On Quasifree States of CAR and Bogoliubov Automorphisms

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Abstract
A necessary and sufficient condition for two quasifree states of CAR to be quasiequivalent is obtained. Quasifree states is characterized as the unique KMS state of a Bogoliubov automorphism of CAR. The structure of the group of all inner Bogoliubov automorphisms of CAR is clarified.

§1. Introduction
A classification of gauge invariant quasifree states of the canonical anticommutation relations (CAR) up to quasi and unitary equivalence is recently obtained by Powers and Størmer [12]. We shall generalize their result to arbitrary quasifree states.

We use the formalism developed earlier [2] and study quasifree state \( \psi \) of a selfdual CAR algebra. It is then shown that \( \psi_1 \) and \( \psi_2 \) are quasiequivalent if and only if \( S_1^{1/2} - S_2^{1/2} \) is in the Hilbert Schmidt class. For a gauge invariant quasifree state \( \varphi_A \) in the paper of Powers and Størmer, \( S = A \otimes (1 - A) \) and hence our result is a direct generalization of Powers and Størmer.

The quasifree primary state \( \psi_s \) for which \( S \) does not have eigenvalue 1 is shown to be the unique KMS state for the one parameter group \( \tau(U(\lambda)) \) of Bogoliubov * automorphisms of CAR, where \( \tau(U(\lambda)) \) corresponds to a unitary transformation \( U(\lambda) = \exp i\lambda H \) of the direct sum of testing function spaces of creation and annihilation operators and \( H \) is related to \( S \) by \( S = (1 + e^{-H})^{-1} \). This is used to simplify some of arguments. A quasifree state \( \psi_S \) is primary unless \( 1/2 \) is an isolated point spectrum of \( S \), has an odd multiplicity and \( S(1 - S) \) is in the

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Hilbert Schmidt class.

It is shown that a Bogoliubov automorphism $\tau(V)$ is inner if and only if $V-1$ is in the trace class and $\det V > 0$ or $V+1$ is in the trace class and $\det V < 0$. It is a $*$ automorphism if and only if $V$ is unitary. A double valued representation of the identity component (i.e. $\det V > 0$) of the group of inner Bogoliubov automorphisms of a CAR algebra by elements of CAR algebra (such that it implements the automorphism) is obtained with a help of bilinear hamiltonians. It is a generalization of the observable algebra introduced by Araki and Wyss [4].

A necessary and sufficient condition for the unitary implementability of a Bogoliubov transformation in a Fock representation is obtained.

In an appendix, a general structure of two projections is presented and an angle operator is introduced. Some of the discussions in the main text can be carried out by introducing a specific basis, although we have avoided this in the present paper. For such a purpose, this general analysis of two projections is useful.

The CAR algebra has been extensively studied by many authors ([4—7, 10, 12—17]) and some of our results such as Theorem 6 and 7 are in these earlier references.

§2. Basic Notations

We quote a few notions concerning a self dual CAR algebra from an earlier paper [2].

Let $K$ be a complex Hilbert space and $H$ be an antiunitary involution (a complex conjugation, $H^2=1$, $\langle Hf, Hg \rangle = \langle g, f \rangle$) on $K$. A self dual CAR algebra $\mathcal{A}_\text{SDC}(K, H)$ over $(K, H)$ is a $*$ algebra generated by $B(f)$, $f \in K$, its conjugate $B(f)^*$, $f \in K$ and an identity which satisfy the following relations: (1) $B(f)$ is (complex) linear in $f$, (2) $B(f)B(g)^* + B(g)^*B(f) = \langle g, f \rangle 1$, and (3) $B(f)^* = B(Hf)$.

If $K$ has a finite dimension, $\mathcal{A}_\text{SDC}(K, H)$ has a finite dimension. Irrespective of the dimension of $K$, $\mathcal{A}_\text{SDC}(K, H)$ has a unique $C^*$ norm and $\overline{\mathcal{A}_\text{SDC}(K, H)}$ denotes its $C^*$ completion.

Any unitary operator $U$ on $K$ commuting with $H$ preserves the
above relations (1)~(3) and hence defines a * automorphism \( \tau(U) \) of \( \mathfrak{S}_{\text{SDC}}(K, r') \) by \( \tau(U)B(f) = B(Uf) \). \( U \) and \( \tau(U) \) shall be called a Bogoliubov transformation and a Bogoliubov * automorphism.

The antilinear transformation

\[
\tau'(r') = \sum_{n=1}^{N} c_n B(f_i^{(o)}) \cdots B(f_i^{(o)}) = \sum_{n=1}^{N} c_n^* B(r f_i^{(o)}) \cdots B(r f_i^{(o)})
\]

also leaves relations (1)~(3) invariant and hence can be extended to a conjugate * automorphism (i.e. antilinear * isomorphism onto itself) which will be denoted by \( \tau'(r') \).

Any projection operator \( P \) on \( K \) satisfying \( r' P r' = 1 - P \) is called a basis projection. There exists a basis projection \( P \) if and only if the dimension of \( K \) is even or infinite. Any two basis projections \( P_1 \) and \( P_2 \) can be transformed to each other by a Bogoliubov transformation \( U : P = UP_1 U^* \).

Any projection \( P \) on \( K \) such that \( P \perp r' P r' \) is called a partial basis projection. \( \dim(1 - P - r' P r') \) is called the \( r' \) codimension of \( P \).

By identifying \( B(f) \) and \( B(r f) \), \( f \in PK \), with creation and annihilation operators on a CAR algebra \( \mathfrak{S}_{\text{CAR}}(K) \) over \( K = PK \), we have a * isomorphism of \( \mathfrak{S}_{\text{SDC}}(K, r') \) with \( \mathfrak{S}_{\text{CAR}}(K) \), where \( P \) is any basis projection.

Here \( \mathfrak{S}_{\text{CAR}}(K) \) is the * algebra generated by creation operators \( (a^f, f) \), \( f \in K \), their conjugates \( (a^f)^* = (f, a) \) (annihilation operators) and an identity, satisfying the following relations: (1) \( (a^f) \) is (complex) linear in \( f \), (2) \( (a^f)(g) + (a^g)(f) = (f, a)(g, a) + (g, a)(f, a) = 0 \), (3) \( (a^f)(g)(a) + (g, a)(a^f) = (g, f)1 \). \( \mathfrak{S}_{\text{CAR}}(K) \) is the completion of \( \mathfrak{S}_{\text{CAR}}(K) \) with respect to its unique \( C^* \) norm.

(A more precise notation will be something like \( B_{\alpha, \gamma}(f) \), \( (a^f_k, f) \) and \( (f, a^f_k) \), which is useful whenever elements of more than one algebras with different \( K, r, \) and \( K_i \) appear at the same time. We shall meet in later sections a case where elements of \( \mathfrak{S}_{\text{SDC}}(K, r') \) and \( \mathfrak{S}_{\text{SDC}}(K, \hat{r}) \), \( \hat{K} = K \oplus K \), \( \hat{r} = r \oplus (-r) \), appear at the same time. In this case, \( B_{\alpha, \gamma}(f), f \in K \) is identified with \( B_{\alpha, \gamma}(f \oplus 0) \) and will be denoted simply as \( B(f) \).)
§3. Quasiequivalence of Quasifree States

Definition 3.1. A state $\varphi$ on $\mathcal{H}_{SDC}(K, \Gamma)$ satisfying the following relation is called a quasifree state:

$$\varphi(B(f_1) \cdots B(f_{2n+1})) = 0,$$

$$\varphi(B(f_1) \cdots B(f_{2n})) = (-1)^{n(n-1)/2} \sum_{s} \varepsilon(s) \prod_{j=1}^{n} \varphi(B(f_{s(j)}B(f_{s(j+n)})),$$

where $n=1, 2, \cdots$, the sum is over all permutations $s$ satisfying

$$s(1) < s(2) < \cdots < s(n),$$

$$s(j) < s(j+n), \quad j = 1, \cdots, n,$$

and $\varepsilon(s)$ is the signature of $s$.

Lemma 3.2. For any state $\varphi$ over $\mathcal{H}_{SDC}(K, \Gamma)$, there exists a bounded operator $S$ on $K$ satisfying

$$\varphi(B(f)^*B(g)) = (f, Sg),$$

$$1 \geq S^* = S \geq 0,$$

$$S + \Gamma S\Gamma = 1.$$

Proof. We have

$$B(f)^*B(f) \leq B(f)^*B(f) + B(f)B(f)^* = \|f\|^2,$$

$$\|B(f)\| = \|B(f)^*B(f)\|^{1/2} \leq \|f\|.$$

Hence (3.3) defines a bounded linear operator $S$ on $K$.

From the positivity of $\varphi$, it follows that $S^* = S \geq 0$. From the anticommutation relations, we have

$$\varphi(B(f)^*B(g)) = (f, g) - \varphi(B(g)B(f)^*)$$

$$= (f, g) - \varphi(B(\Gamma g)^*B(\Gamma f))$$

$$= (f, g) - (\Gamma g, S\Gamma f).$$

Since

$$\Gamma f = (\Gamma(\Gamma h), \Gamma f) = (f, \Gamma h),$$

we have $(\Gamma^* g, S\Gamma f) = (S^* g, \Gamma f) = (f, \Gamma S^* g)$. Hence (3.5) follows.

From $S \geq 0$ and $1 - S = \Gamma S \Gamma$, it follows that $1 - S \geq 0$. Q.E.D.
Lemma 3.3. For any bounded linear operator $S$ satisfying (3.4) and (3.5), there exists a unique quasifree state $\varphi$ satisfying (3.3).

The uniqueness is immediate from (3.1) and (3.2). The existence follows from Lemma 4.6.

Definition 3.4. The unique quasifree state of Lemma 3.3 is denoted $\varphi_s$.

From Lemmas 3.2 and 3.3, $\varphi_s$ exhausts all quasifree states of $\mathcal{F}_{S}(K, \Gamma)$.

Theorem 1. Two quasifree states $\varphi_s$ and $\varphi_{s'}$ give rise to mutually quasiequivalent representations of $\mathcal{F}_{S}(K, \Gamma)$ if and only if $S^{1/2} - (S')^{1/2}$ is in the Hilbert Schmidt class.

The proof will be presented in section 5.

§4. Fock Representation Induced by Quasifree States

Definition 4.1. $\mathcal{F}_s$, $\pi_s$, and $\Omega_s$ denote the Hilbert space, the representation and the cyclic unit vector canonically associated with the quasifree state $\varphi_s$ through the relation

$$\varphi_s(A) = (\Omega_s, \pi_s(A)\Omega_s), \quad A \in \mathcal{F}_{S}(K, \Gamma).$$

Lemma 4.2. Let $\varphi_s$ be a quasifree state. If a Bogoliubov transformation $U$ commutes with $S$, then there exists a unitary operator $T_s(U)$ on $\mathcal{F}_s$ such that

$$T_s(U)\Omega_s = \Omega_s$$

and

$$T_s(U)\pi_s(A)T_s(U)^* = \pi_s(\tau(U)A)$$

for all $A \in \mathcal{F}_{S}(K, \Gamma)$.

Proof. If $[U, S] = 0$, then $\varphi_s(\tau(U)A) = \varphi_s(A)$. Hence

$$T_s(U)\sum c_i \pi_s(A_i)\Omega_s = \sum c_i \pi_s(\tau(U)A_i)\Omega_s,$$

and

$$T_s(U^*)\sum c_i \pi_s(A_i)\Omega_s = \sum c_i \pi_s(\tau(U^*)A_i)\Omega_s.$$
define isometric linear mappings from a dense subset of $\mathcal{D}_s$ into $\mathcal{D}_s$ satisfying

$$T_s(U)T_s(U^*) = T_s(U^*)T_s(U) \subset 1,$$

$$T_s(U) \subset T_s(U^*)^*.$$  

Therefore, the closure of this $T_s(U)$ is unitary and satisfies (4.1) and (4.2).

Note that $T_s(-1)$ is defined for all $S$.

**Lemma 4.3.** Let $P$ be a basis projection. If a state $\varphi$ of $\mathfrak{A}_{SDC}(K, r')$ satisfies

(4.3) $\langle B(f)B(f)^* \rangle = 0, \quad f \in PK$,

then $\varphi = \varphi_P$. The representation $\pi_P$ is irreducible.

**Proof.** By splitting every $B(f)$ as $B(Pf) + B(Prf)^*$ and using commutation relations to bring $B(Pf)$ to the left and $B(Prf)^*$ to the right, any element $A$ in $\mathfrak{A}_{SDC}(K, r')$ can be written as

$$A = \sum_i \mathcal{P}_i B(f_i)^* + \sum_j B(g_j) \mathcal{P}_j' + \lambda$$

where $f_i, g_j \in PK$, $\mathcal{P}_i$ and $\mathcal{P}_j'$ are polynomials. The condition (4.3) implies $\varphi(A) = \lambda$ and hence state $\varphi$ satisfying (4.3) is unique.

From (3.3), $\varphi_P$ satisfies (4.3).

The condition (4.3) may be stated as $\varphi(A^*A) = 0$ whenever $A$ belongs to the closed left ideal $\mathcal{L}$ generated by $B(f)^*$, $f \in PK$. The uniqueness of such state implies that $\mathcal{L}$ is maximal and the unique state $\varphi$ is pure [9]. Q.E.D.

The state $\varphi_P$ is called a Fock state and $\pi_P$ is called a Fock representation. Under the identification of $\mathfrak{A}_{SDC}(K, r')$ with $\mathfrak{A}_{CAR}(PK)$, this coincides with an ordinary definition of the Fock vacuum of CAR and the existence of such state $\varphi_P$ is known. A different choice of the basis projection $P$ produces a different identification $\alpha_P$ of the selfdual CAR algebra with a CAR algebra and correspondingly different Fock state $\varphi_P$. All of them are mutually related by Bogoliubov automorphisms.
**Definition 4.4.** Let $S$ be an operator on $K$. Then $P_s$ denotes the operator on $K \oplus K$ given by a matrix

$$P_s = \begin{pmatrix} S & S^{1/2}(1-S)^{1/2} \\ S^{1/2}(1-S)^{1/2} & 1-S \end{pmatrix}. \tag{4.4}$$

**Lemma 4.5.** If $S$ satisfies (3.4) and (3.5), then $P_s$ is a basis projection on $(\hat{K}, \hat{\Gamma})$ where $\hat{K}=K \oplus K$, $\hat{\Gamma}=\Gamma \oplus (-\Gamma)$.

**Proof.** A direct computation shows $P_s^2=P_s=P_s^*$, $I-P_sI=1-P_s$.

**Lemma 4.6.** Let $S$, $P_s$, $\hat{K}$, $\hat{\Gamma}$ be as in Lemma 4.5. Then the restriction of the Fock state $\varphi_{P_s}$ of $\mathcal{H}_{SDC}(\hat{K}, \hat{\Gamma})$ to $\mathcal{H}_{SDC}(K, \Gamma)$ is the quasifree state $\varphi_s$.

Since $\varphi_{P_s}$ is quasifree, its restriction is also quasifree. $\varphi_{P_s}(B(f)^*)B(g)=(f, P_s g)=(\hat{f}, S\hat{g})$ if $f=\hat{f} \oplus 0$, $g=\hat{g} \oplus 0$. Q.E.D.

**Lemma 4.7.** Let $P$ be a basis projection and

$$\pi_P(B(f))=\pi_P[B([2P-1]f)]T_P(-1), \quad f \in K. \tag{4.5}$$

Then there exists a representation $\pi_P$ of $\mathcal{H}_{SDC}(K, \Gamma)$ on $\mathcal{D}_P$ which is uniquely determined by (4.5). $\mathcal{D}_P$ is cyclic for $\pi_P$ and the corresponding vector state is $\varphi_P$.

**Proof.** It follows from (4.5) that $\pi_P(B(f))$ satisfies relations (1), (2) and (3) for $\mathcal{H}_{SDC}(K, \Gamma)$ in section 2. Hence the existence of a representation $\pi_P$ of $\mathcal{H}_{SDC}(K, \Gamma)$ satisfying (4.5) follows. Since $B(f)$ generates $\mathcal{H}_{SDC}(K, \Gamma)$, $\pi_P$ is unique. By applying $\pi_P(B(f_i))$, $i=1, \cdots, n$ successively on $\mathcal{D}_P$, one can reproduce $\pi_P(B(f_1)) \cdots \pi_P(B(f_n))\mathcal{D}_P$ up to $\pm$ sign and hence $\mathcal{D}_P$ is cyclic. From the same computation, it is seen that the vector state given by $\mathcal{D}_P$ is $\varphi_P$.

**Lemma 4.8.** Let $K_0$ be a $\Gamma$ invariant subset of $K$, $E(K_0)$ be the projection operator for the smallest closed subspace of $K$ containing $K_0$ and

$$R_P(K_0) = \{\pi_P(B(f)); f \in K_0\}''.$$ \tag{4.6}

The following conditions are equivalent.
(1) \( \Omega_\rho \) is cyclic for \( R_\rho(K_0) \),

(2) \( (1 - E(K_0)) \wedge (1 - P) = 0 \),

(3) \( (1 - E(K_0)) \wedge P = 0 \).

Here \( P \wedge P' \) denotes the projection for \( PK \cap P'K \). The following conditions are also equivalent.

(1)' \( \Omega_\rho \) is separating for \( R_\rho(K_0) \),

(2)' \( E(K_0) \wedge (1 - P) = 0 \),

(3)' \( E(K_0) \wedge P = 0 \).

**Proof.** (3) \( \Rightarrow \) (1): As is known, \( \xi_\rho \) is a direct sum of subspaces \( \xi_i^{(n)} \), \( n = 0, 1, \ldots \), such that the set of vectors \( \prod \{ B(f_i) \} \Omega_\rho \), \( f_i \in PK \), is total in \( \xi_i^{(n)} \). (3) implies that \( Pf, f \in K_0 \) is total in \( PK \). [If \( (g, Pf) = 0 \) for all \( f \in K_0 \) and \( g \in PK \), then \( (g, f) = (g, Pf) = 0 \) and hence \( g \in K_0 \cap PK = \{0\} \).] Assume that \( \xi_i^{(n)} \subset R_\rho(K_0) \Omega_\rho \) for \( k < n \). (This is true for \( n = 1 \).) Then

\[
\pi_\rho [B(Pf)] \xi_i^{(n)} \subset \pi_\rho [B(f)] \xi_i^{(n)} - \pi_\rho [B((1 - P)f)] \xi_i^{(n)}.
\]

Therefore, \( B(Pf) \xi_i^{(n)} \) and hence \( \xi_i^{(n+1)} \) are in \( \overline{R_\rho(K_0) \Omega_\rho} \).

(1) \( \Rightarrow \) (2): Suppose that (2) does not hold and \( (1 - E(K_0))g = (1 - P)g = g \neq 0 \). Then \( \pi_\rho [B(g)] \) anticommutes with all \( \pi_\rho [B(f)] \), \( f \in K_0 \) and hence \( \pi_\rho [B(g)] R_\rho(K_0) \Omega_\rho = 0 \). Therefore \( \pi_\rho [B(g)] \) is orthogonal to \( R_\rho(K_0) \Omega_\rho \) and (1) is false.

(2) \( \Rightarrow \) (3): Immediate from \( \Gamma E(K_0) \Gamma = E(K_0) \) and \( \Gamma (1 - P) \Gamma = P \).

To prove the rest, let

(4.7) \( R_\rho(K_0) = \{ \pi_\rho[B(f)] \}; f \in K_0 \} \).''

Then

(4.8) \( R_\rho((2P - 1)K_0) \subset R_\rho(K_0) \).

(3)' \( \Rightarrow \) (1)': We have \( E(K_0) = 1 - E(K_0) \). Hence (3)' implies that \( \Omega_\rho \) is cyclic for \( R_\rho(K_0) \) by (3) \( \Rightarrow \) (1) and so for \( R_\rho((2P - 1)K_0) \). Due to (4.8), this implies (1)'.

(1)' \( \Rightarrow \) (2)': Assume that (2)' does not hold and \( E(K_0)g = (1 - P)g \)
Then $Q=P(B(g))$ is in $R_P(K)$ and $Q\subseteq P=0$. Hence (1)' is false.

(2)'→(3)': Same as (2)→(3).

Q.E.D.

Remark 4.9. It is known that an equality holds in (4.8).

Corollary 4.10. Let

(4.9) $R=\pi_{R_s}(\hat{\mathcal{H}}_{SC}(K,K))$.

Then the following conditions are equivalent.

(1) $\omega_{R_s}$ is cyclic for $R_s$.
(2) $\omega_{R_s}$ is separating for $R_s$.
(3) $S$ does not have an eigenvalue 1.
(4) $S$ does not have an eigenvalue 0.

Proof. Let $Q$ be the projection $1+0$ acting on $\hat{K}=K\oplus K$. Then, by Lemma 4.8, (1) is equivalent to $0=(1-Q)\cap P_s=$ the eigenprojection of $(1-Q)P_s(1-Q)=0\oplus(1-S)$ for an eigenvalue 1 and hence is equivalent to (4). Similarly (2) is equivalent to $0=Q\cap P_s=$ the eigenprojection of $QP_sQ=S\oplus 0$ for an eigenvalue 1 and hence is equivalent to (3). Since $rSr=1-S$, (3)⇔(4). Q.E.D.

If any of the conditions (1)~(4) is satisfied, we can identify $\mathcal{D}_s$ and $\mathcal{D}_s$ with $\mathcal{D}_{R_s}$ and $\mathcal{D}_{R_s}$. In general, $\mathcal{D}_s$ is identified with a subspace of $\mathcal{D}_{R_s}$.

