([0, T]\) of the equation \((2.3)\) satisfying

\[
u = u_0, \ u_t = v_0 \quad \text{on } t = 0 \quad \text{(initial condition)}
\]

and

\[
B_2 u = 0 \quad \text{weakly on } \partial_2 \Omega \times [0, T].
\]

**Theorem 2.** In addition to the assumption of Theorem 1, assume that \(u_0 \in H^1_{loc}(\bar{\Omega} - S)\),

\(2)\) where \(S\) is the boundary of \(\partial_1 \Omega(\partial_2 \Omega)\). Then \(u(x, t)\) also belongs to \(H^1_{loc}(\bar{\Omega} - S)\). Thus the solution satisfies \(B_2 u = 0\)

\(2)\) The space \(H^1_{loc}(\bar{\Omega} - S)\) is the set of functions belonging to locally \(H^1\) in \(\bar{\Omega} - S\).
in the interior of $\partial_2 \Omega$.

**Remark.** Here we have assumed (2.1). Hence if $\partial_1 \Omega$ is an $(n-2)$-dimensional compact manifold, our theorem holds. With difference method, Babaeva and Namazov [1] has shown the existence of the solution for our problem also when $\partial_2 \Omega$ is an $(n-2)$-dimensional compact manifold.

3. Let us consider the space $H=K(\Omega) \times L^2(\Omega)$ with the inner product

$$(U_1, U_2)_H = \sum_{i,j} (a_{ij}(x) \frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_j}) + (v_1, v_2) + \int_{\partial_2 \Omega} \sigma(x) u_1 \bar{u}_2 dS + c_1(u_1, u_2),$$

where $U_i = \{u_i, v_i\}$ ($i=1, 2$) and $c_1$ is a sufficiently large constant depending only on $a_{ij}$ and $\sigma$. We denote by $\|U\|_H$ the $H$-norm of $U$. Obviously the space $H$ is complete and by the well-known interpolation relation (see e.g. [6]) the norm $\|U\|_H$ is equivalent to $\|u\|_1 + \|v\|_0$ ($U = \{u, v\}$).

The formulation in this section is radically due to the book of H.G. Garnir. 3)

Set the operator $A(t)$ in such a way that

$$A(t) = \begin{pmatrix} 0 & 1 \\ -L & -M \end{pmatrix},$$

where $M = 2 \sum_i h_i(x) \frac{\partial}{\partial x_i} + h(x)$ (see (2.3)). Then $A(t)$ is a closed operator from $H$ to itself having the following definition domain

$$D(A(t)) = \{U = \{u, v\} \mid u, v \in K(\Omega), Lu \in L^2(\Omega)$$

and $U$ satisfies the boundary condition $(B_2)$ on $\partial_2 \Omega$ in the sense of Definition 2.2}.

Since $D(A(t))$ is independent of $t$, we write simply by $D(A)$.

**Remark.** Mizohata \textsuperscript{[5]} and Ikawa \textsuperscript{[3]} have set

$$H = H^1_0(\Omega) \times L^2(\Omega)$$

and

$$D(A) = (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$$

for the case of the Dirichlet type boundary condition. They have set also for the case of the Neumann type boundary condition as follows:

$$H = H^1(\Omega) \times L^2(\Omega)$$

and

$$D(A) = \{ U = \{ u, \ v \} \mid u \in H^2(\Omega), \ v \in H^1(\Omega) \text{ and } \frac{d}{dn} u - \langle h, \gamma \rangle v + \sigma u = 0 \text{ on } \partial\Omega \}.$$

**Lemma 1.** There is a positive constant $c_2$ depending only on $A(t)$ and $\sigma(x)$ such that it holds that for any $U \in D(A)$,

$$| (U, A(t)U)_H + (A(t)U, U)_H | \leq c_2 \| U \|_H^2.$$

**Proof.** We easily see

\begin{align*}
(3.3) \quad (U, A(t)U)_H + (A(t)U, U)_H &= \\
&= \sum_{i,j} \left( a_{ij} \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_j} \right) + \left( v, -Lu - Mv \right) \\
&\quad + \int_{\partial \Omega} \sigma uv\,dS + c_1(u, v) \\
&\quad + \sum_{i,j} \left( a_{ij} \frac{\partial v}{\partial x_i}, \frac{\partial u}{\partial x_j} \right) + \left( -Lu - Mv, v \right) \\
&\quad + \int_{\partial \Omega} \sigma vu\,dS + c_1(v, u).
\end{align*}

Since $U$ satisfies the boundary condition $(B_2)$ on $\partial_2 \Omega$ (see Definition 2.2), we have

\begin{align*}
(3.4) \quad \sum_{i,j} \left( a_{ij} \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_j} \right) + \int_{\partial_2 \Omega} \sigma uv\,dS
\end{align*}
Further it is easily seen that
\begin{equation}
(Mv, v) + (v, Mv) = 2 \int_{\partial_{3, \theta}} <h, \gamma> v \bar{v} dS
- 2 \sum_i \left( \frac{\partial h_i}{\partial x_i} v, v \right) + (hv, v) + (v, hv).
\end{equation}
Combining (3.4), (3.5) and (3.3), we have proved the lemma.