Lemma 4.11. If $1/2$ is an eigenvalue of $S$ with an even or 0 or infinite multiplicity, then $R_s$ is a factor.

Proof. First we consider the case where $S$ does not have an eigenvalue $1/2$ (i.e. its multiplicity is 0). We show that $R=(R_s\cup R_s)''$ is irreducible. For this purpose, it is enough to show that $\omega_{R_s}$ is cyclic for $R$ and that there exists a subset $\mathfrak{Q}\subset R$ such that $Q\omega_\mathfrak{Q}=0$ for all $Q\in\mathfrak{Q}$ is equivalent to $\mathfrak{Q}=c\omega_{R_s}$ for some complex number $c$.

Vectors $\pi_{R_s}(\prod_{i=1}^n B(f_i\otimes g_i))\omega_{R_s}$ are total in $\mathcal{D}_{R_s}$. Since $\pi_{R_s}(\prod_{i=1}^n (B(f_i\otimes 0))) \in R_s$ and $\pi_{R_s}(\prod_{i=1}^n B(0\otimes g_i))T_{R_s}(-1)\in R_s$, $\omega_{R_s}$ is cyclic for $R$.

We now take the set of all
A(f) = \pi^p_s(B[(1-S)^{1/2}f \oplus 0]) - \pi^p_s(B[0 \oplus S^{1/2}f])T_r^p_s(-1)

to be \mathcal{O}. The first term is in \mathcal{R}_s and the second term is in \mathcal{R}'_s by (4.8). \ A(f)\Omega_{r^p_s} = 0. \ We shall show that \ A(f)\psi = 0 \ for \ all \ f \in K \ implies \ \psi = c\Omega_{r^p_s}.

Let
\[
\psi_{\pm} = [1 \pm T_{r^p_s}(-1)] \psi.
\]
Since \ T_{r^p_s}(-1) \ anticommutes \ with \ A(f), \ we \ have \ A(f)\psi_{\pm} = 0.

On \ \psi_+, \ T_{r^p_s}(-1) = 1 \ and \ hence \ A(f)\psi_+ = \pi^p_s(B(f'))\psi_+, \ \ f' = (1-S)^{1/2}f \oplus (-S^{1/2}f). \ Obviously \ P_{r^p_s}f' = 0. \ Since \ [(1-S)^{1/2}f]' + \hat{r}[rS^{1/2}f]' = f' \oplus 0 \ and \ -[S^{1/2}f]' + \hat{r}[r(1-S)^{1/2}f]' = 0 \oplus f, \ the \ set \ \{f'; f \in K\} \ coincides \ with \ (1-P_{r^p_s})\hat{K}. \ Therefore \ \psi_+ = c\Omega_{r^p_s} \ by \ Lemma \ 4.3.

On \ \psi_-, \ A(f)\psi_- = \pi^p_s(B(f''))\psi_-, \ \ f'' = (1-S)^{1/2}f \oplus S^{1/2}f. \ A(f)\psi_- = 0 \ for \ all \ f \in K \ implies \ that \ the \ vector \ state \ of \ \psi_-^{ \text{SSDC}(\hat{K}, \hat{r})} \ induced \ by \ \|\psi_-\|^{-1} \ \psi_- \ is \ a \ Fock \ state \ for \ the \ basis \ projection \ \mathcal{P}'_s = 2(S \oplus (1-S)) - \mathcal{P}_s \ provided \ that \ \psi_- \neq 0. \ Here \ (\psi_-, \Omega_{r^p_s}) = 0 \ while, \ from \ the \ equation \ (9.27) \ and \ Theorem \ 6, \ (\psi_-, \Omega_{r^p_s}) \ can \ vanish \ only \ when \ \mathcal{P}'_s(1-P_{r^p_s})\mathcal{P}'_s \ has \ an \ eigenvalue \ 1. \ From \ \mathcal{P}'_sf = f \ and \ \mathcal{P}_sf = 0 \ for \ f = f_1 \oplus f_2, \ we \ have \ (Sf_1) = (1/2)f_1, \ Sf_2 = (1/2)f_2. \ Hence, \ if \ S \ does \ not \ have \ an \ eigenvalue \ 1/2, \ then \ \psi_- = 0.

We now consider the general case where the eigenvalue 1/2 of \ S \ has a nonvanishing multiplicity. \ We shall reduce it to the previous case by Lemma 5.3. \ Let \ E_{1/2} \ be the eigenprojection of \ S \ for an eigenvalue \ 1/2. \ By Lemma 3.3 of \ [2], \ there \ exists \ a \ subprojection \ \mathcal{E} \ of \ E_{1/2} \ such \ that \ E + r\mathcal{E} - E = E_{1/2}. \ Let \ \mathcal{T} \ be \ a \ Hilbert \ Schmidt \ class \ operator \ such \ that \ 0 \leq T \leq 2^{-1} \ and \ (1-E) \ is \ the \ eigenprojection \ of \ \mathcal{T} \ for \ an \ eigenvalue \ 0. \ Let
\[
\hat{S} = S - T + r\mathcal{T}r.
\]
Then \ \hat{S} = \hat{S}^*, \ r\hat{S}r = 1 - \hat{S}, \ \hat{S} \ does \ not \ have \ an \ eigenvalue \ 1/2 \ and
\[
\hat{S}^{1/2} - S^{1/2} = [(1/2 - T)^{1/2} - (1/2)^{1/2}] + r[(1/2 + T)^{1/2} - (1/2)^{1/2}]r.
\]
Since \ (1/2 \pm T)^{1/2} - (1/2)^{1/2} = \pm [(1/2 \pm T)^{1/2} + (1/2)^{1/2}]^{-1}T \ is \ in \ the
Hilbert Schmidt class, $R_S$ and $R_\pi$ are quasiequivalent by Lemma 5.3. We already know that $R_\pi$ is a factor. Therefore $R_S$ is a factor.

Q.E.D.

A full characterization of the case where $R_S$ becomes a factor is given in Theorem 9.

Remark 4.12. From the beginning part of the preceding proof, it follows that $\omega_{R_S}$ is cyclic for $(R_S \cup R_\pi)''$ for any $S$ and hence is separating for the center of $R_S$.

§5. Proof of Theorem 1

The following is Lemma 4.5 of [12], for which we give a different proof.

Lemma 5.1. $S^{1/2} - (S')^{1/2}$ is in the Hilbert Schmidt class if and only if $P_S - P_{S'}$ is in the Hilbert Schmidt class.

Proof. Let $\rho = S^{1/2}$, $\rho' = (S')^{1/2}$. If $\rho - \rho'$ is HS (a Hilbert Schmidt class operator), then all of

\begin{align*}
(5.1) & \quad S - S' = \{\langle \rho - \rho' | \rho + \rho' \rangle + \langle \rho + \rho' | \rho - \rho' \rangle\}/2, \\
(5.2) & \quad (1 - S)^{1/2} - (1 - S') = i(\rho - \rho')i', \\
(5.3) & \quad \rho(1 - S)^{1/2} - \rho'(1 - S')^{1/2} \\
& \qquad = (\rho - \rho') (1 - S)^{1/2} + \rho'((1 - S)^{1/2} - (1 - S')^{1/2}),
\end{align*}

are HS. Hence $P_S - P_{S'}$ is HS.

Conversely, assume $P_S - P_{S'}$ is HS. Then, by Lemma 5.2,

\begin{align*}
(5.4) \quad \|[P_S - Q'] - [P_{S'} - Q']\|_{\text{H.S.}} \leq \|P_S - P_{S'}\|_{\text{H.S.}},
\end{align*}

where $Q' = 0 \oplus 1$ on $K \oplus K$. Since $|P_S - Q'| = S \oplus S$, $|P_{S'} - Q'| = S' \oplus S'$, we have $S^{1/2} - (S')^{1/2}$ in the HS class. Q.E.D.

Lemma 5.2. Let $A$ and $B$ be bounded selfadjoint operators, then

\begin{align*}
(5.5) \quad \|A - B\|_{\text{H.S.}} \geq \|A - |B|\|_{\text{H.S.}}.
\end{align*}

Proof. (5.5) is equivalent to

\begin{align*}
(5.6) \quad \text{tr}\{A^2 + B^2 - AB - BA\} \geq \text{tr}\{A^2 + B^2 - |A| |B| - |B| |A|\}.
\end{align*}
First consider the case, where $A$ has purely discrete spectrum. Let $\psi_\alpha$ be a complete orthonormal set of eigenvectors of $A$ with eigenvalues $\lambda_\alpha$. Then

$$\text{tr} \{ A^2 + B^2 - AB - BA \} = \sum_\alpha \{ \lambda_\alpha^2 + (\psi_\alpha, B^2 \psi_\alpha) - 2\lambda_\alpha (\psi_\alpha, B \psi_\alpha) \},$$

(5.7)  

$$\text{tr} \{ A^2 + B^2 - |A||B| - |B||A| \} = \sum_\alpha \{ \lambda_\alpha^2 + (\psi_\alpha, B^2 \psi_\alpha) - 2\lambda_\alpha (\psi_\alpha, |B| \psi_\alpha) \}. $$

(5.8)  

Since $|B| \geq B \geq -|B|$, $(\psi_\alpha, |B| \psi_\alpha) \geq (\psi_\alpha, B \psi_\alpha)$. Therefore we have (5.6), where $\pm \infty$ is allowed.

For any selfadjoint operator $A$ and $\varepsilon > 0$, there exists a selfadjoint operator $A_\varepsilon$ with purely discrete spectrum such that $\| A - A_\varepsilon \|_{\text{H.S.}} < \varepsilon$. Hence [5]. From (5.5), we have $\| A - |A_\varepsilon| \|_{\text{H.S.}} \leq \| A - A_\varepsilon \|_{\text{H.S.}} < \varepsilon$.

(5.9)  

$$\| A - B \|_{\text{H.S.}} \geq \| A_\varepsilon - B \|_{\text{H.S.}} - \varepsilon$$

$$\geq \| |A_\varepsilon| - |B| \|_{\text{H.S.}} - \varepsilon$$

$$\geq \| A - |B| \|_{\text{H.S.}} - 2\varepsilon.$$  

Since $\varepsilon$ is arbitrary, we have (5.5) for general $A$ and $B$. Q.E.D.

**Lemma 5.3.** If $S^{1/2} - (S')^{1/2}$ is in the Hilbert Schmidt class, then $\varphi_s$ and $\varphi_{s'}$ are quasiequivalent. If $S$ and $S'$ satisfy any of conditions (1) ~ (4) of Corollary 4.10, in addition, then $\pi_s$ and $\pi_{s'}$ are unitarily equivalent.

**Proof.** If $S^{1/2} - (S')^{1/2}$ is HS, then $P_s - P_{s'}$ is HS. Hence by the first half of Theorem 6 (essentially Lemma 9.4), there exists a vector $\xi'$ in $\mathfrak{D}_{P_{s'}}$ such that the vector state of $\xi'$ on $\mathfrak{H}(K, r')$ is $\varphi_{P_{s'}}$, where $\hat{K} = K \oplus K$, $\hat{r'} = r' \oplus (-r')$. Hence $\varphi_s$ and $\varphi_{s'}$ are given as vector states of $\pi_{P_{s'}}(\mathfrak{H}(K, r'))$ by $\Omega_{P_{s'}}$ and $\xi'$ which are separating for the center of $\pi_{P_{s'}}(\mathfrak{H}(K, r'))''$ due to Remark 4.12. Therefore $\varphi_s$ and $\varphi_{s'}$ are quasiequivalent. If both $S$ and $S'$ satisfy conditions in Corollary 4.10, then $\mathfrak{D}_s$ and $\mathfrak{D}_{s'}$ can be both identified with $\mathfrak{D}_{P_{s}}$ and hence $\pi_s$ and $\pi_{s'}$ are unitarily equivalent.

**Lemma 5.4.** Let $A_\varepsilon$ be a sequence of bounded linear operators on a Hilbert space with a strong limit $A$. Then
(5.10) \[ \|A\|_{H.S.} \leq \lim_n \|A_n\|_{H.S.} \]

(Here \( \|C\|_{H.S.} = \left(\sum\|C\psi_i\|^2\right)^{1/2} \) for a complete orthonormal basis \( \{\psi_i\} \) and we allow \(+\infty\). It is independent of the basis.)

**Proof.** We have

\[ \lim_{n} \|A_{n}\|_{H.S.} = \lim_{n} \sum_{i=1}^{N} \|A_{n}\psi_{i}\|^{2} = \sum_{i=1}^{N} \|A\psi_{i}\|^{2}. \]

Since \( N \) is arbitrary, we obtain (5.10). Q.E.D.

**Lemma 5.5.** If \( S \) and \( S' \) satisfy any of conditions (1) – (4) of Corollary 4.10, and if \( P_{S} \rightarrow P_{S'} \) is not in the Hilbert Schmidt class, then \( \pi_{S} \) and \( \pi_{S'} \) are not quasiequivalent.

**Proof.** Let \( Q_{*} \) be an increasing sequence of finite even dimensional projections commuting with \( r' \) and tending to 1 on \( K \). From Lemma 5.4, we have

\[ \lim_{n \rightarrow \infty} \|Q_{*}S_{*}Q_{n}S_{*}Q_{n}\|_{H.S.} = \infty. \]

From (5.4), we have

\[ \lim_{n \rightarrow \infty} \|P_{Q_{*}S_{*}Q_{n}} - P_{Q_{*}S_{*}Q_{n}}\|_{H.S.} = \infty. \]

From Lemma 6.6, we obtain

\[ \lim_{n \rightarrow \infty} \|Q_{*}S_{*}Q_{n}S_{*}Q_{n}\|_{H.S.} = \infty. \]

Therefore, we have

(5.11) \[ \|\pi_{S} - \pi_{S'}\| = 2. \]

Since \( S \) and \( S' \) both satisfy the condition of Corollary 4.10, the representations \( \pi_{S} \) and \( \pi_{S'} \) have cyclic and separating vectors \( \varrho_{S} \) and \( \varrho_{S'} \). If \( \pi_{S} \) and \( \pi_{S'} \) are quasiequivalent, then they are unitarily equivalent. Therefore there exists a separating vector \( \varrho' \) in \( \varrho_{S} \) such that \( (\varrho', \pi_{S}(A)\varrho') = \varrho_{S'}(A) \). Since \( \varrho' \) is cyclic for the commutant, there exists a unitary operator \( W \) in \( \pi_{S}(\mathcal{M}(K, r'))' \) such that \( (W\varrho', \varrho_{S}) \neq 0 \). Then the vector state for \( \varrho''' = W\varrho' \) is again \( \varrho_{S'} \) and we have

(5.12) \[ \|\varrho_{S} - \varrho_{S'}\| \]

\[ \leq \text{tr} |P(\varrho_{S}) - P(\varrho')| = 2 \{1 - \|\varrho_{S}, \varrho''\|^2\}^{1/2} < 2, \]
where \( P(\varphi) \) denote the projection operator on the one dimensional space spanned by \( \varphi \). The contradiction of (5.11) and (5.12) proves the Lemma. \( \text{Q.E.D.} \)

**Proof of Theorem 1.** If \( S^{1/2} - (S')^{1/2} \) is in the Hilbert Schmidt class, then \( \varphi_S \) and \( \varphi_{S'} \) are quasiequivalent by Lemma 5.3.

Now assume that \( S^{1/2} - (S')^{1/2} \) is not in the Hilbert Schmidt class. Let \( E_1 \) and \( E_1' \) be eigenprojections of \( S \) and \( S' \) for an eigenvalue 1. Let \( T \) and \( T' \) be Hilbert Schmidt class operators such that \( 0 \leq T \leq 1 \), \( 0 \leq T' < 1 \) and the eigenprojection of \( T \) and \( T' \) for an eigenvalue 0 are \( 1 - E_1 \) and \( 1 - E_1' \). Let

\[
\hat{S} = S - T^2 + rT^2T \quad \hat{S'} = S' - (T')^2 + r(T')^2T.
\]

Then \( \hat{S} \) and \( \hat{S'} \) have the properties (3.4) and (3.5) and satisfy the condition (3) of Corollary 4.10. Further,

\[
\hat{S}^{1/2} - S^{1/2} = \Gamma T \Gamma - [(1 - T^2)^{1/2} + 1]^{-1} T^2
\]

\[
(\hat{S}')^{1/2} - (S')^{1/2} = \Gamma T' \Gamma - [(1 - (T')^2)^{1/2} + 1]^{-1} (T')^2
\]

are both in the Hilbert Schmidt class. This implies by Lemma 5.3 that \( \varphi_{\hat{S}} \) is quasiequivalent to \( \varphi_S \) and \( \varphi_{\hat{S'}} \) is quasiequivalent to \( \varphi_{S'} \). It also implies that \( (\hat{S}')^{1/2} - (\hat{S})^{1/2} \) is not in the Hilbert Schmidt class.

We can now apply Lemma 5.5 and conclude that \( \varphi_{\hat{S}} \) is not quasiequivalent to \( \varphi_S \) and hence that \( \varphi_{S'} \) is not quasiequivalent to \( \varphi_S \).

\( \text{Q.E.D.} \)

In the present section, we have assumed Lemma 9.4 and Lemma 6.6. We shall prove Lemma 9.4 in the course of our discussion on the unitary implementability of Bogoliubov transformations, although a more direct and hence shorter proof of this Lemma is also possible. We shall prove Lemma 6.6 by using a known structure of KMS states.

**§6. Uniqueness Theorems**

Let \( \tau(\lambda) \) be a continuous one parameter group of automorphisms of a C*-algebra \( \mathfrak{A} \). A state \( \varphi \) of \( \mathfrak{A} \) is said to be a state of finite \( \tau(\lambda) \).
energy if there exists $a$ such that

\begin{equation}
(6.1) \quad \int \varphi(B\tau(\lambda)A)f(\lambda)d\lambda = 0, \quad A, B \in \mathfrak{A}
\end{equation}

whenever $f \in \mathcal{S}$ and

\begin{equation}
(6.2) \quad \tilde{f}(p) = \int f(\lambda)e^{ip\lambda}d\lambda = 0
\end{equation}

for $p \geq a$. When $a$ can be chosen to be 0, $\varphi$ is called $\tau(\lambda)$-vacuum.

A state $\varphi$ is called a KMS state of $\tau(\lambda)$ with inverse temperature $\beta$, if

\begin{equation}
(6.3) \quad \int \varphi(B\tau(\lambda)A)f(\lambda)d\lambda = \int \varphi((\tau(\lambda)A)B)f(\lambda+i\beta)d\lambda
\end{equation}

for $A, B \in \mathfrak{A}$ and $\tilde{f} \in \mathcal{D}$ such that

\begin{equation}
(6.4) \quad f(\lambda) = \frac{1}{2\pi} \int e^{-i\lambda p}\tilde{f}(p)dp.
\end{equation}

(6.3) is referred to as the KMS condition.

**Theorem 2.** Let $U(\lambda)$ be a continuous one parameter group of Bogoliubov transformations. Let $E(p)$ be the spectral projections:

\begin{equation}
(6.5) \quad U(\lambda) = \int e^{ipA}E(dp) = e^{i\lambda H},
\end{equation}

\begin{equation}
H = \int pE(dp).
\end{equation}

Let $E_+ = E((0, \infty))$, $E_0 = E([0])$. Then $\varphi$ is a $\tau(\lambda)$-vacuum if and only if

\begin{equation}
(6.6) \quad \varphi(AB) = \varphi_e(A)\varphi'(B),
\end{equation}

$A \in \mathfrak{A}_{\text{SDC}}((1-E_0)K, \Gamma')$, $B \in \mathfrak{A}_{\text{SDC}}(E_0K, \Gamma')$,

where $\varphi_e$ is a Fock state and $\varphi'$ is an arbitrary state on $\mathfrak{A}_{\text{SDC}}(E_0K, \Gamma')$.

**Proof.** Since $U(\lambda)$ is a Bogoliubov transformation, $\Gamma'E_0\Gamma' = E_0$ and $\Gamma'E_0\Gamma' = 1 - E_0 - E_+$. Namely $E_+$ is a basis projection for $(1-E_0)K$. Let $\varphi_1$ be the restriction of $\varphi$ to $\mathfrak{A}_{\text{SDC}}((1-E_0)K, \Gamma') = \mathfrak{A}$.

Next we have

\[ \int \tau(U(\lambda))B(g)f(\lambda)d\lambda = B(\tilde{f}(H)g). \]
If \( \tilde{f} \) runs over all \( \tilde{f} \in \mathcal{S} \) such that \( \tilde{f}(p) = 0 \) for \( p \geq 0 \), then the set of \( \tilde{f}(H)g, g \in K \) is a dense subset of \( E_K \). Hence (6.1) requires
\[
\varphi_1(B(f)^*B(f)) = 0
\]
for all \( f \in E_K \). By Lemma 4.3, this implies \( \varphi_1 = \varphi_{E_+} \).

Let \( \pi_\varphi \) be the representation of \( \overline{\mathfrak{s}}_{\text{Soc}}(K, \Gamma) \) and \( \Omega_\varphi \) be a cyclic vector associated with \( \varphi \). If \( h_j \in E_0 K, \|h_j\|^2 = 2 \) and \( T^*h_j = h_j, j = 1, \ldots, n \), then the vector states of \( \mathfrak{A} \) by \( \psi = \pi_\varphi(\prod_{j=1}^n B(h_j)) \Omega_\varphi \) are the same Fock states \( \varphi_{E_+} \). Since the union of \( \pi_\varphi(\mathfrak{A}) \psi \) for all such \( \psi \) is total in \( \mathfrak{S}_\varphi \), \( \pi_\varphi \mid \mathfrak{A} \) is quasiequivalent to the Fock representation \( \pi_{E_+} \). Hence, by Lemma 4.2 for \( U = -1 \) and \( S = E_+ \) and by the irreducibility of \( \pi_{E_+} \), there exists \( T = \pi_\varphi(\mathfrak{A})'' \) (corresponding to \( T_\varphi(\cdot (-1)) \) such that \( T\Omega_\varphi = \Omega_\varphi, T^*T = T, T^{\*2} = 1 \) and \( T\pi_\varphi(A)T^* = \pi_\varphi(\tau(-1)A) \) for \( A \in \mathfrak{A} \). Let \( \pi_\varphi(B(h)) = \pi_\varphi(B(h)) T \) for \( h \in E_0 K \). We have \( \pi_\varphi(B(h)) \in \pi_\varphi(\mathfrak{A})' \). Hence \( \pi_\varphi(B(h)) = \pi_\varphi(B(h)) T \) commutes with \( T \). Therefore \( \pi_\varphi(B(h)) \) generates a representation of \( \overline{\mathfrak{s}}_{\text{Soc}}(E_0 K, \Gamma') \), which we denote by \( \pi' \). More explicitly, \( \pi'_\varphi(C) = \pi_\varphi(C) \left( 1 + \frac{T^*}{2} \right) + \pi_\varphi(\tau(-1)C) \left( 1 - \frac{T^*}{2} \right) \). Let \( \varphi_2 \) be the restriction of \( \varphi \) to \( \overline{\mathfrak{s}}_{\text{Soc}}(E_0 K, \Gamma') \). Since \( T\Omega_\varphi = \Omega_\varphi, \varphi_2 \) is the vector state given by \( \varphi_2(C) = (\Omega_\varphi, \pi_\varphi(C) \Omega_\varphi) \). Since \( \Omega_\varphi \) gives rise to a pure state of \( \mathfrak{A} \), we have \( \varphi(AC) = (\Omega_\varphi, \pi_\varphi(A) \pi'_\varphi(C) \Omega_\varphi) = \varphi_{E_+}(A) \varphi_2(C) \) for \( A \in \mathfrak{A} \) and \( C \in \overline{\mathfrak{s}}_{\text{Soc}}(E_0 K, \Gamma') \).

Conversely, if \( \varphi_2 \) is a state on \( \overline{\mathfrak{s}}_{\text{Soc}}(E_0 K, \Gamma') \) and \( \mathfrak{S}_2, \Omega_2, \pi_2 \) is canonically associated with it, then
\[
\pi(AB) = \pi_{E_+}(A) \otimes \pi_2(B) + \pi_{E_+}(\tau(-1)B) \otimes \pi_2(\tau(-1)A) / 2
\]
\[
+ \pi_{E_+}(A) T_{E_+}(-1) \otimes (\pi_2(B) - \pi_2(\tau(-1)B)) / 2
\]
on \( \mathfrak{S}_2 \otimes H_2 \) uniquely extends to a representation of \( \overline{\mathfrak{s}}_{\text{Soc}}(K, \Gamma') \) and \( \Omega = \pi_{E_+} \otimes \Omega_2 \) satisfies \( (\Omega, \pi(AB) \Omega) = \varphi_{E_+}(A) \varphi_2(B) \). Further, \( \tau(U(\lambda)) \) leaves the vector state by \( \Omega \) invariant, and is unitarily implementable by an operator \( T_\varphi(U(\lambda)) \otimes 1 \), whose generator is known to be positive semidefinite for \( HE_{1+} \geq 0 \). Hence (6.1) is satisfied.

**Lemma 6.1.** If the dimension of \( K \) is finite or even and infinite, \( \varphi_{1/2} \) is the unique state of \( \overline{\mathfrak{s}}_{\text{Soc}}(K, \Gamma') \) satisfying
\[
(6.7) \quad \varphi_{1/2}(AB) = \varphi_{1/2}(BA).
\]
Proof. \( \varphi_{1/2} \) satisfies (6.7) due to (3.1), (3.2) and

\[
(6.8) \quad \varphi_{1/2} (B(f)^* B(g)) = \varphi_{1/2} (B(g), B(f)^*) = (f, g)/2.
\]

Let \( \{f_a\} \) be a \( \Gamma' \) invariant orthonormal basis of \( K \). (Such basis exists). Any element in \( \mathcal{A}_{SDC}(K, \Gamma') \) is a polynomial of \( B(f_a) \). Since \( B(f_a)^2 = 1/2 \) and \( B(f_a) \) anticommutes with other \( B(f_b) \), it is enough to deduce the value \( \varphi(B(f_{a_1}) \cdots B(f_{a_n})) \) uniquely from (6.7) when \( \alpha_1 \cdots \alpha_n \) are distinct. If \( n \equiv 0 \mod 2 \) is even, then \( B(f_{a_1}) \cdots B(f_{a_n}) = -B(f_{a_n}) B(f_{a_{n-1}}) \cdots \) implies that \( \varphi(\Pi B(f_{a_k})) = 0 \). If \( n \) is odd and if there is \( \beta \) distinct from all \( \alpha \), then

\[
B(f_{a_1}) \cdots B(f_{a_n}) = 2B(f_{a_1}) \cdots B(f_{a_n}) B(f_\beta)^2
\]

implies again that \( \varphi(\Pi B(f_{a_k})) = 0 \). If \( \dim K \) is even or infinite, this shows the uniqueness. Q.E.D.

\( \varphi_{1/2} \) is called the central state. Existence of such \( \varphi_{1/2} \) follows from Lemma 3.3. If \( \dim K = 2n \), \( \varphi_{1/2} \) is the trace of a full matrix algebra divided by \( 2^n \).

**Corollary 6.2.** For any \( * \) automorphism \( \tau \) of \( \mathcal{A}_{SDC}(K, \Gamma') \), \( \varphi_{1/2} \) is invariant and there exists a unitary operator \( T_{1/2}(\tau) \) on \( H_{1/2} \) such that

\[
T_{1/2}(\tau) \Omega_{1/2} = \Omega_{1/2}, \quad T_{1/2}(\tau) \pi_{1/2}(A) T_{1/2}(\tau)^* = \pi_{1/2}(\tau A),
\]

\[
T_{1/2}(\tau_1) T_{1/2}(\tau_2) = T_{1/2}(\tau_1 \tau_2).
\]

**Theorem 3.** Let \( U(\lambda) \) be as in the previous theorem. Then a KMS state of \( \tau(U(\lambda)) \) with inverse temperature \( \beta \) is unique and is given by a quasifree state \( \varphi_\beta \) with

\[
(6.9) \quad S = (1+e^{-\beta H})^{-1},
\]

provided that \( R_\beta \) is a factor.

Proof. It is known that any KMS state has a central decomposition as an integral over primary KMS states. Hence it is enough to prove the uniqueness of primary KMS state.

Let \( \varphi_1 \) be a primary KMS state and \( \varphi(A) = (\varphi_1(A) + \varphi_1(\tau(-1)A))/2 \). Then \( \varphi \) is again a KMS state and has the property that \( \varphi(Q) = 0 \) for any odd polynomial \( Q \) of \( B(f) \).
Let $\hat{\Omega}_\varphi$, $\pi_\varphi$, $\Omega_\varphi$ be canonically associated with $\varphi$, $R=\pi_\varphi(\overline{\mathfrak{U}}_{\text{SDC}}(K, r'))''$. Since $\varphi(\tau(-1)A)=\varphi(A)$ by construction, there exists a unitary operator $T_\varphi(-1)$ such that $T_\varphi(-1)\pi_\varphi(A)\Omega_\varphi=\pi_\varphi(\tau(-1)A)\Omega_\varphi$.

A $\tau(\lambda)$ KMS state is known to be $\tau(\lambda)$ invariant. Let $T_\varphi(U(\lambda))$ be the unitary operator determined by $T_\varphi(U(\lambda))\pi_\varphi(A)\Omega_\varphi=\pi_\varphi(\tau(U(\lambda))A)\Omega_\varphi$. Let $T_\varphi(U(\lambda))=e^{i\theta}, \Delta=e^{-i\theta/2}$.

Since $\tau(-1)$ commutes with $\tau(U(\lambda))$, $T_\varphi(-1)$ commutes with $T_\varphi(U(\lambda))$ and $\Delta$. $\Omega_\varphi$ is cyclic for $R$ by construction.

The KMS condition implies that $\Omega_\varphi$ is separating. Further, there exists an antiunitary involution $J$ (a complex conjugation) on $\mathfrak{H}$ such that

\begin{align}
J\Omega_\varphi &= \Omega_\varphi, \quad JRF = R', \quad [J, e^{i\theta}] = 0, \\
JA\Omega_\varphi &= \Delta A^*\Omega_\varphi, \quad A \in \mathfrak{A},
\end{align}

where $\mathfrak{A}$ is a dense * subalgebra of $R$ consisting of all $\int_{\pi_\varphi(\tau(U(\lambda)))} A f(\lambda) d\lambda$ with $A \in \pi_\varphi(\overline{\mathfrak{U}}_{\text{SDC}}(K, r'))$ and $f(\lambda) = (2\pi)^{-1} \int f(\rho) \exp(-ipd\rho), \quad f \in \mathfrak{O}$. From the commutativity of $\Delta$ and $T_\varphi(-1)$ we have

\begin{align}
T_\varphi(-1)^*JT_\varphi(-1)A\Omega_\varphi &= T_\varphi(-1)^*\Delta(T_\varphi(-1)AT_\varphi(-1))^*\Omega_\varphi = \Delta A^*\Omega_\varphi = JA\Omega_\varphi.
\end{align}

Hence $J$ commutes with $T_\varphi(-1)$.

Let

\begin{align}
\pi_\varphi'(A) &= J\pi_\varphi(\tau(r')A)J.
\end{align}

It is another representation of $\overline{\mathfrak{U}}_{\text{SDC}}(K, r')$ such that the closure of $\pi_\varphi'(\overline{\mathfrak{U}}_{\text{SDC}}(K, r'))$ is $R'$.

Let $\hat{K}=K \oplus K$, $\hat{r}=r' \oplus (-r')$ and consider the representation $\hat{\pi}$ of $\overline{\mathfrak{U}}_{\text{SDC}}(\hat{K}, \hat{r})$ generated by

\begin{align}
\hat{\pi}(B(\mathfrak{f} \oplus \mathfrak{g})) &= \pi_\varphi(B(\mathfrak{f}')) + \pi_\varphi'(B(\mathfrak{g}))T_\varphi(-1).
\end{align}

It is easily verified that $\hat{\pi}(B(h))$ satisfies the relations (1), (2), (3) for selfdual CAR algebra and hence determines a representation of $\overline{\mathfrak{U}}_{\text{SDC}}(\hat{K}, \hat{r})$.

Let $E(\cdot)$ be the spectral projection of $U(\lambda)$ in (6.5) and $g \in K$ be such that $\|E(d\rho)g\|^2$ has a compact support. Then $\pi_\varphi(B(g)) \in \mathfrak{A}$. 

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By an analytic continuation of \( T_\varphi(U(\lambda)) \pi_\varphi(B(g)) \Omega_\varphi = \pi_\varphi(B[U(\lambda)g]) \Omega_\varphi \), we obtain
\[
(6.14) \quad \Delta \pi_\varphi(B(g)) \Omega_\varphi = \pi_\varphi(B(e^{-\alpha H/2}g)) \Omega_\varphi.
\]
Hence,
\[
(6.15) \quad \hat{\pi}(B(f \otimes g)) \Omega_\varphi = \pi_\varphi(B(f + e^{-\alpha H/2}g)) \Omega_\varphi.
\]

Let \( P \) be the projection on the subspace of \( \hat{K} \) spanned by elements of the form
\[
(6.16) \quad h_1(f) = e^{-\alpha H/2}f \otimes e^{-\alpha H/2}f, \quad f \in K,
\]
where \( H_+ = HE_+ \) and \( H_- = H_+ - H \). Then \( \hat{r}P \hat{K} \) is spanned by
\[
(6.17) \quad h_2(f) = e^{-\alpha H/2}f \otimes (-e^{-\alpha H/2}f)
\]
which is orthogonal to (6.16). Further,
\[
(6.18) \quad h_1(e^{-\alpha H/2}f) + h_2(e^{-\alpha H/2}f) = (e^{-\alpha H_+} + e^{-\alpha H_-})f \otimes 0.
\]

Since \( e^{-\alpha H_\pm} \) are mutually commuting positive selfadjoint operators, their sum has a dense range and hence (6.18) is dense in \( K \otimes 0 \). Similarly, \( h_1(e^{-\alpha H/2}f) - h_2(e^{-\alpha H/2}f) \) is dense in \( 0 \otimes K \). Hence the sum \( h_1(f_1) + h_2(f_2) \) is dense in \( \hat{K} \) and we have \( \hat{r}P \hat{r} = 1 - P \). Therefore, \( P \) is a basis projection.

(6.15) shows that
\[
(6.15) \quad \hat{\pi}(B(h)) \Omega_\varphi = 0
\]
for \( h = h_2(f) \) in a dense subset of \( (1 - P) \hat{K} \) and hence for all \( h \) in \( (1 - P) \hat{K} \). Hence the vector state of \( \mathcal{H}_{\text{SDC}}(\hat{K}, \Gamma) \) given by \( \Omega_\varphi \) is unique (a Fock state \( \hat{\varphi}_\varphi \)) by Lemma 4.3. Then its restriction to \( \mathcal{H}_{\text{SDC}}(K, \Gamma) \) is also unique.

Since
\[
(6.20) \quad P = \begin{pmatrix}
(1 + e^{-\alpha H})^{-1} & (1 + e^{-\alpha H})^{-1} e^{-\alpha H/2} \\
(1 + e^{-\alpha H})^{-1} e^{-\alpha H/2} & (1 + e^{-\alpha H})^{-1} e^{-\alpha H}
\end{pmatrix}.
\]
We have
\[
(6.21) \quad \varphi = \varphi_\xi, \quad S = (1 + e^{-\alpha H})^{-1}
\]
Since \( R=R_s \) is a factor by assumption and since a primary KMS state is an extremal KMS state, we have \( \varphi_1=\varphi=\varphi_s \). Therefore the uniqueness is proved.

It remains to show that \( \varphi_s \) given by (6.21) is actually a KMS state. The KMS condition amounts to

\[
(\langle AA^\gamma A^{-1}\rangle\Omega_\varphi, \langle BB^* A^{-1}\rangle\Omega_\varphi) = (B\Omega_\varphi, A\Omega_\varphi),
\]

for \( A, B \) in \( \mathcal{A} \). Hence we only have to prove the antiunitarity of \( J \) defined by (6.11).

Let \( \varepsilon \) be the Bogoliubov transformation on \( (\hat{K}=K\oplus K, \hat{\Gamma}=\Gamma\oplus -\Gamma) \) given by the matrix \( \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \). Then \( \varepsilon P \varepsilon = 1 - P \) and hence the continuous extension of \( J_0 \) by \( \hat{\tau}(C)\Omega_\varphi = \hat{\tau}(\hat{\Gamma})\tau(\varepsilon)C\Omega_\varphi, C \in \mathcal{F}_{SDC}(\hat{K}, \hat{\Gamma}) \) defines obviously an antiunitary operator \( J_0 \). If we restrict \( C \) to \( \mathcal{F}_{SDC}(K, \Gamma) \) \( (K \oplus 0 \subset \hat{K}) \), we have

\[
(6.22) \quad J_0 \pi_\varphi(B(f_1)) \cdots \pi_\varphi(B(f_n))\Omega_\varphi
= (-1)^{n(n-1)/2}(-i)^n\pi_\varphi(B(\Gamma f_1)) \cdots \pi_\varphi(B(\Gamma f_n))\Omega_\varphi
= (-i)^n(\pi_\varphi(B(f_1))^* A^{-1}) \cdots (\pi_\varphi(B(f_n))^* A^{-1})\Omega_\varphi,
\]

where \( f_i \) is any element in \( K \) such that \( ||E(\lambda)f_i|| \) has a compact support. Hence

\[
(6.23) \quad J = \alpha J_0,
\]

where \( \alpha \) is a function of \( T_\varphi(-1) \), being \( =1 \) if \( T_\varphi(-1) = 1 \) and \( =i \) if \( T_\varphi(-1) = -1 \). Since \( \alpha \) is unitary, \( J \) is antiunitary. Q.E.D.

**Corollary 6.3.** Assume that \( \dim K \) is not odd. If \( S-1/2 \) is of finite rank and \( S \) does not have an eigenvalue \( 1 \), we have for \( A \in \mathcal{F}_{SDC}(K, \Gamma) \)

\[
(6.24) \quad \varphi_s(A) = \varphi_{1/2}(e^{-,(B,HB)/3}A) / \varphi_{1/2}(e^{-,(B,HB)/3})
= (\det 2S)^{1/2} \varphi_{1/2}(e^{-,(B,HB)/3}A)
\]

where \( (B, HB) \) is defined in Lemma 7.3 and \( H=\log S(1-S)^{-1} \).

**Proof.** The left hand side is a unique KMS state for \( U(\lambda) = e^{iH\lambda} \) with \( \beta=1 \). Since \( S-1/2 \) is of finite rank and \( \Gamma S\Gamma = 1 - S \), an eigenvalue \( 1/2 \) of \( S \) has an infinite or even multiplicity (according as \( K \) has an infinite or finite even dimension). Hence we have only to show that
the right hand side is a KMS state. Since \( S(1-S)^{-1}-1 \) is of finite rank due to the assumption on \( S \), \( H \) is of finite rank. It is hermitian and satisfies \( iH^r = -H \). Hence, there exists \( Q = \langle B, H | B | \rangle / 2 \in \mathcal{S}(K, \mathcal{S}) \).

We then have from (6.7)

\[
\varphi_{1/2}(e^{-\phi}AB) = \varphi_{1/2}(e^{-\phi}B'A), \quad B' = e^{\phi}Be^{-\phi}.
\]

Since \( e^{\phi}B(f)e^{-\phi} = B(e^{\phi}f) \), \( B' \) is an analytic continuation of \( \tau(U(\lambda))B \) to \( \lambda = -i \). Hence the right hand side of (6.24) is a KMS state for \( U(\lambda) \) with \( \beta = 1 \).

The normalization factor is computed by (8.25). Q.E.D.

**Lemma 6.4.** Let \( \Sigma = \Sigma_1 \otimes \Sigma_2 \) and \( R = \mathcal{B}(\Sigma_1) \otimes 1 \). Let \( \Omega \) be a unit vector, cyclic and separating for \( R \), and \( J \) be an antiunitary involution satisfying \( JRJ = R' \) and \( J\Omega = \Omega \). Assume that \( \langle \Omega, Aj(A)\Omega \rangle \geq 0 \) for all \( A \in R \) where \( j(A) = JAJ \). Then there exists a standard diagonal expansion \((3), \text{Definition 2.2})

\[
\Omega = \sum \lambda_i \phi_{i1} \otimes \phi_{i2}
\]

such that \( \lambda_i > 0 \) and

\[
J(\phi_{i1} \otimes \phi_{i2}) = (\phi_{i1} \otimes \phi_{i2}).
\]

**Proof.** Let

\[
\Omega = \sum \lambda'_i \psi_{i1} \otimes \psi_{i2}, \quad \lambda'_i > 0
\]

be a standard diagonal expansion of \( \Omega \) and let \( J_0 \) be an antiunitary involution defined by

\[
J_0 \sum c_{ij} \psi_{i1} \otimes \psi_{i2} = \sum c_{ij}^* \psi_{j1} \otimes \psi_{j2}.
\]

Let \( W = J_0 J \).

Then \( W \) is unitary and satisfies \( W\Omega = \Omega, WRW^* = R \). Hence there exists a unitary \( U_1 \) in \( \mathcal{B}(\Sigma_1) \) such that \( WAW^* = (U_1 \otimes 1)A(U_1^* \otimes 1) \) for all \( A \in R \). Since \( (U_1^* \otimes 1)W \) is in \( R' \), it can be written as \( 1 \otimes U_2 \).

Then \( W = U_1 \otimes U_2 \).

Let \( \rho_1 \) and \( \rho_2 \) be the unique trace class operators on \( \Sigma_1 \) and \( \Sigma_2 \) satisfying

\[
\text{tr}(\rho_1 A_1) = \langle \Omega, (A_1 \otimes 1)\Omega \rangle,
\]

\[
\text{tr}(\rho_2 A_2) = \langle \Omega, (A_2 \otimes 1)\Omega \rangle.
\]
From $W\Omega=\Omega$ and $W=U_{1}\otimes U_{2}$, we have $[\rho_{\nu}, U_{\nu}]=0, \nu=1, 2$.

Let $\rho_{\nu}=\sum_{i}x_{i}P_{\nu}(x)$ be the spectral resolution of $\rho_{\nu}$. Then $P_{\nu}(x)=\sum_{s}P(x_{s})$ where $P(x_{s})$ denotes the minimal projection of $\mathcal{B}(S_{s})$ corresponding to $x_{s}$ and the sum extends over those $i$ such that $(\lambda_{i})^{2}=x$.

Let

$$\Phi_{1s}=\sum_{i}u_{si}^{*}x_{si},$$

be a complete orthonormal set of eigenvectors of $U_{1}$ belonging to eigenvalues $e^{i\theta_{i}}$. Since $[P_{1}(x), U_{1}]=0$ and each $P_{1}(x)$ has a finite dimension, $U_{1}$ has a purely discrete spectrum and we can chose $u_{si}$ such that $u_{si}u_{sj}\neq 0$ only if $i=j$.

Let

$$\Phi_{2s}=\sum_{i}u_{si}^{*}x_{si}.$$

Since $(u_{si})$ is unitary, we have (6.25) where $\lambda_{i}=\lambda_{i}$ if $u_{si}\neq 0$.