**Lemma 2.** If \( \lambda \) is real and \( |\lambda| \geq c_2 \), we have for any \( U \in D(A) \)
\begin{equation}
\|(\lambda I - A(t))U\|_H \geq (|\lambda| - c_2)\|U\|_H.
\end{equation}

**Proof.** We easily see
\begin{equation}
\|(\lambda I - A(t))U\|_H^2 \geq \lambda^2\|U\|_H^2 - \lambda\{(U, A(t)U)_H + (A(t)U, U)_H\}.
\end{equation}
By Lemma 1 we get
\begin{equation}
\|(\lambda I - A(t))U\|_H^2 \geq (\lambda^2 - |\lambda| c_2)\|U\|_H^2
\geq \{(|\lambda| - c_2)^2 + c_2(|\lambda| - c_2)\}\|U\|_H^2.
\end{equation}
Now set for any \( \varphi, \phi \in K(\Omega) \)
\begin{equation}
B_i[\varphi, \phi] = \sum_{i, j} \left( a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \right)
+ \left( \sum_i b_i \frac{\partial \varphi}{\partial x_i} + c \varphi, \phi \right) + \int_{\partial_{3, \theta}} \sigma \varphi \bar{\phi} dS
- \lambda \int_{\partial_{3, \theta}} <h, \gamma> \varphi \bar{\phi} dS
+ \lambda (M \varphi, \phi) + \lambda^2 (\varphi, \phi).
\end{equation}
Then using the interpolation relation for the trace of functions (see e.g. [6]), we see that there is a positive constant \( c_3 \) such that if \( \lambda \) is real and \( |\lambda| \geq c_3 \), it holds for any \( \varphi \in K(\Omega) \),
\begin{equation}
|B_i[\varphi, \phi]| \geq c_3^{-1}\|\varphi\|_{1}^2.
\end{equation}
It is easily seen that
\[ |B_i[\varphi, \psi]\| \leq c_i\|\varphi\|_1\|\psi\|_1 \] for any \( \varphi, \psi \in K(\Omega) \).

Hence by the theorem of Lax-Milgram we have the following

**Lemma 3.** For any given anti-linear functional \( l \) on \( K(\Omega) \) there is a unique solution \( u \in K(\Omega) \) of the equation
\[ B_i[u, \varphi] = l(\varphi) \] for any \( \varphi \in K(\Omega) \).

From Lemma 3 we immediately see that

**Lemma 4.** If \( \lambda \) is real and \( |\lambda| \geq c_3 \), then for any \( F \in H \) there is a unique solution \( U \in D(A) \) of the equation
\[ (\lambda I - A(t))U = F. \] (3.7)

**Proof.** Put \( U = \{u, v\} \) and \( F = \{f, g\} \). Then the equation (3.7) is equivalent to
\[ v = \lambda u - f \] (3.8)
and
\[ Lu + \lambda(\lambda + M)u = g + (\lambda + M)f. \]

Let us put in Lemma 3
\[ l(\varphi) = ((\lambda + M)f + g, \varphi) - \sum_{\partial\Omega} h, \varphi \int_{\partial\Omega} f dS. \]

Then \( l \) satisfies the assumption of Lemma 3 by the well-known inequality. Thus there is a \( u \in K(\Omega) \) such that \( B_i[u, \varphi] = l(\varphi) \) for any \( \varphi \in K(\Omega) \). In particular, taking \( \varphi \) as in \( C_0^\infty(\Omega) \), we see that (3.9) holds and \( Lu \in L^2(\Omega) \). Hence we get from (3.6), (3.8) and (3.9)
\[ \sum_{i,j} \left( a_{ij} \frac{\partial u}{\partial x_i}, \frac{\partial \varphi}{\partial x_j} \right) + \int_{\partial\Omega} u \varphi dS \]
\[ - \int_{\partial\Omega} h, \varphi \int_{\partial\Omega} f dS \]
\[ = \left( - \left\{ \sum_{i,j} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial}{\partial x_j} \right) \right\} u, \varphi \right) \text{ for any } \varphi \in K(\Omega). \]
By Definition 2.2, this equality implies that $U$ satisfies the boundary condition $(B_2)$ on $\partial_2 \Omega$. Thus $U \in D(A)$. Therefore we have completed the proof.