From $W\Omega=\Omega$ and (6.25), we have

$$U_{2}\Phi_{2s}=e^{-i\theta_{s}}\Phi_{2s}.$$

Since $J=J_{0}W$, we have

$$J(\Phi_{1s}\otimes\Phi_{2s})=\varepsilon_{1s}\Phi_{1s}\otimes\Phi_{2s},$$

where $\varepsilon_{1s}=e^{i(\theta_{s}-\theta_{i})}$. Since $J^{2}=1$, we have $(\varepsilon_{1s})^{2}=1$. Therefore $\varepsilon_{1s}=\varepsilon_{1b}=\pm 1$.

Let $A_{sl}\in\mathcal{B}(S_{s})$ be defined by $A_{sl}^{*}c_{j}s_{l}^{*}=c_{s}^{*}A_{sl}^{*}s_{l}e_{i}s_{l}^{*}$. From $(\Omega, A\Omega)\geq 0$ with $A=A_{sl}^{*}\otimes I_{l}$, we have $A\Omega=\lambda_{l}s_{l}^{*}\otimes \Phi_{2s}^{*}+\lambda_{i}s_{i}^{*}\otimes \Phi_{2s}$, $JAJ=JAJ=\varepsilon_{b}(\lambda_{b}s_{b}^{*}\otimes \Phi_{2s}^{*}+\lambda_{i}s_{i}^{*}\otimes \Phi_{2s})$, and hence $2\lambda_{b}\lambda_{i}\varepsilon_{b}\geq 0$. From this we have $\varepsilon_{b}\geq 0$ and hence (6.26) holds.

Lemma 6.5. Let $R$ be a type $I$ factor, $\Omega$ and $\Omega'$ be cyclic and separating unit vectors and $J$ be an antiunitary involution such that $JR=J'\Omega$, $J\Omega=\Omega$, $J\Omega'=\Omega'$, $(\Omega, A\Omega)\geq 0$ and $(\Omega', A\Omega')\geq 0$ for all $A\in R$ where $j(A)=JAJ$. Let $\varphi$ and $\varphi'$ be the vector states of $R$ given by $\Omega$ and $\Omega'$. Then

$$\|\varphi-\varphi'\|\geq 2(1-|\langle \Omega, \Omega' \rangle|).$$
Proof. Since $R$ is a type I factor, we can identify the Hilbert space and $R$ as follows:

$$
\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 ,
$$

$$
R = \mathcal{B}(\mathcal{H}_1) \otimes 1 .
$$

Let

$$
\mathcal{Q} = \sum \lambda_i \Theta_i \otimes \Theta_{i^*} ,
$$

$$
\mathcal{Q}' = \sum \lambda_i' \Theta_{i'} \otimes \Theta_{i'^*} ,
$$

be the standard diagonal expansions of $\mathcal{Q}$ and $\mathcal{Q}'$ given by the previous Lemma.

From (6.26) and antiunitarity of $J$, we have

$$
(\phi_{ij}, \phi_{ij}^*) (\phi_{jl}, \phi_{jl}^*) = (\phi_{ij}, \phi_{ij}^*) (\phi_{kl}, \phi_{kl}^*) .
$$

Since the matrices $u_{ij} = (\phi_{ij}, \phi_{ij})$ and $u_{ij} = (\phi_{ij}, \phi_{ij})$ are unitary, there exists $u_{ij} \neq 0$. Setting $\epsilon = u_{ij}/u_{ij}^*$, we have $u_{ij} = \epsilon u_{ij}$, where $\epsilon$ is common for all $j, l$. From the unitarity, we have $|\epsilon| = 1$. From (6.35), we have $\epsilon = \epsilon^*$. Hence $\epsilon = \pm 1$.

We now have

$$
(\mathcal{Q}, \mathcal{Q}') = \sum \lambda_i \lambda_i' (\Theta_{ii}^*, \Theta_{ij}^*) (\Theta_{ij}, \Theta_{ij}^*)
$$

$$
= \epsilon \sum \lambda_i \lambda_i' (\Theta_{ii}^*, \Theta_{ij}^*)
$$

Let

$$
\rho = \sum \lambda_i^2 P(\Theta_{ii}) ,
$$

$$
\rho' = \sum (\lambda_i')^2 P(\Theta_{ii}) .
$$

Then $\phi(A) = \text{tr} \rho A$ and $\phi'(A) = \text{tr} \rho' A$. We now have

$$
\|\phi - \phi'\| = \sup_{|A| \leq 1} |\phi(A) - \phi'(A)| = \text{tr} |\rho - \rho'|
$$

$$
\geq \text{tr} (\rho^{1/2} - (\rho')^{1/2})^2 = 2(1 - \text{tr} \rho^{1/2} \rho'^{1/2})
$$

where the inequality is due to Lemma 4.1 of [12]. From (6.36), we have

$$
\text{tr} \rho^{1/2} (\rho')^{1/2} = |(\mathcal{Q}, \mathcal{Q}')| .
$$

From (6.39) and (6.40), we have (6.34). Q.E.D.

Lemma 6.6. Assume that $\dim K$ is finite and even. Let $\varphi_8$ and
\(\varphi_s\) be two quasifree states of \(\mathcal{M}_{SDO}(K, \Gamma)\). Then

\[
(6.41) \quad \|\varphi_s - \varphi_{s'}\| \geq 2 \left[ 1 - \det \left( 1 - (P_s - P_{s'})^2 \right) \right]^{1/8}.
\]

**Proof.** Let \(\widehat{K} = K \oplus K\), \(\widehat{\Gamma} = \Gamma \oplus -\Gamma\).

First consider the case where \(S\) and \(S'\) do not have an eigenvalue \(1\). In this case we can show

\[
(6.42) \quad P_s \cap (1 - P_{s'}) = 0,
\]
as follows:

Let \(g = g_1 \oplus g_2\), \(P_s g = g\), \(P_s g_2 = 0\). Then \(S^{1/2} g_2 = (1 - S)^{1/2} g_1\) and \((S')^{1/2} g_1 = -(1 - S')^{1/2} g_2\). Since \(S\) and \(S'\) do not have an eigenvalue \(1\), the same holds for \(1 - S = \Gamma SF\) and for \(1 - S' = \Gamma S'T\). Therefore \(S\), \(S'\), \((1 - S)\) and \((1 - S')\) have their inverses. We have \(S^{-1/2} (1 - S)^{1/2} + (1 - S')^{-1/2} (S')^{1/2}\) \(g_1 = 0\). From \(S^{-1/2} (1 - S)^{1/2} > 0\) and \((1 - S')^{-1/2} (S')^{1/2} > 0\), we have \(g_1 = 0\). Similarly we have \((1 - S)^{-1/2} S^{1/2} + (S')^{-1/2} (1 - S')^{1/2}\) \(g_2 = 0\) and hence \(g_2 = 0\). This proves \((6.42)\).

Let \(\varepsilon = \left( \begin{array}{cl} 0 & i \\ -i & 0 \end{array} \right)\). Then \(\varepsilon^* = \varepsilon^{-1} = \varepsilon\), \([\widehat{\Gamma}, \varepsilon] = 0\) and \(\varepsilon P_s \varepsilon = 1 - P_s\) for any \(S\). Consequently, \(\widehat{\Gamma}\varepsilon\) commutes with \(P_s\) and \(P_{s'}\), and hence anti-commutes with \(H(P_{s'}/P_s)\) defined as in \((9.2)\).

Let \(J_s\) be an antiunitary involution on \(\mathcal{S}_{P_s}\) defined by

\[
(6.43) \quad J_0 \pi_{P_s}(C) \mathcal{O}_{P_s} = \pi_{P_s}(\tau(\widehat{\Gamma}) \tau(\varepsilon) C) \mathcal{O}_{P_s}
\]
for \(C \in \mathcal{M}_{SDO}(\widehat{K}, \widehat{\Gamma})\). Let \(J = \alpha J_0\) where \(\alpha\) is a function of \(T_{P_s}(-1) = T_s(-1)\) being \(-1\) if \(T_s(-1) = 1\) and \(= i\) if \(T_s(-1) = -1\). For a finite even dimensional \(K\), \(\pi_{P_s}(\mathcal{M}(K, \Gamma))\) is always a factor and hence the proof of Theorem 3 is applicable where \(H = \log \{S(1 - S)^{-1}\}\) and \(\beta = 1\). From \((6.11)\) we have

\[
(6.44) \quad (\mathcal{O}_{P_s}, A(JAJ) \mathcal{O}_{P_s}) = (A^* \mathcal{O}_{P_s}, A^* \mathcal{O}_{P_s}) \geq 0
\]
for \(A \in \pi_{P_s}(\mathcal{M}(K \oplus \widehat{\Gamma}, \varepsilon))\), where \(A = e^{-\theta t/2} \geq 0\).

Let \(Q\) be defined as in \((9.4)\) where \(n = 0\). Then \(\pi_{P_s}(Q)\) commutes with \(J_0\) and \(T_{P_s}(-1)\). Hence \(Q' = \pi_{P_s}(Q)^* \mathcal{O}_{P_s}\) is invariant under \(J_0\) and \(T_{P_s}(-1)\). Furthermore, we have

\[
(6.45) \quad \varphi_{P_s}(C) = (\varphi', \pi_{P_s}(C) \varphi'),
\]
where \(\varphi'\) is a quasi-free state of \(\mathcal{M}_{SDO}(K, \Gamma)\) with \(\varphi \).
(6.46) \[ J_\varphi \pi_p(C) \varphi' = \pi_p(\tau(\hat{r}^*) \tau(\varphi) C) \varphi', \]
(6.47) \[ T_p(-1) \pi_p(C) \varphi' = \pi_p(\tau(-1) C) \varphi', \]
for \( C \in \mathfrak{A}_{\text{SDC}}(\hat{K}, \hat{r}) \). Therefore

(6.48) \[ \langle \varphi', A(JAJ) \varphi' \rangle = \langle A^* \varphi', A^* \varphi' \rangle \geq 0 \]
for \( A \in \pi_p(\mathfrak{A}_{\text{SDC}}(K \oplus 0, \hat{r})) \), where \( A = e^{-\theta r/2} \) denotes the \( A \) in the proof of Theorem 3 corresponding to \( S' \).

We can now apply the previous Lemma and obtain

(6.49) \[ \| \varphi_{r_1} - \varphi_{r_2} \| \geq 2(1 - |\varphi_{r_2}, \varphi'\rangle). \]

From (9.8), we obtain (6.41).

The general case, where one or both of \( S \) and \( S' \) have an eigenvalue 1, can be obtained by taking a limit. Q.E.D.

§7. Bilinear Hamiltonian

Lemma 7.1. There exists a derivation \( \delta(H) \) on \( \mathfrak{A}_{\text{SDC}}(K, r) \)
satisfying

(7.1) \[ \delta(H)B(f) = B(Hf) \]
if and only if \( H \) is a bounded linear operator on \( K \) satisfying

(7.2) \[ H^* = -r H. \]

If (7.2) holds, (7.1) uniquely defines \( \delta(H) \). It is a * derivation of 
\( \mathfrak{A}_{\text{SDC}}(K, r) \) if and only if

(7.3) \[ H^* = -H. \]

Proof. For the first part, we have to check the condition that
\( \delta(H) \) is consistent with the relations (1) and (2) for the definition of
\( \mathfrak{A}_{\text{SDC}}(K, r) \). For the condition (1), it is necessary and sufficient that
\( H \) is linear. For the condition (2), it is necessary and sufficient that
(7.2) holds. From (7.2) it follows that \( H^* \) is defined on all \( K \) and
hence \( H \) must be bounded.

For the second part, the uniqueness of \( \delta(H) \) is immediate. The
relation (3) for \( \mathfrak{A}_{\text{SDC}}(K, r) \) implies that \( (\delta(H)B(f))^* = \delta(H)B(f)^* \)
if and only if \( \Gamma H = H \Gamma \). Under (7.2), this is equivalent to (7.3).

**Lemma 7.2.** The * derivation \( \delta(H) \) is the infinitesimal generator of the Bogoliubov automorphism \( \tau(e^{iH}) \).

**Proof.** From (7.3) it follows that \( e^{iH} \) is unitary. From (7.2) and (7.3), it follows that \([H, \Gamma] = 0\) and hence \([e^{iH}, \Gamma] = 0\). Hence \( e^{iH} \) is a Bogoliubov transformation. The rest is immediate.

**Lemma 7.3.** Let \( H \) be a finite rank operator on \( K \) and

\[
Hh = \sum_{i=1}^{r} f_i(g_i, h), \quad h \in K.
\]

Let

\[
(B, HB) = \sum_{i=1}^{r} B(f_i)B(g_i)^*.
\]

\( (B, HB) \) does not depend on the choice of \( f_i \) and \( g_i \) for a given \( H \), is linear in \( H \) and satisfies

\[
(B, HB)^* = (B, H^*B).
\]

In addition, it satisfies

\[
(B, HB) = (B, \alpha(H)B) + \frac{1}{2} \text{tr} H,
\]

\[
\alpha(H) = \frac{1}{2}(H - \Gamma H^* \Gamma).
\]

\( H = \alpha(H') \) satisfies (7.2) for any \( H' \).

If \( H \) satisfies (7.2), then \( \alpha(H) = H \), \( \text{tr} H = 0 \) and

\[
[(B, HB), A] = 2\alpha(H)A, \quad A \in \mathfrak{sdg}(K, \Gamma),
\]

\[
\varphi_s((B, HB)) = -\text{tr} SH,
\]

\[
(1/4) \|H\|_{\mathfrak{m}} \leq \|(B, HB)\| \leq \|H\|_{\mathfrak{m}},
\]

\[
\tau(U) (B, HB) = (B, UHU^{-1}B),
\]

\[
\delta(Hi) (B, HB) = (B, [H, H] B).
\]

Here \( \varphi_s \) is a quasifree state and \( \|H\|_{\mathfrak{m}} = \text{tr}[\{(H^*H)^{1/2}\}]. \)

(The formulae (7.12) and (7.13) hold for a general \( H \) not satisfying (7.2).)

**Proof.** For \( (B, HB) \) defined by (7.5), we have (cf. [2])

\[
\text{tr} SH,
\]

\[
(1/4) \|H\|_{\mathfrak{m}} \leq \|(B, HB)\| \leq \|H\|_{\mathfrak{m}},
\]

\[
\tau(U) (B, HB) = (B, UHU^{-1}B),
\]

\[
\delta(Hi) (B, HB) = (B, [H, H] B).
\]
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(7.14) \([\langle B, HB \rangle, B(f)] = B(2\alpha(H)f)\),

(7.15) \(\varphi_{2}\langle B, HB \rangle = \text{tr} H - \text{tr} SH\).

For the central state, \(S=1/2\) and

(7.16) \(\varphi_{1/2}\langle B, HB \rangle = \text{tr} H/2\).

If \(K\) has an infinite or even dimension, \(\mathfrak{N}_{\text{BDC}}(K, \Gamma)\) is known to have the trivial center. Hence (7.14) determines \((B, HB)\) up to a constant and (7.16) fixes that constant. Even if the dimension of \(K\) is odd, we can make this argument by imbedding \(\mathfrak{N}_{\text{BDC}}(K, \Gamma)\) in \(\mathfrak{N}_{\text{BDC}}(K', \Gamma')\) with a bigger \(K'\) with an even dimension.

This argument shows that \((B, HB)\) is independent of the way in which \(H\) is expressed in (7.4) and also that (7.7) holds because \(\alpha(H)\) has trace 0 and both sides of (7.7) satisfy (7.14) and (7.16). Note that \(\text{tr} \Gamma H^* \Gamma = \sum_i \langle r^a e_i, \Gamma H^* \Gamma e_i \rangle = \sum_i \langle H^* \Gamma e_i, \Gamma e_i \rangle = \sum_i \langle \Gamma e_i, H \Gamma e_i \rangle = \text{tr} H\).

The linearity of \((B, HB)\) in \(H\), (7.6), (7.12) and (7.13) follow from the definition (7.5). (7.9) and (7.10) follow from (7.14) and (7.15).

If \(H\) satisfies (7.2) and is selfadjoint, it has the spectral decomposition

(7.17) \(H = \sum_{\lambda} \lambda E_{\lambda}\),

where

(7.18) \(r E_{\lambda} \Gamma = E_{-\lambda}\).

Hence we have a partial basis projection \(\sum_{\lambda>0} E_{\lambda} = E_+\) and an orthonormal basis \(f_i\) in \(E_+ K\) such that

(7.19) \(\langle B, HB \rangle = \sum_i \lambda_i \langle B(f_i) B(f_i)^* - B(f_i) B(f_i) \rangle\).

Since \(\|B(f) B(f)^* - B(f) B(f)\| \leq \|B(f) B(f)^* + B(f) B(f)\| = \|f\|^4\), we have

(7.20) \(\|\langle B, HB \rangle\| \leq \sum_i \lambda_i = \frac{1}{2} \|H\|_{\text{tr}}\).

On the other hand, \(\varphi_{2}\langle (B, HB) \rangle = -\sum_i \lambda_i = -\frac{1}{2} \|H\|_{\text{tr}}\) for \(S=E_+ + (1/2)E_0\). Hence, for a selfadjoint \(H\) satisfying (7.2), we have
(7.21) \[ \| (B, HB) \| = \| H \|_\tau / 2. \]

If \( H = H_1 + iH_2, \) \( H_1^* = H_1, \) \( H_2^* = H_2, \) then consider the polar decomposition \( H = |H| U \) where \( U_1^* = U_1, \) \( U_1^2 = 1. \) Then \( (\text{tr} H_1 U_1)^* = \text{tr} U_1 H_1 \) and \( \text{tr} H_1 U_1 \) is real and we have

(7.22) \[ \| H \|_\tau = \sup_{\| \varphi \| \leq 1} |\text{tr} HQ| \geq |\text{tr} HU_1| \geq |\text{tr} |H| | 1 = \| H \|_\tau. \]

Hence

(7.23) \[ \| H_1 \|_\tau + \| H_2 \|_\tau \geq \| H \|_\tau \geq \max (\| H_1 \|_\tau, \| H_2 \|_\tau). \]

On the other hand, for any operator \( A = A_L + iA_R, \) \( A_1^* = A_1, \) \( A_2^* = A_2, \) we have \( \| A \| = \sup_{\| \varphi \| \leq 1} |\langle \varphi, A\varphi \rangle| \sup_{\| \varphi \| \leq 1} |\langle \varphi, A\varphi \rangle| \sup_{\| \varphi \| \leq 1} |\langle \varphi, A_1\varphi \rangle| = \| A_1 \|. \)

Hence

(7.24) \[ \| A_1 \| + \| A_2 \| \geq \| A \| \geq \max (\| A_1 \|, \| A_2 \|). \]

By combining (7.21), (7.23) and (7.24), we have (7.11). Q.E.D.

**Lemma 7.4.** Let \( H \) be a trace class operator and \( H_n \) be a sequence of finite rank operators such that \( \| H - H_n \|_\tau \to 0 \) as \( n \to \infty. \) Then \( (B, HB) \) has a limit \( (B, HB) \) in \( \mathcal{F}_{SDC}(K, \tau) \) independent of the sequence for a given \( H. \) It is linear in \( H, \) and satisfies (7.6) and (7.7). If \( H \) satisfies (7.2), \( (B, HB) \) satisfies (7.9), (7.10), (7.11), (7.12) and (7.13).

**Proof.** From (7.11) and (7.7), the convergence and the uniqueness follow. The rest follows from the corresponding properties for \( H_n. \)

**Theorem 4.** The derivation \( \delta(H) \) can be extended to an inner derivation of \( \mathcal{F}_{SDC}(K, \tau) \) if and only if \( H \) is in the trace class.

**Proof.** “If” part follows from Lemma 7.4 and (7.9). \( \delta(H) \) can be extended to an inner derivation if and only if \( \delta(i(H^* + H)) \) and \( \delta(H^* - H) \) can be extended to inner *-derivations. For an inner *-derivation \( \delta(H), \tau(e^{i\lambda}) \) for all real \( \lambda \) must be an inner automorphism by Lemma 7.2. From later result in Theorem 5 this implies that either \( e^{i\lambda} - 1 \) is in the trace class or \( e^{i\lambda} + 1 \) is in the trace class. In either case, the selfadjoint operator \( iH \) must have purely discrete spectrum.
If $e^{\lambda t}+1$ is in the trace class, then the eigenvalues $x_j$ of $iH$ can have an accumulation point only at $\frac{\pi}{\lambda}(2n+1)$, $n=0, \pm 1, \pm 2, \cdots$ which can happen only for a countable number of $\lambda$. For other values of $\lambda$, $e^{\lambda t}-1$ must be in the trace class and hence $iS$ can have an accumulation point only at $2\pi\lambda^{-1}$. This, first of all, excludes the other possibility and further implies that $i$ can have an accumulation point of eigenvalues only at $0$. From the condition $\|e^{\lambda t}-1\|_\tau=\sum_i\{2(1-\cos x_i)\}^{1/2}<\infty$, and the inequality $1-\cos x\geq x^2/3$ for $|x|\leq 1$, we obtain $\|H\|_\tau<\infty$. Q.E.D.

§8. Inner Bogoliubov Transformations

Definition 8.1. $\mathcal{T}_\pm$ denotes the set of invertible bounded linear operators $V$ on $K$ such that $V-1$ is in the trace class, $\det V=\pm 1$, respectively, and

$$ (8.1) \quad \Gamma V^*\Gamma=V^{-1}. $$

$\mathcal{T}_\pm$ is equipped with an operator multiplication, an adjoint operation $*$ and a topology induced by spheres $\{V':\|V'-V\|_\tau\leq \varepsilon\}$.

$\mathcal{T}_+$ and $\mathcal{T}_+\cup\mathcal{T}_-$ are topological groups and $\mathcal{T}_+$ is connected.

Since $V-1$ is compact, it has a (Jordan) expansion:

$$ (8.2) \quad V-1=V_d+\sum_{\lambda\in\mathcal{A}}E_\lambda(\lambda-1+N_\lambda) $$

where $\mathcal{A}$ is a bounded open set containing $1$, $(V_d-\lambda)^{-1}$ is holomorphic for $\lambda\not\in\mathcal{A}$,

$$ (8.3) \quad E_\lambda E_{\lambda'}=\delta_{\lambda\lambda'}E_{\lambda'}, $$

$$ (8.4) \quad E_{\lambda'}N_\lambda=N_\lambda E_{\lambda'}=\delta_{\lambda\lambda'}N_\lambda, $$

$$ (8.5) \quad N_\lambda^{\dim \delta_1}=0, \quad \dim E_\lambda<\infty, $$

$$ (8.6) \quad V_dE_\lambda=E_\lambda V_d=0 \quad (\lambda\not\in\mathcal{A}), $$

$$ (8.7) \quad \lim_{n\to\infty}(V_d/r)^*r=0, \quad r=\text{sup}_{\lambda\in\mathcal{A}}|\lambda|. $$

$V_d$, $E_\lambda$ and $N_\lambda$ are uniquely determined by these properties and are given by

$$ (8.8) \quad E_\lambda=\lim_{\rho\to0}(2\pi)^{-1}\int_0^{2\pi}\rho e^{i\theta}(\lambda+\rho e^{i\theta}-V)^{-1}d\theta, $$

where $\lambda\not\in\mathcal{A}$.
(8.9) \[ N_\lambda = E_\lambda (V - \lambda) \]

and (8.2). The \( \det V \) may be computed by

(8.10) \[ \det V = \exp \text{tr} \log V \]

(8.11) \[ \log V = (1 - \sum_{\lambda \in \mathcal{A}} E_\lambda) \log (1 + V_d) \]

\[ + \sum_{\lambda \in \mathcal{A}} E_\lambda [\log \lambda + \log (1 + \lambda^{-1} N_\lambda)], \]

where \( \log (1 + V_d) \) and \( \log (1 + \lambda^{-1} N_\lambda) \) are defined by power series, which converge due to (8.7) and (8.5), and we take \( \Delta \) such that \( r < 1 \). Since \( V \) is invertible, \( \lambda \neq 0 \).

For \( V \) satisfying (8.1), the uniqueness implies

(8.12) \[ \tau E^* \tau = E_{(\lambda^{-1})} \]

(8.13) \[ (\lambda + \tau N^*_\lambda \tau) (\lambda^{-1} + N(\lambda^{-1})) = 1, \]

(8.14) \[ (1 + \tau V^*_\lambda \tau) (1 + V_d) = 1, \]

where (8.14) holds if \( \Delta \) is invariant under \( \lambda \rightarrow \lambda^{-1} \). If we choose branches of \( \log \lambda \) in (8.11) such that \( \log \lambda + \log \lambda^{-1} = 0 \) for \( \lambda \neq -1 \) and \( \log (-1) = i\pi \), then we have

(8.15) \[ \log V + \tau (\log V)^* \tau = 2\pi i E_{-1}. \]

Hence we have

(8.16) \[ \det V = (-1)^{\dim E_{-1}}. \]

Thus the condition \( \det V = \pm 1 \) can be replaced by \( \dim E_{-1} = \text{even or odd} \).

**Lemma 8.2.** If \( H \) is an operator in the trace class satisfying (7.2), then \( e^H \in \mathcal{S}_+ \). If \( V \) is a normal operator in \( \mathcal{S}_+ \) or \( V \in \mathcal{S}_+ \) does not have an eigenvalue \(-1\), then there exists a trace class operator \( H \) satisfying (7.2) such that \( V = e^H \). If \( V > 0 \), \( H \) can be chosen hermitian and if \( V \) is unitary, \( iH \) can be chosen hermitian.

**Proof.** The first part is immediate. Let \( V \in \mathcal{S}_+ \) be normal. We then have \( N_{-1} = 0 \), \( \dim E_{-1} = \text{even} \), \( E^*_1 = E_{-1} \), \( \tau E_{-1} \tau = E_{-1} \). Hence there exists a subprojection \( F \) of \( E_{-1} \), which satisfies \( \tau F \tau = E_{-1} \), \( F^* = F \) = \( F^2 \). \( H = \log V - 2\pi i F \) satisfies \( e^H = Ve^{-2\pi i F} = V \) and (7.2) due to (8.15).
$H$ is obviously in the trace class. If $V \in \mathcal{S}_+$ does not have an eigenvalue $-1$, then $H - \log V$ has the required property.

If $V > 0$, then we can choose $\log \lambda$ to be real and for this choice of the branch of $\log$, $H$ is hermitian. If $V$ is unitary, we may take $|\text{Im} \log \lambda| < \pi$ for $\lambda \neq -1$, and for this choice of the branch of $\log$, $iH$ is hermitian.

**Lemma 8.3.** There exists a covering group $\mathcal{T}^*_+ \times \mathcal{T}^*_+$ of $\mathcal{T}^*_+$ equipped with adjoint operation and a *-homomorphic homeomorphism $\pi$ of $\mathcal{T}^*_+$ onto $\mathcal{T}^*_+$ such that $\pi$ is 2 to 1 and the loop $\{ \exp 2\pi i \lambda E; 0 \leq \lambda \leq 1 \}$ for an odd dimensional partial basis projection $E$ gives an element of $\mathcal{T}^*_+$ different from 1. There exists a homeomorphic *-isomorphism $Q$ from $\mathcal{T}^*_+$ into $\mathcal{S}_{\text{spec}}(K, r)$ such that

\[
(8.17) \quad Q(g) B(f) Q(g)^{-1} = B(\pi(g) f),
\]

\[
(8.18) \quad \varphi_{1/2}(Q(g)) = \exp \frac{1}{2} \text{tr} \log [(1 + \pi(g))/2],
\]

\[
(8.19) \quad \|Q(g)\| = \exp \frac{1}{4} \text{tr} |\log |\pi(g)||,
\]

\[
(8.20) \quad \min(\|Q(g) - 1\|, \|Q(g) + 1\|) \leq \|Q(g)\| - 1 + \frac{1}{2} \text{tr}|(\pi(g) |\pi(g)|^{-1})^{1/2} - 1|.
\]

Here the branch in $(8.18)$ is to be determined by the analytic continuation of $(\det [(1 + e^{iu})/2])^{1/2}$ from $z = 0$ to $1$ if $\pi(g) = e^u$ and if the continuous inverse image of the path $\{e^{iu}; 0 \leq u \leq 1\}$ ends at $g$. It is to be determined by the continuity for other $g$.

**Proof.** Let $\mathcal{S}$ be the set of trace class operators satisfying $(7.2)$, equipped with a topology induced by spheres $\{H'; \|H' - H\|_{tr} < \epsilon\}$. Let $\mathcal{S}_0$ be the set of $V \in \mathcal{S}_+$ such that $-1$ is not an eigenvalue of $V$.

For any $V \in \mathcal{S}_0$, we see from the Jordan expansion $(8.2)$ that the following $H_f(V)$ satisfies $(7.2)$ and $V = e^{iH_f(V)}$ for each path $f$ from $z = 0$ to $z = 1$ avoiding zeroes of $\det(1 + z(V - 1))$.

\[
(8.21) \quad H_f(V) = \frac{1}{2} \{H_0(V) - iH_0(V) * I\},
\]

\[
(8.22) \quad H_0(V) = \int_f (1 + z(V - 1))^{-1} (V - 1) dz.
\]
For $V$ in the sphere $|V-1|<1$, we can take $\gamma$ to be the interval $[0, 1]$ on real axis. Then $H_{\gamma}(V)\rightarrow V$ is a one-to-one homeomorphism of an open neighbourhood of 0 in $\Sigma$ onto an open neighbourhood of 1 in $\mathcal{F}_+$.

Let $V\in \mathcal{F}_+$. Let $V^*V=e^{\lambda}I$, $H_1\in \Sigma$, $H_1^*=H_1$. Let $|V| = \exp(1/2)H_1$, $U=V|V|^{-1}=e^{H_2}$, $H_2\in \Sigma$, $H_2^*=-H_2$. (Since $V$ is invertible, $U$ is unitary.) Let $V(z)=e^{H_2}|V|$. $V(z)\in \mathcal{F}_+$ for all complex number $z$. Since $V(z)$ is an entire function of $z$ and $V(0)$ does not have an eigenvalue $-1$, $\det(V(z)+1)=0$ has isolated roots. Hence $\mathcal{F}_0$ and $\exp \Sigma$, which contains $\mathcal{F}_0$, are dense in $\mathcal{F}_+$.

For $H\in \Sigma$, define
\[
(8.23) \quad \hat{Q}(H) = \exp \frac{1}{2} (B, HB).
\]

From Lemma 4.3, we have
\[
(8.24) \quad \hat{Q}(H)B(f)\hat{Q}(H)^{-1}=B(e^Hf).
\]

Obviously $\hat{Q}(H)^* = \hat{Q}(H^*)$.

Let $H\in \Sigma$ be selfadjoint. Then, in the Jordan expansion $H=H_d + \sum_{k=0}^{d-1} E_\lambda \lambda$, $E_+ = \sum_{\lambda>0} E_\lambda$ is a partial basis projection and $\hat{Q}(H)$ belongs to $\overline{\mathcal{U}_SD}(K', \Gamma')$ for $K'=(E_+ + \Gamma E_+ \Gamma')K$. By identifying $\overline{\mathcal{U}_SD}(K', \Gamma')$ with $\overline{\mathcal{U}_GAR}(E_+ K)$, $(B, HB)=2(a', H_+ a) - \text{tr} H_+$ where $H_+=HE_+$ and $(B, H+B)=(a', H_+ a)$ is $Q_\phi(H_+)$ in the notation of [4]. By using the formula (12.3) of [4] with $\rho=1/2$, we have
\[
(8.25) \quad \varphi_{1/2}(Q(H)) = \exp \left\{ \text{tr} \log \left[ (1+e^{H_+})/2 \right] - \frac{1}{2} \text{tr} H_+ \right\}
\]
\[
= \exp \frac{1}{2} \text{tr} \log \cosh(H_+/2)
\]
\[
= \exp \frac{1}{2} \text{tr} \log \left[ (1+e^{H_+})/2 \right] + \log \left[ (1+e^{-H_+})/2 \right]
\]
\[
= \exp \frac{1}{2} \text{tr} \log \left[ (1+e^{H})/2 \right].
\]

Note that the central state of $\overline{\mathcal{U}_SD}(K', \Gamma')$ is the same as the restriction of the central state of $\overline{\mathcal{U}_SD}(K, \Gamma)$ to $\overline{\mathcal{U}_SD}(K', \Gamma')$.

Let $V(z)$ be holomorphic in $z$ and $V(z)\in \mathcal{F}_+$. Then
\[(8.26) \quad \text{det} \left( \frac{1+V}{2} \right) = \exp \text{tr} \log \left( \frac{1+V}{2} \right) \]

for $V=V(z)$ is holomorphic in $z$ and have zeros of an even order unless it is identically 0. Hence its square root is locally holomorphic at every $z$. We define

\[(8.27) \quad f(H) = \exp \frac{1}{2} \text{tr} \log \left( \frac{1+e^{i\theta}}{2} \right), \]

where the value is the analytic continuation from $f(0)=1$ and does not depend on the path of the analytic continuation.

By setting $H=H_1+zH_2$ in (8.25) and making an analytic continuation from real $z$ to $z=i$, we have

\[(8.28) \quad \varphi_{1/2}(\hat{Q}(H)) = f(H) \]

for all $H \in \Sigma$. If $H \in \Sigma$ is selfadjoint, we have from (7.21)

\[(8.29) \quad \|Q(H)\| \leq \exp \frac{1}{2} \|B, HB\| = \exp \frac{1}{4} \text{tr} |H|. \]

On the other hand, we can consider the Fock state $\varphi_{E_{E'}}$ of $\overline{\mathbb{S}}_{\text{Soc}}(K', \Gamma')$ and use (12.3) of [4] with $\rho=1$, we have

\[(8.30) \quad \varphi_{E_{E'}}(\hat{Q}(H)) = \exp \frac{1}{2} \text{tr} H_s - \exp \frac{1}{4} \text{tr} |H|. \]

Therefore

\[(8.31) \quad \|\hat{Q}(H)\| = \exp \frac{1}{4} \text{tr} |H|. \quad (H^s=H) \]

Since $\hat{Q}(H)^{-1} = \hat{Q}(-H)$ and $\hat{Q}(H) > 0$, we have

\[(8.32) \quad \exp \frac{1}{4} \text{tr} |H| \geq \hat{Q}(H) \geq \exp -\frac{1}{4} \text{tr} |H|. \quad (H^*=H) \]

Let $iH \in \Sigma$ be selfadjoint. Then $\hat{Q}(H)$ as well as $e^a$ are unitary. If $E$ is a one dimensional partial basis projection and $H=i\lambda(E-E^*E^T)$, then $(B, HB)$ has the spectrum $\{i\lambda, -i\lambda\}$ and hence $\|\hat{Q}(H) - 1\| = \frac{1}{2} \text{tr} |e^{i\lambda/2} - 1|$. If $H=\sum H_i$, $H_i^* = -H_i$, $H_iH_i = H_iH_i = 0$, then $\hat{Q}(H) = \Pi\hat{Q}(H_i)$, each $\hat{Q}(H_i)$ is unitary and $\|\hat{Q}(H) - 1\| \leq \sum \|\hat{Q}(H_i) - 1\|.$
(Here we have used $\hat{Q}(H) - 1 = \sum_j (\Pi_{k>j} \hat{Q}(H_k))(\hat{Q}(H_j) - 1)$.) Hence we have

\[(8.33) \quad \|\hat{Q}(H) - 1\| \leq \frac{1}{2} \text{tr}|e^{it/2} - 1|, \quad (H^* = -H)\]

Let $V = e^{it} = e^{it} \in \mathcal{S}_r, \ H, H' \in \Sigma$. We show that $\hat{Q}(H) = \pm \hat{Q}(H')$. Let $r > 0$ be sufficiently small such that $\lambda + 2n\pi \neq x$ for any $n, 0 < |x| \leq r$ and $|\lambda| > r$ in the following Jordan expansion of $H$:

\[(8.34) \quad H = \sum_{|\lambda| > r} E_{\lambda}(H)(\lambda + N_{\lambda}(H)) + H_r.\]

Define

\[(8.35) \quad E_{\lambda}(H_0) = \sum_{\lambda \neq \pi} E_{\lambda + 2\pi \lambda}(H) \quad (\lambda \neq \pi i), \]

\[(8.36) \quad N_{\lambda}(H_0) = \sum_{\lambda \neq \pi} N_{\lambda + 2\pi \lambda}(H) \quad (\lambda \neq \pi i), \]

\[(8.37) \quad E_{\lambda^i}(H_0) = \sum_{\lambda \neq \pi} E_{\lambda + 2\pi \lambda^i}(H), \]

\[(8.38) \quad N_{\lambda^i}(H_0) = \sum_{\lambda \neq \pi} N_{\lambda + 2\pi \lambda^i}(H), \]

\[(8.39) \quad H_0 = \sum_{|\lambda| > r} E_{\lambda}(H_0)(\lambda + N_{\lambda}(H_0)) + H_r + \sum_{n=0} N_{2\pi n}(H_0),\]

where the sum $\sum'$ is over $\lambda$ such that $|\text{Im}\lambda| \leq \pi$ and $\lambda \neq (\pi i - \rho), \rho > 0$. Similar definitions are made for $H', H_0', H^0, H^0'$.

If $E$ is of a finite rank, $E^*rE = E^*E^* = 0$ and $E^2 = E$, then we have

\[(8.40) \quad f \left[2\pi z i (E - \Gamma E^* \Gamma')\right] = (\cos \pi z) \dim E.\]

Hence $\hat{Q}(2\pi i (E - \Gamma E^* \Gamma')) = (-1)^{\dim E}$. If $H_1$ and $H_2$ commute with each other, we have

\[(8.41) \quad \hat{Q}(H_1) \hat{Q}(H_2) = \hat{Q}(H_1 + H_2).\]

(8.40) and (8.41) implies

\[(8.42) \quad \hat{Q}(H) = \pm \hat{Q}(H_0), \quad \hat{Q}(H') = \pm \hat{Q}(H'_0).\]

From $e^{it} = e^{it'}$, we have $e^{it} = e^{it'}$ and hence

\[(8.43) \quad H_0 - H_0 = H_0' - H_0', \]

\[(8.44) \quad H_0 = \pi i [E_{\lambda^i}(H_0) - E_{\lambda^i}(H_0')], \]

\[(8.45) \quad H_0' = \pi i [E_{\lambda^i}(H_0') - E_{\lambda^i}(H_0)].\]
(8.46) \[ E_{\eta_1}(H_0) + E_{\eta_2}(H_0) = E_{\eta_1}(H_0') + E_{\eta_2}(H_0'). \]

Since \([H_0, H_0'] = [H_0', H_0] = 0\), we would obtain \(\hat{Q}(H) = \pm \hat{Q}(H')\) by (8.41), (8.42) and (8.43) if we can prove \(\hat{Q}(H_0) = \pm \hat{Q}(H_0')\).

From (8.24), \(\hat{Q}(H_0)\hat{Q}(H_0')^{-1}\) commutes with all \(B(f)\). The algebra \(\mathfrak{sl}_{2\mathbb{C}}(K, \tau)\), which is isomorphic to CAR \(C^*\) algebra, is known to be simple [8] and in particular has a trivial center. Therefore \(\hat{Q}(H_0)\hat{Q}(H_0')^{-1} = c\mathbf{1}\) for some complex number \(c\).

On the other hand
\[
\hat{Q}(H_0)^2 = \hat{Q} \left[ 2\pi i (E_{\eta_1}(H_0) - E_{\eta_2}(H_0)) \right] = (-1)^{\dim E_{\eta_1}(H_0)}
\]
and \(\hat{Q}(H_0')^2 = (-1)^{\dim E_{\eta_1}(H_0')}\). From (8.46), \(\dim E_{\eta_1}(H_0) = \dim E_{\eta_2}(H_0')\) and hence \(\hat{Q}(H_0)^2 = \hat{Q}(H_0')^2\). Therefore \(c^2 = \pm 1\) and \(\hat{Q}(H_0) = \pm \hat{Q}(H_0')\). This completes the proof of \(\hat{Q}(H) = \pm \hat{Q}(H')\).

From the above argument, \(\hat{Q}(H_{\tau}(V)) = \pm \hat{Q}(H_{\tau'}(V))\) for any \(\tau\) and \(\tau'\) where \(H_{\tau}(V)\) is defined by (8.21).

Let \(e^{i\epsilon_1}, e^{i\epsilon_2} \in \mathcal{T}_0\) and \(V(z) = e^{i\epsilon_1}e^{i\epsilon_2}\). Let \(z(t), 0 \leq t \leq 1\) be a path between \(z(0) = 0\) and \(z(1) = 1\) avoiding zeroes of \(\det(V(z) + 1)\). For each \(0 \leq t \leq 1\), there exists an open interval \(I_t\) containing \(t\) and a fixed path \(\tau_t\) such that \(\det(1 + z'(V[z(t')] - 1)) \neq 0\) for \(z' \in \tau_t\) and \(t' \in I_t\). \(H_{\tau_t}(V(z))\) defined by (8.21) is holomorphic in \(z\) at \(z(t')\), \(t' \in I_t\). The equality between two analytic functions of \(z\)

(8.47) \[ \hat{Q}(H_{\tau_t}(V(z))) = \pm \hat{Q}(zH_1)\hat{Q}(zH_2) \]

hold for all \(z = z(t'), t' \in I_t\), if it holds for \(z = z(t), t\) in some dense subset of an open interval in \(I_t\), by the continuity and an analytic continuation.

The formula (8.47) holds with the plus sign if \(|z|\) is sufficiently small by the Baker Haussdorf formula and Lemma 7.3. Since \([0, 1]\) is covered by a finite number of \(I_t\), (8.47) holds for all \(z = z(t), 0 \leq t \leq 1\). Hence, if \(e^{i\epsilon_1}e^{i\epsilon_2} = e^{i\epsilon} \in \mathcal{T}_0\), we have

(8.48) \[ \hat{Q}(H_{\tau_1})\hat{Q}(H_2) = \pm \hat{Q}(H). \]

Let \(V = e^{i\epsilon} \in \mathcal{T}_0\). Then
\[(8.49) \quad \hat{Q}(H) = \pm \hat{Q}(H_2) \hat{Q}(H_1)\]

where \(|V| = e^{|t|}, V|V|^{-1} = e^{|t|} \). Since \(\hat{Q}(H_2)\) is unitary, we obtain from (8.31)

\[(8.50) \quad \|\hat{Q}(H)\| = \exp \frac{1}{2} \operatorname{tr} \log |V|\]

where \(\log |V|\) is hermitian.