Let us rewrite by new $c_2$ the maximum of $c_2$ and $c_3$. Then combining Lemmas 2 and 4, we obtain

**Lemma 5.** If $|\lambda| \geq c_2$, then it holds that

$$||(\lambda I - A(t))^{-1}||_H \leq \frac{1}{|\lambda| - c_2}.$$ 

4. In this section we shall prove that $D(A)$ is dense in $H$. Let us denote by $C^\infty_{C_0}(\mathbb{R}^n)$ the set of $C^\infty$ functions on $x_n \leq 0$ having compact support there. Then we have

**Lemma 6.** For $u$ in $C^\infty_{C_0}(\mathbb{R}^n)$ there is a sequence $\{\varphi_i\}$ in $C^\infty_{C_0}(\mathbb{R}^n)$ such that

(i) $\varphi_i \to u$ in $H^1(\mathbb{R}^n)$

(ii) $\frac{\partial}{\partial x_n} \varphi_i = 0$ on $x_n = 0$

and

(iii) if $u(x', x_n)^4 = 0$ for the fixed $x'$ and any $x_n$, then each $\varphi_i(x', x_n)$ vanishes also for the $x'$ and any $x_n$.

The proof of Lemma 6 is familiar, so it is sufficient to show construction of $\varphi_i$. The functions $\varphi_i$ are given as follows;

$$\varphi_i(x', x_n) = \int_{x_n}^{x_n} \alpha_i(s) \frac{\partial u}{\partial x_n}(x', s) ds,$$

where

$$\alpha_i(s) = \begin{cases} 
0 & \text{if } s > - \frac{1}{i} \\
1 & \text{if } s < - \frac{2}{i}.
\end{cases}$$

For the bounded function $\sigma(x')$ on $x_n = 0$, let us take a new sequence

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4) Put $x' = (x_1, \ldots, x_{n-1})$. 

\{\varphi_i \exp(x_n \sigma(x'))\}. Then we have also the following

**Lemma 7.** For \(u \in C^\omega_{(1)}(\mathbb{R}^n)\) there is a sequence \(\{\varphi_i\} \subset C^\omega_{(1)}(\mathbb{R}^n)\) such that the properties (i), (iii) in Lemma 6 hold and in the place of (ii) the following property holds:

\[(\frac{\partial}{\partial x_n} + \sigma(x')) \varphi_i = 0 \text{ on } x_n = 0.

Generalizing Lemma 7, we prove the following

**Lemma 8.** For any \(u \in K(\Omega)\), there is a sequence \(\{\varphi_i\} \subset C^\omega(\Omega)\) such that \(\frac{d}{dn} + \sigma(x)\varphi_i = 0\) on \(\partial_2 \Omega\) such that each \(\varphi_i\) vanishes in a neighborhood of \(\overline{\partial_1 \Omega}\) and \(\varphi_i \rightarrow u\) in \(H^1(\Omega)\).

**Proof.** We may assume that \(u \in C^\omega(\Omega)\) and \(u = 0\) in a neighborhood of \(\overline{\partial_1 \Omega}\). For each point \(P\) in \(\Omega\) let us take an open neighborhood \(U(P)\) in such a way that

\[U(P) \subset \Omega \quad \text{for } P \in \Omega,
\]

\[u = 0 \text{ in } U(P) \quad \text{for } P \in \overline{\partial_1 \Omega}
\]

and

\[U(P) \cap \overline{\partial_2 \Omega} = \emptyset \quad \text{for } P \in \overline{\partial_2 \Omega}.
\]

Since (2.1) holds from our assumption, such a selection of \(U(P)\) is possible.