We also have

\[
\min(\|\hat{Q}(H) - 1\|, \|\hat{Q}(H) + 1\|) \\
\leq \|\pm \hat{Q}(H_2) \hat{Q}(H_1) - 1\| \\
\leq \|\hat{Q}(H_1) - 1\| + \|\mp \hat{Q}(H_2) - 1\|.
\]

Hence we obtain from (8.32) and (8.33)

\[(8.51) \quad \min(\|\hat{Q}(H) - 1\|, \|\hat{Q}(H) + 1\|) \\
\leq \|\hat{Q}(H)\| - 1 + \frac{1}{2} \operatorname{tr} |e^{\hat{Q}(H_2)} - 1|.
\]

Let \(V_* = e^{n_*} \in \mathcal{T}_*, H_* \in \Sigma\) and \(\lim \|V_* - V\|_{tr} = 0, V \in \mathcal{T}_*\). Then \(\operatorname{tr} |\log |V_*||\) is bounded. Since \(\|V_* V_*^{-1} - 1\|_{tr} \to 0\) as \(n, m \to \infty\), there exists \(\varepsilon_* = \pm 1\) such that

\[(8.52) [\varepsilon_* \hat{Q}(H_*)] [\varepsilon_* \hat{Q}(H_{**})]^{-1} \to 1\]

as \(n, m \to \infty\) due to (8.51). Since \(\|\hat{Q}(H_*)\|\) is bounded due to (8.50), we have

\[(8.53) \quad \|\varepsilon_* \hat{Q}(H_*) - \varepsilon_* \hat{Q}(H_{**})\| \to 0,\]

as \(n, m \to \infty\). Hence there exists a limit of \(\varepsilon_* \hat{Q}(H_*)\) as \(n \to \infty\). The limit does not depend on the choice of \(H_*\) and \(\varepsilon_*\) except for a factor \(\pm 1\). We shall write the limit as \(Q(V, \varepsilon)\) where \(\varepsilon = \pm 1\) and \(Q(V, 1) = -Q(V, -1)\). The properties \(Q(S)^* = Q(S^*)\), (8.24), (8.28), (8.48), (8.50) and (8.51) extends to \(Q(V, \varepsilon)\) by the continuity.

Let \(\mathcal{S}_*\) be the abstract group with an involution \(*\), which is * isomorphic to the group of operators \(Q(V, \varepsilon)\), \(Q\) be the * isomorphism from \(g \in \mathcal{S}_*\) to the corresponding \(Q(V, \varepsilon)\) and \(\pi(g) = V\) if \(Q(g) = Q(V, \varepsilon)\). From (8.24), \(Q(V, \varepsilon_1) = Q(V_2, \varepsilon_2)\) only if \(V_1 = V_2\). Hence \(\pi\) is well defined. From \(Q(V, \varepsilon)^* = \pm Q(V^*, \varepsilon')\) and (8.48) for \(Q(V, \varepsilon)\), \(\pi\) is a * homomorphism. From (8.51), it is a homeomorphism. Since
Quasifree States of CAR and Bogoliubov Automorphisms

Theorem 5. Let $K$ be an infinite dimensional Hilbert space. A Bogoliubov transformation $\tau(U)$ of $\mathfrak{A}_{SDC}(K, \Gamma)$ is inner if and only if $U \in \mathcal{S}_+$ or $-U \in \mathcal{S}_-$. 

Proof of "if" part.

If $U \in \mathcal{S}_+$, then $Q(\pi^{-1}U)$ is unitary and induces the desired automorphism $\tau(U)$ on $\mathfrak{A}_{SDC}(K, \Gamma)$.

If $U_1 \in \mathcal{S}_{a_1}$, $U_2 \in \mathcal{S}_{a_2}$, then $U_1 U_2 \in \mathcal{S}_{a_1 a_2}$. Hence it is enough to show that $\tau(U)$ is inner for at least one unitary $U$ in $\mathcal{S}_-$. Let $\{e_a\}$ be a $\Gamma$ invariant orthonormal basis of $K$. A complete orthogonal family of $\Gamma$ invariant vectors $f_a$ can be obtained inductively by picking up $f$ in $\{f_{a}; \beta < a_0\}$ and defining $f_{a+1} = f + \Gamma f$, $f_{a+1} = i(f - \Gamma f)$.

Let $U$ be defined by the requirement of linearity, boundedness and $U e_a = e_a$, $U e_{-a} = -e_a$ for all $a \neq 0$. Then $-U \in \mathcal{S}_-$ and $\tau(U)$ is implementable by $\sqrt{2} B(e_0)$. Q.E.D.

To prove the "only if" part, we need some preparations.

Let $K_a$ be a $\Gamma$ invariant finite even dimensional subspace of $K$, $\mathfrak{A}_a = \mathfrak{A}_{SDC}(K_a, \Gamma)$ and $\mathfrak{A}_a$ be the set of elements in $\mathfrak{A}_{SDC}(K, \Gamma)$ commuting with every element of $\mathfrak{A}_a$. We know the following properties.

(a) $\cup \mathfrak{A}_a$ is dense in $\mathfrak{A}_{SDC}(K, \Gamma)$.
(b) Let $\varphi_1$ and $\varphi_2$ be states of $\mathfrak{A}_a$ and $\mathfrak{A}_a^\perp$.

Then there exists a state $\varphi$ of $\mathfrak{A}_{SDC}(K, \Gamma)$ such that $\varphi|_{\mathfrak{A}_a} = \varphi_1$, $\varphi|_{\mathfrak{A}_a^\perp} = \varphi_2$.

Property (b) follows from the fact that $\mathfrak{A}_a$ is a full matrix algebra.

Lemma 8.4. Let $U$ be a unitary element of $\mathfrak{A}_{SDC}(K, \Gamma)$. Then there exists a unitary $V$, in $\mathfrak{A}_a$ such that $\lim_{n \to \infty} V - U = 0$.

Proof. From (a), there exists $A_n \in \mathfrak{A}_a$ such that $\lim_{n \to \infty} A_n - U = 0$. Let $|| A_n - U || = \varepsilon_n$. Then $|| A_n || \leq || U || + \varepsilon_n = 1 + \varepsilon_n$ and $|| A_n^* A_n - 1 || \leq || A_n^* - U^* || || A_n || + || U^* || || A_n - U || \leq \varepsilon_n (2 + \varepsilon_n)$.

Hence $|| A_n ||^{-1} \leq [1 - \varepsilon_n (2 + \varepsilon_n)]^{-1/2}$ and $|| A_n ||^{-1} - 1 \leq [1 - \varepsilon_n (2 + \varepsilon_n)]^{-1/2} - 1$ provided that $\varepsilon_n (2 + \varepsilon_n) \leq 1$. ($|| A_n || = (A_n^* A_n)^{1/2}$.)
Let $V_n = A_n A_n^{-1}$. Then $V_n$ is isometric and
\[
\|V_n - U\| \leq \|A_n - U\| + \|A_n^{-1} - 1\| \\
\leq \epsilon_n \left[1 - \epsilon_n (2 + \epsilon_n)\right]^{-1/2} + \left[1 - \epsilon_n (2 + \epsilon_n)\right]^{-1/2} - 1 \to 0.
\]
Therefore $\lim\|V_n - U\| = 0$.

We now have $\lim\|V_n V_n^* - 1\| = 0$ due to $UU^* = 1$. Since $V_n V_n^*$ is a projection, $\|V_n V_n^* - 1\| = 1$ unless $V_n V_n^* = 1$. Hence $V_n$ is unitary for sufficiently large $n$. Q.E.D.

**Lemma 8.5.** Let $U$ be a unitary element of $\mathcal{F}_{\text{SDC}}(K, \Gamma)$ and $U_n$ be unitary element in $\mathfrak{A}_n$ such that $UAU^{-1} = U_n A U_n^{-1}$ for all $A \in \mathfrak{A}_n$. Then there exists a complex number $\lambda_n$ such that $|\lambda_n| = 1$ and $\lim \lambda_n U_n = U$.

**Proof.** Let $U_n^o = U_n^{-1} U$. We have $U_n^o \in \mathfrak{A}_n$. Let $V_n$ be as given by Lemma 8.4. We then have
\[
\lim\|U_n^o - U_n^{-1} V_n\| = 0.
\]

Let $\varphi_{1n}$ and $\varphi_{2n}$ be states of $\mathfrak{A}_n$ and $\mathfrak{A}_n^c$ and let $\varphi_n$ be a state of $\mathcal{F}_{\text{SDC}}(K, \Gamma)$ such that $\varphi_n|\mathfrak{A}_n = \varphi_{1n}$, $\varphi_n|\mathfrak{A}_n^c = \varphi_{2n}$. We have
\[
\sup_{\varphi_{2n}} |\varphi_{2n}(U_n^o) - \varphi_{1n}(U_n^{-1} V_n)| \\
\leq \sup_{\varphi} |\varphi(U_n^o - U_n^{-1} V_n)| = \|U_n^o - U_n^{-1} V_n\| \to 0.
\]

Let $\lambda_n = \varphi_{1n}(U_n^{-1} V_n)$ for a fixed sequence $\varphi_{1n}$. Then
\[
\|U_n^o - \lambda_n\| = \sup_{\varphi_{2n}} |\varphi_{2n}(U_n^o - \lambda_n)| \to 0.
\]
Therefore $\lim\|U - \lambda_n U_n\| = 0$. Q.E.D.

**Proof of "only if" part of Theorem 5.** Let $U$ be a Bogoliubov transformation which can be implemented by a unitary $W$ in $\mathcal{F}_{\text{SDC}}(K, \Gamma)$: $WAW^* = \tau(U) A$. Any inner *-automorphism is unitarily implementable in any representation. From Theorem 8, we see that $U-1$ or $U+1$ is in the Hilbert Schmidt class. In either case, $U$ has a purely discrete spectrum.

First consider the case where multiplicities of eigenvalues 1 and $-1$ of $U$ are not odd. Then there exists $\Gamma'$ invariant finite even dimen-
sional spectral projections $E_\pm$ of $U$ such that $\lim E_\pm = 1$. Let $U_\pm = E_\pm U$. Let $W_\pm$ be a unitary element of $\mathcal{U}_{\text{SDO}}(E_\pm K, \Gamma')$ such that $W_\pm AW_\pm^* = \tau(U_\pm)A$ for $A \in \mathcal{U}_{\text{SDO}}(E_\pm K, \Gamma')$. By Lemma 8.5, there exists complex numbers $\lambda_\pm$ such that $\lim \lambda_\pm W_\pm = W$.

Let $U = e^{it\theta}$, $\|H\| \leq \pi$, $H^* = H$, $\Gamma H \Gamma = -H$, and $E_\pm$ be a basis projection such that $[E_\pm, H] = 0$, $E_\pm H \leq 0$. If multiplicities of eigenvalues 1 and $-1$ of $U$ are not odd, such $H$ and $E_\pm$ exists. Let $E_- = 1 - E_+$, $H_\pm = E_\pm H$.

By Lemma 8.3, $W_\pm$ is proportional to $Q(iE_\pm H)$ and by Lemma 9.2,

$$\varphi_{E_\pm}(W_\pm) = c \exp(i/2) \text{tr}(E_\pm H_\pm),$$

where $|c| = 1$ is common for two equations. Since $[U, E_{\pm}] = 0$, $W_{E_{\pm}}$ must be a multiple of $\varphi_{E_{\pm}}$ by Lemma 4.3. Therefore

$$\lim_{\pm \to \infty} \lambda_\pm \exp(i/2) \text{tr}(E_\pm H_\pm) = c'$$

where $c'$ is common for $\pm$. This implies

$$\lim \exp(i/2) \text{tr}E_\pm(H_+ - H_-) = 1.$$

From $\Gamma H \Gamma = -H_-$, $[\Gamma, E_\pm] = 0$, we have $\text{tr} E_\pm H_- = -\text{tr} E_\pm H_+$. Therefore

$$\lim \exp i \text{tr} E_\pm H_- = 1.$$

Since $0 \leq H_\pm \leq \pi$ and $E_\pm$ can be chosen to pick up (an increasing sequence of) any finite number of eigenvectors of $H_\pm$, this implies that $H_\pm$ must be in the trace class. Therefore $U \in \mathcal{T}_+$ in this case.

In order to consider a general case, we again use Theorem 8. If both $\dim E_1$ and $\dim E_{-1}$ are finite, then either 1 or $-1$ is an accumulation point of the spectrum of $U$. Then there exists a Bogoliubov transformation $U' \in \mathcal{T}_+$ which commutes with $U$ such that $UU'$ has an infinite multiplicity for an eigenvalue 1 or $-1$. Since we know already that $U'$ is inner, it is sufficient to consider the case where either $\dim E_1$ or $\dim E_{-1}$ is infinite.

We now consider a case where the dimension of the eigenprojection $E_1$ of $U$ for an eigenvalue 1 is finite and odd. Let $Q_{E_1}(-1)$
\[ = \prod_i \{ \sqrt{2} B(f_i) \} \text{ where } \{f_i\} \text{ is any complete orthonormal set of } \Gamma \text{ invariant vectors in } E, \]

Then \( Q_{\chi}(\chi) \) is unitary and implement the Bogoliubov automorphism \( \tau(U_i) \) for \( U_i \) which is 1 on \( E, K \) and \(-1\) on \( (1-E, K) \). Since \( UU_i \) has no eigenvalue \(-1\) and an infinite multiplicity for an eigenvalue 1, \( \tau(UU_i) \) is inner only if \( UU_i \in \mathcal{T}_+ \).

Since \(-U_i \in \mathcal{T}_- \), this implies \(-U \in \mathcal{T}_- \).

Finally we consider a case where the dimension of the eigenprojection \( E, \) of \( U \) for an eigenvalue \(-1\) is finite and odd. As before \( \tau(U_i) \) is inner for \( U_i \) which is 1 on \( E, K \) and \(-1\) on \( (1-E, K) \). Since \( UU_i \) has no eigenvalue 1 and an infinite multiplicity for an eigenvalue \(-1\), it is not inner.

Q.E.D.

§9. Unitary Implementable Bogoliubov Transformations

Lemma 9.1. Let \( P \) and \( P' \) be basis projections. Let \( \sin \theta = |P-P'| \), \( 0 \leq \theta \leq \pi/2 \). Let \( E_{\pi/2} = P \wedge (1-P') + (1-P) \wedge P' \), \( E_0 = P \wedge P' + (1-P) \wedge (1-P') \). Let

\[ F_+ = \frac{1}{2} (1-E_{\pi/2} - E_0 \pm i (\sin \theta \cos \theta)^{-1} [P, P']) , \]

\[ F_- = \theta (F_+ - F_-). \]

Let \( e_1 \cdots e_n \) be an orthonormal basis of \( \{ P \wedge (1-P') \} K \ (n \leq \infty) \) and \( U \) be a unitary operator, determined by the requirement that \( Ue_i = \chi e_i \), \( UU e_i e_j = e_i \), \( Uf = f \) for \( f \in (1-E_{\pi/2}) K \). Assume that \( |P-P'| \) is in the trace class, Let

\[ \hat{R}(P'/P) = e^{iH(P'/P)} U. \]

\[ Q = \left\{ \exp \left( \frac{i}{2} (B, H(P'/P)B) \right) \prod_{j=1}^n (B(e_j) - B(\chi e_j)) \right\}. \]

Then \( \hat{R}(P'/P) \in \mathcal{T}_+^\sigma, \sigma = (-1)^n, Q \) is unitary and

\[ \hat{R}(P'/P) P \hat{R}(P'/P)^* = P', \]

\[ QB(f) Q^* = B((-1)^n \hat{R}(P'/P)f), \]

\[ \varphi_r(QAQ^*) = \varphi_r(A) \]

\[ \varphi_r(Q) = (\det \cos \theta)^{1/4}, \]
where the positive quartic root is taken and $A \in \mathcal{U}_{SDC}(K, \Gamma)$.

**Proof.** Since $(P - P')^2$ commutes with $P$ and $P'$, $\theta$ commutes with $P$ and $P'$. $\theta$ also commutes with $\Gamma$. $E_0$ and $E_{\pi/2}$ are spectral projections of $\theta$ for the eigenvalues $0$ and $\pi/2$. From $[P, P']^2 = -\sin^2 \theta \cos^2 \theta$, it follows that $F_+ F_- = F_- F_+ = 0$. Because $P$ and $P'$ are basis projections, $\Gamma F_\pm \Gamma = F_\pm$. Namely $F_\pm$ are partial basis projections.

If $|P - P'|$ is in the trace class, then $\theta$ is also in the trace class and hence $H(P'/P) \in \mathcal{S}_+$. $Q$ is unitary. $(9.6)$ is immediate for $f \in (1 - E_{\pi/2})K$, $f = e_i$ and $f = \Gamma e_i$ and hence for all $f$. From $(9.5)$ and $(9.6)$, we have

$$
\phi_p(Q^* A Q) = \phi_p(\tau( (-1)^s \hat{R}(P'/P)^*) A)
= \phi_p(\tau(\hat{R}(P'/P)^*) A) = \phi_{p, \theta}(A).
$$

By Definition 3.1, $(3.3)$ and $(g, Pe_k) = (g, Pf) = 0$ for $g = e_i$ and $f = \Gamma e_i$, $k \neq j$ and $f \in (1 - E_0)K$, we have for $n \neq 0$,

$$
\phi_p(Q) = 0.
$$

Since $Q \in \mathcal{U}_{SDC}(1 - E_0)K, \Gamma'$, we can compute $(9.8)$ by using the Fock state $\phi_{p(1 - E_0)}$ on $\mathcal{U}_{SDC}(1 - E_0)K, \Gamma'$. Hence we may assume $E_0 = 0$ without loss of generality. If $n = 0$ and $E_0 = 0$, there exists a basis projection $E$ of $(K, \Gamma')$ commuting with $P$ and $P'$ and a unitary operator $u$ such that $[u, P] = [u, P'] = 0$ and $u E u^* = 1 - E$, due to Lemma A. Then it follows that $\text{tr} E H(P'/P) = (1/2) \text{tr} H(P'/P) = 0$.

We can identify $(B, H(P'/P)B)$ in $\mathcal{U}_{SDC}(1 - E_0)K, \Gamma'$ with $2(a^t, E H(P'/P)a)$ in $\mathcal{U}_{GAR}(E K)$ and use the formula for $\langle e^x \rangle$ in the Appendix C of [4] where $K = \text{i} H(P'/P)E$ ($(a^t K a)$ is written as $[K]$ in [4]) and $\rho = (1 - P)E$. We have

$$
\phi_p(Q) = \exp \{ \text{tr} \log (1 + (e^x - 1) \rho) \}
= \exp \text{tr}_e \log (1 + (1 - P) (e^x - 1) (1 - P))
= \exp \text{tr}_{e(1 - P)} \log (\cos \theta)
= \exp \frac{1}{4} \text{tr} \log (\cos \theta) = \det (\cos \theta)^{1/4}.
$$
where the positive root is to be taken.

Q.E.D.

Lemma 9.2. Let \( g \in T_+ \) and \( P \) be a basis projection. Then

\[
(9.12) \quad \varphi_r(Q(g)) = \text{det}_r \left( P \pi(g)^{-1} P \right)^{1/2},
\]

where \( \text{det}_r \) is the determinant taken on the space \( PK \), the branch of the square root is to be determined by an analytic continuation from the value 1 for \( g = 1 \) and the continuity.

Proof. First we consider the case where \( \pi(g) = e^{it} \), \( H^* = H \), \( \gamma H* \gamma = -H \), \( H \) is in the trace class and the continuous inverse image \( \pi^{-1}(e^{it}) \), \( 0 \leq t \leq 1 \) connects 1 and \( g \). Then \( \pi(g) \) is a Bogoliubov transformation. Let

\[
(9.13) \quad P' = \pi(g) P \pi(g)^*.
\]

which is again a basis projection. Since \( H \) is in the trace class

\[
\| P' - P \|_r \leq \sum_{n=1}^{\infty} (n!)^{-1} \left\| \prod_{n=1}^{\infty} [H, P] \right\|_r \leq \sum_{n=1}^{\infty} (n!)^{-1} 2^n \| H \|_r \| H \|^n \| P \| < \infty.
\]

Let

\[
(9.14) \quad \widehat{R}(P/P')^* \pi(g) = V.
\]

\( V \) commutes with the basis projection \( P \) by (9.13) and (9.5). Since \( V \in T_+ \cup T_- \), this implies \( \text{det} V = +1 \) and hence \( V \in T_+ \). Let \( g' \in T_+ \) be such that \( \pi(g') = V \). Let \( V = e^{it} \) where \( H'^* = H' \), \( \gamma H' \gamma = -H' \), \( [P, H'] = 0 \). We then have \( Q(g') = \pm \exp \frac{i}{2} (B, H'B) \). Under the identification of \( \mathfrak{H}_{\text{SDG}}(K, \gamma) \) with \( \mathfrak{H}_{\text{OAR}}(PK) \), \( (B, H'B) = 2(\alpha^*, H'Pa) - \text{tr}(H'P) \). Therefore

\[
(9.15) \quad \varphi_r(AQ(g')) = \pm \varphi_r(A) \exp -\frac{i}{2} \text{tr}(H'P).
\]

By substituting \( A = \widehat{R}(P/P') \), we obtain

\[
(9.16) \quad \varphi_r(Q(g)) = \pm \text{det} (\cos \theta)^{1/4} \exp -\frac{1}{2} \text{tr}(\log V) P
\]

\[= \pm \text{det}_r \left( (\cos \theta) V^{-1} P \right)^{1/2}.\]

Substituting \( \cos \theta V^{-1}P = P \cos \theta V^{-1}P \) and \( P \cos \theta = \widehat{R}(P/P')^*P \), we obtain
By absorbing ± to the ambiguity in the branch of square root, we obtain (9.12).

By substituting \( g_*(z) \) such that \( \pi(g_*) = \exp i(H_f^{(a)} + zH_s^{(a)}) \), \( H_f^{(a)*} = H_f^{(a)} \), \( \Gamma' H_f^{(a)} \Gamma' = -H_f^{(a)*} \), making analytic continuation in \( z \) from \( z = 0 \) to \( i \) and taking limit of \( n \to \infty \) in the trace class norm, one obtains (9.12) for most general \( g \).

Q.E.D.

Remark. The formula (9.12) can be also obtained by the following method. Consider the case where \( \pi(g) - 1 \) and \( S - (1/2) \) are of finite rank and \( S \) does not have eigenvalues 0 and 1. Then from (6.24), and (8.18), we have

\[
\varphi_s(Q(g)) = \varphi_{1/2}[Q(g) \exp -\frac{1}{2}(B, \log \{S(1-S)^{-1}\}B)] \det(2S)^{1/2} = \pm \det(2S)^{1/2} \exp -\frac{1}{2} \text{tr} \log [(1 + \pi(g)(1-S)S^{-1})/2] = \pm \exp -\frac{1}{2} \text{tr} \log [S + \pi(g)(1-S)].
\]

We can now allow \( S \) to take eigenvalues 0 and 1 and to be not of finite rank. (9.18) holds by continuity. If \( S \) is the projection \( P \), we have

\[
\varphi_s(Q(g)) = \det [P + (1-P)\pi(g)(1-P)]^{1/2}.
\]

Since \( \delta'(A^*A) = \det A \) and \( \Gamma \pi(g)^* \Gamma' = \pi(g)^{-1} \), we obtain (9.12).

Conversely (9.18) can be obtained from (9.12) by

\[
\varphi_s(Q(g)) = \det \rho_s(P_s \pi(g)^{-1}P_s)^{1/2} = \det (1 - P_s + P_s \pi(g)^{-1}P_s)^{1/2}
\]

where \( \pi(g) \) is understood as \( \pi(g_0) \) on \( \hat{\mathcal{A}} \). It can be checked easily that this coincides with the above expression.

Note that the formula (12.3) of [4] is a special case of (9.18), where \( S = 1 - \rho \).

Lemma 9.3. Let \( P \) be a basis projection, \( g_* \) be in \( \mathcal{T}_* \), \( V \) be a Bogoliubov transformation \( P_\pi(g_*) P_{\pi(g_*)}^* \), \( P' = VPV^* \). Assume that \( \pi(g_*) \) is unitary,
(9.20) \( P \wedge (1 - P') = 0, \)

(9.21) \( \lim_{n \to \infty} \pi(g_n) = V, \)

(9.22) \( \lim_{n \to \infty} \| P_n - P' \|_{\text{H.S.}} = 0, \)

where \( \| A \|_{\text{H.S.}} = \| A^* A \|_\text{tr}. \) Let \( z_n \) be such that

(9.23) \( \varphi_p(Q(g_n)) = z_n | \varphi_p(Q(g_n)) |. \)

Then

(9.24) \( Q_p(V) = \lim_{n \to \infty} \pi_p(Q(g_n)) z_n^{-1} \)

exists and does not depend on the sequence \( g_n \) for a given \( V. \) It satisfies

(9.25) \( Q_p(V^*) = Q_p(V)^* \)

(9.26) \( Q_p(V) \pi_p(B(f)) = \pi_p(B(V f)) Q_p(V) \)

(9.27) \( (\Omega_p, Q_p(V) \Omega_p) = \det_p(PP' P)^{1/4} > 0. \)

If \( H \in \sum \) and \( H^* = -H, \) then \( Q_p(e^H) = \hat{Q}(H) \det(P e^H P)^{-1/2} \)

\( | \det(P e^H P) |^{1/2}. \)

Proof. Since \( \pi(g_n) - 1 \) is in the trace class, \( (P_n - P) \) is in the trace class and hence is in the H.S. class. From (9.22) \( (P' - P) \) is also in the H.S. class. Hence \( P (1 - P') P = P (P' - P)^2 P \) is in the trace class, and \( (1 - P') P = (P - P') P \) is in the H.S. class. From (9.22), it follows

(9.28) \( \| (1 - P_n) P - (1 - P') P \|_{\text{H.S.}} = \| (P_n - P') P \|_{\text{H.S.}} \to 0. \)

Hence \( \| (1 - P_n) P \|_{\text{H.S.}} \) is uniformly bounded.

We now have

(9.29) \[ \| PP_n P - PP' P \|_{\text{tr}} = \| \{ P (1 - P_n) - P (1 - P') \} (1 - P_n) P \]

\[ + P (1 - P') \{ (1 - P_n) P - (1 - P') P \} \|_{\text{tr}} \leq \| (1 - P_n) P - (1 - P') P \|_{\text{H.S.}} \{ \| (1 - P_n) P \|_{\text{H.S.}} \}

\[ + \| (1 - P') P \|_{\text{H.S.}} \to 0. \]

We also note that (9.22) implies \( \| P_n - P \|_{\text{H.S.}} \to 0 \) as \( n, m \to \infty. \) Hence \( P_{nm} \equiv \pi(g_n) P \pi(g_m) \) satisfies \( \| P_{nm} - P \|_{\text{H.S.}} = \| P_n - P \|_{\text{H.S.}} \to 0 \) as \( n, m \to \infty. \) Therefore
(9.30) \[ \| PP_{nm} P - P \|_u = \| P (P_{nm} - P)^2 P \|_u \to 0 \]
as \( n, m \to \infty \).

Let

\[ Q_s = \chi^{-1}_s \pi_P(Q(g_s)), \quad \Psi_s = Q_s \Omega_P. \]

We obtain, from (9.29),

\[ \langle \Omega_P, \Psi_s \rangle = \{ \det \pi(P\pi(g_s)^{-1} P) \}^{1/2} \]
\[ = \{ \det \pi(P\pi(g_s) P) \det \pi(P\pi(g_s)^* P) \}^{1/4} \]
\[ = [ \det \pi(PP' P) ]^{1/4} \to [ \det \pi(PP' P) ]^{1/4} \]

and, from (9.30),

\[ (9.33) \quad | (\Psi_s, \Psi_m) | = \det \pi(PP_{nm} P)^{1/4} \to 1. \]

Due to (9.20), \( c = [ \det \pi(PP' P) ]^{1/4} \neq 0 \).

Let \( \exp i \theta_{nm} = (\Psi_s, \Psi_m) / | (\Psi_s, \Psi_m) | \). If \( | 1 - (\Psi_s, \Psi_m) | < \varepsilon^2 / 2 \), then \( \| (\exp i \theta_{nm}) \Psi_s - \Psi_m \| < \varepsilon \) and hence \( | e^{i \theta_{nm}} (\Omega_P, \Psi_s) - (\Omega_P, \Psi_m) | < \varepsilon \). If \( | (\Omega_P, \Psi_s) - c | < \varepsilon \) and \( | (\Omega_P, \Psi_m) - c | < \varepsilon \) in addition, then \( | e^{i \theta_{nm}} - 1 | c < 3 \varepsilon \) and hence \( \| \Psi_s - \Psi_m \| < (1 + 3/c) \varepsilon \).

Therefore \( \Psi_s \) is a Cauchy sequence and has a strong limit \( \Psi(V) \).

Let \( \Psi = \pi_P(A) \Omega_P, \ A \in \mathcal{A}_{S\mathcal{D}}(K, I) \). Then

\[ Q_s \pi_P(A) \Omega_P = \pi_P(\pi(g_s) A) Q_s \Omega_P \]

\[ \| Q_s \pi_P(A) \Omega_P - \pi_P(\pi(V) A) \Psi(V) \| \]
\[ \leq \| A \| \| \Psi_s - \Psi(V) \| + \| \pi(g_s) A - \pi(V) A \| \to 0, \]

where (9.21) is used for the second term. Hence \( Q_s \) has a strong limit \( Q_s(V) \), which satisfies (9.27) due to (9.32). It also satisfies

\[ (9.35) \quad Q_s(V) \pi_P(A) \Omega_P = \pi_P(\pi(V) A) Q_s(V) \Omega_P. \]

(9.27) implies \( Q_s(V) \Omega_P \neq 0 \). Since \( \pi_P \) is irreducible, (9.35) implies that the range of \( Q_s(V) \) is the whole space. As a strong limit of unitary \( Q_s \), \( Q_s(V) \) is isometric and hence is unitary. (9.35) implies (9.26), which uniquely determines the unitary operator \( Q_s(V) \) up to a multiplicative constant for a given \( V \). The constant is unique due to (9.27). Hence \( Q_s(V) \) does not depend on the sequence.

(9.26) and (9.27) are satisfied when \( Q_s(V^*)^* \) is substituted into
Lemma 9.4. Let $P$ be a basis projection, $V$ be a Bogoliubov transformation and $P'=V^*PV$. If $(P'-P)$ is in the Hilbert Schmidt class, $V$ is unitarily implementable in the Fock representation $\pi_P$.

Proof. $E_{\pi_P}=P\wedge(1-P')+(1-P)\wedge P'$ is the spectral projection of $(P-P')^2$ for an eigenvalue 1 and hence has a finite dimension. Let $e_1\cdots e_s$ be an orthonormal basis of $(P\wedge(1-P'))K$. Let $U$ be a unitary operator determined by the requirements $Ue_j=re_j$, $Ue'_j=e_j$, $Uf=f$ for $f\in(1-E_{\pi_P})K$. Then $U$ is a Bogoliubov transformation such that $U-1$ is of finite rank. We have $\det U=(-1)^s$. Hence $\tau((-1)^sU)$ is inner and hence is unitarily implementable.

We now consider $V_1=Vu$, $P''=V_1^*PV_1$. Then $v=|P''-P|$ is in the H.S. class and $P\wedge(1-P'')=(1-P)\wedge P''=0$. There exists a monotonically increasing sequence of a finite dimensional spectral projection $E_\alpha$ of $v$ such that $\lim E_\alpha=1-E_\alpha$ where $E_\alpha$ is the eigenprojection of $v$ for an eigenvalue 0. Consider $R(P''/P)=(1-v^2)^{1/2}-(1-v^2)^{-1/2} \cdot [P, P'']$. Then consider $U_\alpha=(1-E_\alpha)+R(P''/P)E_\alpha$. We have

$$\|U_\alpha-R(P''/P)\|_{\text{H.S.}} \leq \|(1-E_\alpha)(1-v^2)^{1/2}-(1-v^2)^{-1/2} [P, P'']\|_{\text{H.S.}} \rightarrow 0$$

where $E_\alpha$ commutes with $P$ and $P''$ and $[P, P''] \|_2=(1-v^2)v^2$ is in the trace class. Hence there exists $Q_P(R(P''/P))$ on $H_P$ which implements $\tau(R(P''/P))$.

We now consider $V_2=VUR(P''/P)$. It commutes with $P$ and hence $\varphi_P$ is invariant under $\tau(V_2)$. Hence it is unitarily implementable in $\pi_P$. Q.E.D.

Theorem 6. Two Fock states $\varphi_P$ and $\varphi_{P'}$ are unitarily equivalent if and only if $(P-P')$ is in the Hilbert Schmidt class.

Proof. First assume that $P-P'$ is in the Hilbert Schmidt class. Then there exists a Bogoliubov transformation $V$ bringing $P'$ to $P$, which is unitarily implementable by Lemma 9.4.

Now assume that $P-P'$ is not in the Hilbert Schmidt class. Then
$(P-P')^2$ is not in the trace class. Since $P$ commutes with $(P-P')^2$ and $\Gamma P(P-P')^2\Gamma = (1-P)(P-P')^2$, $(1-P)(P-P')^2 = (1-P)P'(1-P)$ is not in the trace class.

By Lemma A, there exists a partial basis projection $E$ and a partial isometry $u$ such that $[E, P] = [E, P'] = 0$, $(P-P')^2 (E + \Gamma E \Gamma) = (P-P')^2$, $[u, P] = [u, P'] = 0$, $u^* u = E$ and $u^* u = \Gamma E \Gamma$. Then $u E u^* = \Gamma E \Gamma$, $u E (1-P) P' (1-P) u^* = \Gamma E \Gamma (1-P) P' (1-P)$ and $(E + \Gamma E \Gamma) \cdot (1-P) P' (1-P) = (1-P) P' (1-P)$. Hence, if $(P-P')^2$ is not in the trace class, then $E (1-P) P' (1-P)$ is not in the trace class.

As a consequence, there exists an infinite number of unit vectors $e_j \in E (1-P) K$, $j = 1, 2, \cdots$ such that $(e_j, e_k) = (e_j, P'e_k) = 0$ for $j \neq k$ and $\sum_j (e_j, P'e_j) = \infty$. This is proved as follows:

If $E (1-P) P' (1-P)$ has a continuous spectrum $\Xi$, then take a number $\delta > 0$ such that $\Xi \cap (\delta, 1) \neq \phi$ and take an infinite number of mutually disjoint interval $\Delta_j$ in $[\delta, 1]$ with $\Delta_j \cap \Xi, \neq \phi$. Take any unit vector $e_j$ from $E (\Delta_j) K$ where $E (\Delta_j)$ is the spectral projector of $(1-P) P' (1-P)$ for $\Delta_j$. $(e_j, e_k) = (e_j, P'e_k) = 0$ for $j \neq k$ is automatic.

If $E (1-P) P' (1-P)$ has a purely discrete spectrum, then take $e_j$ to be a complete orthonormal set of eigenvectors of $E (1-P) P' (1-P)$ in $E (1-P) K$. Then $e_j \in (1-P) K$, $(e_j, P'e_k) = (e_j, E (1-P) P' (1-P) e_k) = 0$ for $j \neq k$ and $\sum_j (e_j, P'e_j) = \infty$.

Since $e_j \in E K$, $E K \perp \Gamma E K$ and $[E, P'] = 0$, we have $(e_j, \Gamma e_k) = (e_j, P' \Gamma e_k) = 0$ for any $j$ and $k$.

Let $P_j$ be the projection on the space spanned by $\Gamma e_j$, $U_\lambda (\alpha) = \exp i \lambda (P_{\alpha} - \Gamma P_{\alpha} \Gamma)$, $U^{(\alpha)}(\alpha) = \prod_{k=1}^{\alpha} U_k(\lambda)$ and $U(\lambda) = \prod U_\lambda (\alpha)$. We have

\begin{align*}
(9.36) \quad \det P U^{(\alpha)}(\lambda) P &= \exp i \lambda, \\
(9.37) \quad \det P' U^{(\alpha)}(\lambda) P' = \prod_{j=1}^{\alpha} \det \{ P' \exp i \lambda (P_j - \Gamma P_j \Gamma) P' \} \\
&= (\exp i \lambda)^2 \prod_{j=1}^{\alpha} [1 + (e^{-i\lambda} - 1) (e_j P' e_j)]^2.
\end{align*}

From (9.36), it follows that $P e U^{(\alpha)}(\lambda) e^{-i\lambda}$ has a strong limit $Q e (U(\lambda))$. It also follows from the proof of Lemma 9.3 that approach to the limit is uniform locally in $\lambda$ and hence $Q e (U(\lambda))$ is continuous.
in \( \lambda \). Hence \((\phi, Q_p(U(\lambda))\phi) \neq 0\) for sufficiently small \( \lambda \) for a given \( \phi \). Thus for \( \phi \in S_p \), there exists \( \lambda \) such that

\[
\lim_{n \to \infty} (\phi, Q_p(U(\lambda))e^{-i\lambda n} \phi) \neq 0
\]

and \( \cos \lambda \neq 1 \).

Let \( g_\ast \in \mathcal{D}_* \) be such that \( \pi(g_\ast) = U(\lambda) \). Due to (9.37), it is necessary for the existence of a nonzero limit

\[
\lim_{n \to \infty} \varphi_{pr}(Q(g_\ast)e^{-i\lambda n}) \neq 0
\]

that

\[
\sum (e^{-i\lambda} - 1) (e, P' e) < \infty.
\]

This implies that \( \varphi_p \) and \( \varphi_{pr} \) cannot be unitarily equivalent. Q.E.D.

**Theorem 7.** A Bogoliubov automorphism \( \tau(V) \) is unitarily implementable in the Fock representation \( \pi_p \) if and only if \( (1 - P)VP \) is in the Hilbert Schmidt class.

**Proof.** We note that

\[
\varphi_p(\tau(V)A) = \varphi_{pr}(A).
\]

Hence, if \( V \) is unitarily implementable, \( |V^*PV - P| \) is in the Hilbert Schmidt class. Hence \( P|V^*PV - P|^2 = \langle PV^*(1 - P) \rangle \langle (1 - P)VP \rangle \) is in the trace class, which implies \( (1 - P)VP \) is in the Hilbert Schmidt class.

Conversely, if \( (1 - P)VP \) is in the trace class, then \( P|V^*PV - P|^2 \) and \( r\langle P|V^*PV - P|^2 \rangle r = (1 - P)|V^*PV - P|^2 \) are both in the trace class. Hence \( V^*PV - P \) is in the Hilbert Schmidt class and \( V \) is unitarily implementable for \( \pi_p \). Q.E.D.

**Theorem 8.** A Bogoliubov automorphism \( \tau(U) \) is unitarily implementable for all Fock representations if and only if \( U - 1 \) or \( U + 1 \) is in the Hilbert Schmidt class, where \( \dim K \neq \text{odd} \).

**Proof.** "If" part is immediate from Theorem 7. We may assume that \( \dim K \) is infinite. [The case where \( \dim K = \text{odd} \) is not considered because there is no Fock representation].
For “only if” part, we have to show that if \( PU(1-P) \) is in the Hilbert Schmidt class for all basis projection \( P \), then \( U-1 \) or \( U+1 \) is in the Hilbert Schmidt class where \( U \) satisfies \( U^* = U^{-1}, \Gamma U \Gamma = U \).

Let \( A \) be any measurable subset of \( \{e^{i\theta}; 0 < \theta < \pi \} \) and \( E \) be any subprojection of the spectral projection of \( U \) for the set \( A \) such that \( [U, E] = 0 \). Assume that \( E \) has an infinite dimension and \( E = 1 - E \Gamma E \Gamma \) has an infinite or an even dimension. Let \( E = E_1 + \Gamma E \Gamma \).

There exists an antiunitary involution \( T \) (a complex conjugation) on \( EK \), commuting with the spectral projections of \( UE_1 \) and with \( \Gamma \). Let \( P \) be the subprojection of \( E \) for the subspace spanned by \( f + i\Gamma Tf, f \in E_K \). Then \( (E-P)K \) is spanned by \( f - i\Gamma Tf, f \in E_K \) and \( \Gamma P \Gamma = E - P \). Hence there exists a basis projection \( P \geq P \).

Since \( U(f + i\Gamma Tf) = Uf + i\Gamma T^* Uf \), we have

\[
(1-P)U(f + i\Gamma Tf) = \frac{1}{2}([U - U^*])f - i\Gamma T(U - U^*)f, \]

Therefore

\[
\| (1-P)U(f + i\Gamma Tf) \|^2/\|f + i\Gamma Tf\|^2 = \| (U - U^*)f/2\|^2/\|f\|^2.
\]

Since \( (1-P)UP = (1-P)U P \) must be in the Hilbert Schmidt class, \( (U - U^*)E \), must be in the Hilbert Schmidt class. [Note that \( 2^{-1/2} (f_j + i\Gamma T f_j) \) is an orthonormal basis of \( P \) if \( f_j \) is an orthonormal basis of \( E_K \).]

In order that \( (U - U^*)E \) is in the H.S. class for any \( E \), it is necessary that \( U \) has a purely discrete spectrum and its accumulation points are at most 1 and \(-1\).

Next assume that \( Uf_j = e^{i\alpha_j} f_j, Ug_j = e^{i\beta_j} g_j, j = 1, 2, \cdots, 0 \leq \alpha_j \leq \pi, 0 \leq \beta_j \leq \pi, |\alpha_j - \beta_j| \geq \alpha(>0) \), \( (f_j, f_s) = (g_j, g_s) = \delta_{js} \) and \( (\Gamma g_j, g_s) = (\Gamma f_j, f_s) = 0 \). Further assume that the orthogonal complement of the set of all \( f_j, g_j, \Gamma f_j, \Gamma g_j, j = 1, 2, \cdots \) has an infinite or even dimension.

Let \( P \) be the subspace spanned by \( (f_j + g_j) \) and \( (\Gamma f_j - \Gamma g_j) \), \( j = 1, 2, \cdots \). Then there exists a basis projection \( P \geq P \). We have

\[
(1-P)U(f_j + g_j) = (e^{i\alpha_j} - e^{i\beta_j})(f_j - g_j)/2.
\]
Therefore,
\[\|(1-P)UP\|_{h.s.}^2 \geq \sum_j \|(1-P)U(f_j+g_j)\|^2/2\]
\[\geq \sum_j \sin^2 [(\alpha_j - \beta_j)/2] \]
\[\geq \sum_j \sin^2 (\alpha_j/2) = \infty.\]

Thus, the spectrum of \(U\) cannot have more than one accumulation points nor points with an infinite multiplicity.

From the above two conclusion, we see that \(U-1\) or \(U+1\) must be compact.

If \(U+1\) is compact, then \((-U)-1\) is compact and \(\tau(-1)\) is unitarily implementable in all Fock representation.

If \(U-1\) is compact and an eigenvalue 1 has a finite multiplicity, there exists an infinite number of eigenvectors \(f_i\) of \(U\) belonging to an eigenvalue \(e^{i\alpha_i}\) such that \(0<\alpha_i<\pi\) and \(\sum \alpha_j<\infty\). Let \(E\) be the projection for the subspace spanned by all \(f_i\) and \(rf_i\), and \(W=UE + (1-E)\). Then \(\tau(W)\) is an inner Bogoliubov automorphism by Lemma 8.3 and an eigenvalue 1 of \(UW^*\) has an infinite multiplicity.