Now there is a finite point set \(\{P_1, \ldots, P_N\}\) and the union of \(U(P_k)\) \((1 \leq k \leq N)\) covers \(\Omega\). Let the function \(\alpha_k\) be in \(C^\omega_0(U(P_k))\) such that \(\sum_{k=1}^{N} \alpha_k = 1\) in \(\Omega\). We assume that the points \(P_1, \ldots, P_N(1 \leq N)\) are in \(\partial_2 \Omega\). Each subdomain \(U(P_k) \cap \Omega(1 \leq k \leq N)\) can be mapped in a one to one \(C^\omega\) way into \(y_n \leq 0^5\) such that the operator \(\frac{d}{dn}\) on \(U(P_k)\) is transformed into \(\frac{\partial}{\partial y_n}\). Applying Lemma 8 for \(\alpha_k u\) on \(y_n \leq 0\), we can find a sequence \(\{\varphi_i^{(k)}\}\) \(\subset C^\omega(\Omega)(1 \leq k \leq N')\) having the following property that

\[5) \text{We denote by } (y_1, \ldots, y_n) \text{ the new coordinate.} \]
\[ \varphi_i^{(k)} = 0 \quad \text{in a neighborhood of } \partial_1 \Omega, \]
\[(4.1) \quad \left( \frac{d}{dn} + \sigma(x) \right) \varphi_i^{(k)} = 0 \quad \text{on } \partial_2 \Omega \]
and
\[ \varphi_i^{(k)} \to \alpha_k u \quad \text{in } H^1(\Omega). \]

Setting
\[(4.2) \quad \varphi_i = \sum_{k=1}^{N'} \varphi_i^{(k)} + \sum_{k=N'+1}^{N} \alpha_k u, \]
we easily see
\[ \varphi_i \to u \quad \text{in } H(\Omega). \]

The other properties of \( \{ \varphi_i \} \) are obvious from (4.1) and (4.2). Hence we have finished the proof.

Finally we have

**Lemma 9.** The definition domain \( D(A) \) (see (3.2)) is dense in \( H \).

**Proof.** Let the vector function \( \{ u, v \} \) be in \( H(=K(\Omega) \times L^2(\Omega)) \). First we take a sequence \( \{ v_i \} \subset C_0^0(\Omega) \) converging to \( v \) in \( L^2(\Omega) \). Secondly we set \( u_i = \varphi_i \) for the sequence \( \{ \varphi_i \} \) in Lemma 8. Obviously, \( \{ u_i, v_i \} \to \{ u, v \} \) in \( H \). Since \( \left( \frac{d}{dn} + \sigma(x) \right) u_i = 0 \) on \( \partial_2 \Omega \), we see that (2.7) holds by Green's formula. Thus each \( \{ u_i, v_i \} \) satisfies the boundary condition \( (B_2) \) on \( \partial_2 \Omega \). Hence \( D(A) \) is dense in \( H \).

5. In virtue of Lemma 5 and 9, we can apply the theory of evolution equations quite similarly as in \( [3] \) and \( [5] \) as follows. Suppose that \( F(t) = \{ 0, f(t) \} \) is in \( D(A) \) and \( F(t), A F(t) \in \mathcal{E}(H)[0, T] \). Then for any given \( U_0 = \{ u_0, v_0 \} \in D(A) \), there is a unique solution \( U(t) = \{ u(t), v(t) \} \in D(A) \cap \mathcal{E}(H)[0, T] \) of the equation
\[(5.1) \quad \frac{d}{dt} U(t) = AU(t) + F(t) \quad \text{ in } 0 < t \leq T \]
with the initial condition \( U(0) = U_0 \). The equation (5.1) is equivalent to
Since \( v = u_t \) and (2.7) holds, we see that \( u \) satisfies the boundary condition (2.5) weakly on \( \partial_2 \Omega \times [0, T] \) (see Definition 2.1). Hence Theorem 1 in Section 2 has been shown.

The statement of Theorem 2 is proved quite similarly as in Theorem 1, if we add to the definition domain \( D(A) \) the condition that \( u \in H^1_{10c} (\bar{\Omega} - S) \).

Finally, we show the energy inequality for \( U(t) \in D(A) \cap C^1(H)[0, T] \). It is easily seen that from Lemma 1

\[
\frac{d}{dt} \| U(t) \|^2_H = (U'(t), U(t))_H + (U(t), U'(t))_H \\
= (AU(t) + F(t), U(t))_H + (U(t), AU(t) + F(t))_H \\
\leq 2c_2 \| U(t) \|^2_H + 2 \| U(t) \|_H \| F(t) \|_H.
\]

From this it follows

\[
\| U(t) \|_H \leq e^{c_2t} \left( \| U(0) \|_H + \int_0^t \| F(s) \|_H \, ds \right).
\]

Hence

\[
\| u(t) \|_1 + \| u'(t) \|_0 \leq Ce^{c_2t} \left( \| u(0) \|_1 + \| v(0) \|_0 + \int_0^t \| f(s) \|_0 \, ds \right).
\]

**References**