Thus we may restrict our attention to the case where \(U-1\) is compact and an eigenvalue 1 of \(U\) has an infinite multiplicity. In this case \(\Delta\) can be taken the whole set \(\{e^{i\theta}; 0<\theta<\pi\}\) and hence \(U-U^*\) is in the H.S. class. This implies that \(U-1\) is in the H.S. class.

Q.E.D.

§10. Pseudo Fock States

Lemma 10.1. Let \(P\) be a partial basis projection with the \(\Gamma\) codimension 1. Let \(e_0\) be a fixed \(\Gamma\) invariant unit vector in \((1-P-\Gamma PrK).\) Let \(\pi_P\) on \(\mathcal{H}_P\) be the Fock representation of \(\mathfrak{A}_{SDC}(PK+\Gamma PK, \Gamma)\). Then there exists an irreducible representation \(\pi_{(P, e_0)}\) of \(\mathfrak{A}_{SDC}(K, \Gamma)\) on \(\mathcal{H}_P\) uniquely determined by the following requirements:

\[\pi_{(P, e_0)}(B(f)) = 2^{-1/2}(e_0, f)T_\tau(-1) + \pi_P[B Pf + \Gamma Pf] .\]

Proof. Since \(\pi_{(P, e_0)}(B(f))\) given by (10.1) satisfies the defining properties (1), (2), (3) of a selfdual CAR algebra, it automatically has
a unique extension to the whole $\mathfrak{I}_{SDC}(K, \Gamma')$.

**Definition 10.2.** A pseudo Fock state $\varphi_{(P, e_0)}$ of $\mathfrak{I}_{SDC}(K, \Gamma')$ is defined by

\[ \varphi_{(P, e_0)}(A) = (\Omega_P, \pi_{(P, e_0)}(A) \Omega_P), \]

where $P$, $e_0$ and $\pi_{(P, e_0)}$ are given in Lemma 10.1 and $\Omega_P$ is the cyclic vector corresponding to the Fock state $\varphi_P$ of $\mathfrak{I}_{SDC}(PK + \Gamma PK, \Gamma')$.

**Lemma 10.3.** Let $P$ be a partial basis projection with an $\Gamma$ codimension 1 and

\[ S = (1/2) (1 + P - r Pr'), \]

Then

\[ \varphi_s = (1/2) \{ \varphi_{(P, e_0)} + \varphi_{(P, -e_0)} \}. \]

Pure states $\varphi_{(P, e_0)}$ and $\varphi_{(P, -e_0)}$ are not unitarily equivalent. $R_s$ is not a factor and its center is generated by $\pi_s(B(e_0))_T(-1)$, where $T_P(-1)$ is a unitary operator in $\pi_s(\mathfrak{A}(PK + \Gamma PK, \Gamma'))$ satisfying $T_P(-1) \pi_s(B(f)) T_P(-1) = \pi_s(B(-f))$ for $f \in PK + \Gamma PK$.

**Proof.** Any element $A$ in $\mathfrak{I}_{SDC}(K, \Gamma')$ can be written as $A_1 + A_2 B(e_0) = A$ where $A_1$ and $A_2$ are in $\mathfrak{I}_{SDC}(PK + \Gamma PK, \Gamma')$. Both sides of (10.4) give $\varphi_{(P, e_0)}(A)$ and hence (10.4) holds.

If $A \in \mathfrak{I}_{SDC}(PK + \Gamma PK, \Gamma')$, then $\pi_{(P, e_0)}(A) = \pi_{(P, -e_0)}(A)$. The set of all such $\pi_{(P, e_0)}(A)$ is irreducible by Lemma 4.3. Therefore any unitary operator satisfying $W \pi_{(P, e_0)}(A) W^* = \pi_{(P, -e_0)}(A)$ must be a multiple of the identity. However, $\pi_{(P, -e_0)}(B(e_0)) = -2^{-1/2} T_P(-1) \neq \pi_{(P, e_0)}(B(e_0))$. Therefore $\pi_{(P, e_0)}$ and $\pi_{(P, -e_0)}$ are not unitarily equivalent.

From this, it follows that

\[ \mathcal{D}_s = \mathcal{D}_{\varphi_{(P, e_0)}} \oplus \mathcal{D}_{\varphi_{(P, -e_0)}} \]

(10.6)

\[ \Omega_s = 2^{-1/2} (\Omega_{\varphi_{(P, e_0)}} \oplus \Omega_{\varphi_{(P, -e_0)}}) \]

(10.7)

\[ \pi_s(B(f)) = \pi_{\varphi_{(P, e_0)}}(B(f)) \oplus \pi_{\varphi_{(P, -e_0)}}(B(f)) \]

and $\pi_s(B(e_0))(T_P(-1) \oplus T_P(-1))$, which is $2^{-1/2}$ on $\mathcal{D}_{\varphi_{(P, e_0)}}$ and $-2^{-2/1}$ on $H_{\varphi_{(P, -e_0)}}$, generates the center of $R_s$. The operator $T_P(-1) \oplus T_P(-1)$,
which belongs to $\pi_{\Lambda}(PK+\Gamma PK, \Gamma')^\prime$, can be characterized up to a multiplicative constant by its anticommutation property with $B(f)$, $f \in PK+\Gamma PK$.

**Theorem 9.** Let $E$ be a partial basis projection with a finite odd $\Gamma'$ codimension and $T$ be a Hilbert Schmidt operator such that $TE = ET = T$. Let

\begin{equation}
S = T + 1 - \Gamma T \Gamma + \left(\frac{1}{2}\right)(1 - E - \Gamma E \Gamma').
\end{equation}

Then $R_s$ is not a factor. Conversely, if $R_s$ is not a factor, then $S$ is of the form given by (10.8).

**Proof.** Let $e_1 \cdots e_{2n+1}$ be a complete orthonormal system of $\Gamma$ invariant vectors in $(1 - E - \Gamma E \Gamma')K$ and $E_0$ be the projection on the subspace spanned by $e_{2j} + ie_{2j+1}$, $j = 1, \cdots, n$. By setting $E_1 = E + E_0$, $T_1 = T + (1/2)E_0$, we obtain a case where the partial basis projection $E_1$ has a $\Gamma'$-codimension 1.

Let

\begin{equation}
S' = \Gamma E_1 \Gamma' + (1/2)(1 - E_1 - \Gamma E_1 \Gamma').
\end{equation}

Then $S^{1/2} - (S')^{1/2}$ is in the Hilbert Schmidt class and hence $R_s$ and $R_s'$ are $*$ isomorphic. By Lemma 10.3, where we set $P = \Gamma E_1 \Gamma'$, $R_s'$ is not a factor and hence $R_s$ is not a factor.

If $S$ is of the form given by (10.8) where the $\Gamma'$ codimension of $E$ is finite and even and $T$ is as before. Then the same argument as above shows that $R_s$ is $*$ isomorphic to $R_s'$ where $S' = \Gamma E_1 \Gamma'$ is a basis projection. Hence $R_s$ is a factor.

If $S$ is not of the form given by (10.8) where the $\Gamma'$-codimension of $E$ is finite, then $S^{1/2} - (S')^{1/2}$ is not in the Hilbert Schmidt class. Let $P_s' = 2(S \oplus (1 - S)) - P_s$. Then $P_s' - P_s$ is not in the Hilbert Schmidt class. In the proof of Lemma 4.11, $\psi_-$, if nonvanishing, is a vector giving a vector states $\varphi_{P_s'}$ in the representation space associated with $\varphi_{P_s}$. By Theorem 6, we have $\psi_- = 0$ and hence from the proof of Lemma 4.11, $R_s$ must be a factor. Q.E.D.
Appendix: Angle between Two Projections

We state a result concerning an angle operator between two projections which is essentially taken from [1]. If one of two projections has either dimension 1 or codimension 1, then the nonzero eigenvalue of the angle operator coincides with the geometrical angle between corresponding subspaces.

Theorem 10. Let $P_1$ and $P_2$ be projection operators on a complex Hilbert space $K$. Let $\theta(P_1, P_2)$ be defined by

\begin{align}
\theta(P_1, P_2) &= 0, \\
\sin \theta(P_1, P_2) &= |P_1 - P_2|.
\end{align}

Let $E(0)$ and $E(\pi/2)$ denote eigenprojections of $\theta(P_1, P_2)$ for eigenvalues 0 and $\pi/2$, $E = E(0) + E(\pi/2)$, and

\begin{align}
v_1 &= \cos \theta(P_1, P_2), \\
v_2 &= \sin \theta(P_1, P_2).
\end{align}

Let

\begin{align}
R(P_1/P_2) &= v_1 + v_1^{-1}[P_1, P_2], \\
I(P_1, P_2) &= v_1^{-1}(P_1 + P_2 - 1).
\end{align}

Let

\begin{align}
u_{11}(P_1/P_2) &= P_1(1 - E), \\
u_{22}(P_1/P_2) &= (1 - P_1)(1 - E), \\
u_{12}(P_1/P_2) &= (v_1 v_2)^{-1} P_1 P_2 (1 - P_1), \\
u_{21}(P_1/P_2) &= (v_1 v_2)^{-1} (1 - P_1) P_2 P_1.
\end{align}

Let $P \sqcap P'$ denote the projection on $PK \cap P'K$ if $P$ and $P'$ are projections. Let $\mathcal{B}$ be the von Neumann algebra $\{P_1, P_2\}''$ generated by $P_1$ and $P_2$ and $\mathcal{Z}$ be its center $\mathcal{B} \cap \mathcal{B}'$.

Then $\mathcal{Z}$ is generated by $\theta(P_1, P_2) = \theta(P_2, P_1)$, $EP_1$ and $EP_2$. $\mathcal{B}$ is generated by its center $\mathcal{Z}$ and $u_{ij}(P_1/P_2)$, $i, j = 1, 2$ satisfying

\begin{align}
u_{ij}(P_1/P_2)^* &= u_{ji}(P_1/P_2), \\
u_{ij}(P_1/P_2) u_{ij}(P_1/P_2) &= \delta_{ij} u_{ij}(P_1/P_2).
\end{align}
\( R E \) is commutative and is generated by four minimal projections \( P_1 \wedge P_2, P_1 \wedge (1-P_2), (1-P_1) \wedge P_2 \) and \((1-P_1) \wedge (1-P_2)\) where

\[
\begin{align*}
E(0) &= P_1 \wedge P_2 + (1-P_1) \wedge (1-P_2), \\
E(\pi/2) &= P_1 \wedge (1-P_2) + (1-P_1) \wedge P_2.
\end{align*}
\]

\( R(1-E) \) is a tensor product of the center \( \mathfrak{B}(1-E) \) and the type \( I_2 \) factor generated by the matrix unit \( u_{ij}(P_i/P_j) \). Relative to this matrix unit, we have

\[
\begin{align*}
(A.12) & \quad P_1(1-E) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
(A.13) & \quad P_2(1-E) = \begin{pmatrix} v_1^2 & v_1 v_2 \\ v_1 v_2 & v_2^2 \end{pmatrix}, \\
(A.14) & \quad R(P_1/P_2)(1-E) = \begin{pmatrix} v_1 & v_2 \\ -v_2 & v_1 \end{pmatrix}, \\
(A.15) & \quad I(P_1, P_2)(1-E) = \begin{pmatrix} v_1 & v_2 \\ v_2 & -v_1 \end{pmatrix}.
\end{align*}
\]

The operator \( R(P_1/P_2) \) satisfies

\[
\begin{align*}
(A.16) & \quad R(P_1/P_2)^* = R(P_2/P_1), \\
(A.17) & \quad R(P_1/P_2)R(P_1/P_2)^* = R(P_1/P_2)^* R(P_1/P_2) = 1 - E(\pi/2), \\
(A.18) & \quad R(P_1/P_2) P_2 R(P_1/P_2)^* = P_1 - P_1 \wedge (1-P_2), \\
(A.19) & \quad R(P_1/P_2)^* P_1 R(P_1/P_2) = P_2 - P_2 \wedge (1-P_1).
\end{align*}
\]

The operator \( I(U_1, U_2) \) satisfies

\[
\begin{align*}
(A.20) & \quad I(P_1, P_2)^* = I(P_1, P_2) = I(P_1, P_1), \\
(A.21) & \quad I(P_1, P_2)^* - 1 - E(\pi/2), \\
(A.22) & \quad I(P_1, P_2) u_{ij}(P_1/P_2) I(P_1, P_2) = u_{ij}(P_2/P_1).
\end{align*}
\]

Proof. Since \(-1 \leq P_1 - P_2 \leq 1\), we have \(0 \leq |P_1 - P_2| \leq 1\) and hence \(\theta(P_1, P_2)\) is uniquely well defined by \((A.2)\) and \((A.1)\). By a direct calculation,

\[
\begin{align*}
(A.23) & \quad [(P_1 - P_2)^\delta, P_1] = [(P_1 - P_2)^\delta, P_2] = 0
\end{align*}
\]

and hence \(\theta(P_1, P_2) \in \mathfrak{B}\).

If \((P_1 - P_2)f = f\), then \((f, P_1 f) \leq \|f\|\) and \((f, P_2 f) \geq 0\) imply
\[ \|P_2 f\| = 0 \] and hence \( P_2 f = 0 \), \( P_1 f = f \). If \( (P_1 - P_2) f = -f \), we obtain \( P_2 f = f, P_1 f = 0 \). Converses are obviously true. Hence we have

\[ (A.24) \quad E(\pi/2) = P_1 \setminus (1 - P_2) + (1 - P_1) \setminus P_2. \]

Next assume \( P_1 f = P_2 f \). Let \( g_1 = P_1 f, g_2 = (1 - P_1) f \). Then \( P_1 g_1 = g_1 = P_2 g_1 \) and \( (1 - P_1) g_2 = g_2 = (1 - P_1) g_2 \). Hence \( g_1 \in (P_1 \setminus P_2) K, g_2 \in ((1 - P_1) \setminus (1 - P_2)) K \) and \( f = g_1 + g_2 \). Conversely, such \( f \) satisfies \( (P_1 - P_2) f = 0 \). Hence

\[ (A.25) \quad E(0) = P_1 \setminus P_2 + (1 - P_1) \setminus (1 - P_2). \]

Obviously \( P_1 \setminus P_2, (1 - P_1) \setminus P_2, P_1 \setminus (1 - P_2) \) and \( (1 - P_1) \setminus (1 - P_2) \) belong to \( \mathcal{F} \).

From \( (A.23), (A.24) \) and definitions, we have

\[ (A.26) \quad u_{ij} (P_1/P_2) E = E u_{ij} (P_1/P_2) = 0. \]

By using identities

\[ (A.27) \quad P_1 (P_1 - P_2)^* = P_1 (1 - P_2) = P_1 - P_1 P_2 P_1 \]
\[ (A.28) \quad P_2 (P_1 - P_2)^* = P_2 (1 - P_1) = P_2 - P_2 P_1 P_2 \]
\[ (A.29) \quad (1 - P_1) (P_1 - P_2)^* = (1 - P_1) P_2 (1 - P_1) \]
\[ (A.30) \quad (1 - P_2) (P_1 - P_2)^* = (1 - P_2) P_1 (1 - P_2) \]

we obtain \( (A.10) \) and \( (A.11) \). This also shows that \( u_{ij} (P_1/P_2) \) are everywhere defined bounded operators. [The range of \( P_1 P_2 (1 - P_1) \) and \( (1 - P_1) P_2 P_1 \) is in \( (1 - E) K \), where \( (v_1 v_2)^{-1} \) is uniquely defined].

By using \( (A.27) \) and \( (A.29) \), we have \( (A.12), (A.14) \) and \( (A.15) \) are immediate from the definition. \( (A.16) \sim (A.22) \) are obtained from \( (A.12) \sim (A.15) \).

\( \mathfrak{F} \) is generated by \( P_1 \) and \( P_2 \) and hence by \( \theta(P_1, P_2), EP_1, EP_2 \) and \( u_{ij} (P_1/P_2) \). Since \( E u_{ij} (P_1/P_2) = 0 \), \( \mathfrak{F} E \) is generated by \( E \theta(P_1, P_2), EP_1, EP_2 \) and hence as is stated in the Lemma.

On \( (1 - E) K \), \( u_{ij} (P_1/P_2) \) generates a type \( I_2 \) factor and hence \( \mathfrak{F} (1 - E) \) is as is stated in the Lemma and \( \mathfrak{J} \) is generated by \( \theta(P_1, P_2), EP_1 \) and \( EP_2 \). Q.E.D.

As an immediate application of Theorem 10, we have
Theorem 11. Let $P_1$ and $P_2$ be basis projections on $K$ relative to $\Gamma$. Then

\begin{align}
(A.31) \quad & \Gamma \theta(P_1, P_2) \Gamma = \theta(P_1, P_2), \\
(A.32) \quad & \Gamma R(P_1/P_2) \Gamma = R(P_1/P_2), \\
(A.33) \quad & \Gamma I(P_1, P_2) \Gamma = -I(P_1, P_2), \\
(A.34) \quad & \Gamma u_{ij}(P_1/P_2) \Gamma = -u_{ji}(P_1/P_2) \quad (i \neq j), \\
(A.35) \quad & \Gamma u_{ii}(P_1/P_2) \Gamma = u_{ii}(P_1/P_2) \quad (i \neq j), \\
(A.36) \quad & \Gamma (P_1 \wedge P_2) \Gamma = (1 - P_1) \wedge (1 - P_2), \\
(A.37) \quad & \Gamma (P_1 \wedge (1 - P_2)) \Gamma = (1 - P_1) \wedge P_2.
\end{align}

There exists an antiunitary involution $T$ which commutes with $\theta(P_1, P_2)$, $u_{ij}(P_1, P_2)$ and $\Gamma$.

The linear operator $\hat{R}(P_1, P_2)$ defined by

\begin{align}
(A.38a) \quad & \hat{R}(P_1/P_2) E(\pi/2) = T E(\pi/2), \\
(A.38b) \quad & \hat{R}(P_1/P_2)(1 - E(\pi/2)) = R(P_1/P_2),
\end{align}

is unitary, commutes with $\theta(P_1, P_2)$ and $\Gamma$ and satisfies

\begin{align}
(A.39) \quad & \hat{R}(P_1, P_2) P_1 \hat{R}(P_1, P_2)^* = P_1.
\end{align}

$\Gamma$ on $(1 - E)K$ is given by

\begin{align}
(A.40) \quad & \Gamma(1 - E) = T \varepsilon (u_{12}(P_1/P_2) - u_{21}(P_1/P_2))
\end{align}

where $\varepsilon$ is a linear operator, commutes with $\theta(P_1, P_2)$, $u_{ij}(P_1, P_2)$, $T$ and $\Gamma$ and satisfies $\varepsilon^* = -\varepsilon$, $\varepsilon^2 = E - 1$. The multiplicity of $\theta(P_1, P_2)$ at any point in $(0, \pi/2)$ is a multiple of 4.

Proof. From $\Gamma P_i \Gamma = 1 - P_i$ and definitions, we obtain (A.31) ~ (A.37). We shall prove the existence of the operator $T$ and its property.

Let $e_i$ be any $\Gamma$ invariant vector. Let $K(e_i)$ be a closed real linear space generated by $\{ \sum \mathcal{P}_{ij} u_{ij}(P_1/P_2) + \mathcal{P} E \} e_i$ where $\mathcal{P}_{ij}$ and $\mathcal{P}$ are any bounded selfadjoint operator in $\mathcal{B}$. Then $K(e_i) + i K(e_i)$ is a closed subspace of $K$, containing $e_i$ and invariant under $\Gamma$ and $\mathcal{P}$. Furthermore, for any $\varphi_1$ and $\varphi_2$ in $K(e_i)$, $(\varphi_1, \varphi_2)$ is real. Note that
\((e_i, \mathcal{P}_{ij} u_{ij}(P_1/P_2)e_i)=0\) if \(i \neq j\) due to (A. 34).

If mutually orthogonal subspaces \(K(e_v) + iK(e_v)\) having such properties are given for \(\nu < \nu_0\), then by choosing any \(\Gamma\) invariant vector \(e_{v_{0}}\) in \((\bigcup_{\nu < \nu_0} \{K(e_v) + iK(e_v)\})^1\), we can obtain \(K(e_{v_{0}}) + iK(e_{v_{0}})\), which is orthogonal to \(K(e_v) + iK(e_v), \nu < \nu_0\) and has such properties. By induction, the total Hilbert space is a direct sum of such \(K(e_v) + iK(e_v)\). Let

\[ T \sum(f_v + ig_v) = \sum(f_v - ig_v) \]

for \(f_v, g_v \in K(e_v)\). Then \(T\) is an antiunitary involution commuting with \(\theta(P_1, P_2), u_{ij}(P_1, P_2)\) and \(\Gamma\).

The statements concerning \(\widehat{R}(P_1/P_2)\) and \(\varepsilon\) are immediate where \(\varepsilon\) is defined by \(\Gamma T(u_{22}(P_1/P_2) - u_{22}(P_1/P_2))\). Since \(T\varepsilon\) restricted to \((1-E)K\) is an antiunitary operator, commuting with \(\theta(P_1, P_2)\) and \(P_1\) and satisfying \((T\varepsilon)^2 = -(1-E), \theta(P_1, P_2)\) restricted to \(P_1(1-E)\) has an even multiplicity. Since \(\theta(P_1, P_2)\) restricted to \(1-P_1\) has the same multiplicity as \(\theta(P_1, P_2)\) restricted to \(P_1 \) due to \(\Gamma P_1 \Gamma = 1 - P_1\), and \([\theta(P_1, P_2), \Gamma] = 0\), the multiplicity of \(\theta(P_1, P_2)\) at any point in \((0, \pi/2)\) must be a multiple of 4. Q.E.D.

**Lemma A.** Let \(P\) and \(P'\) be basis projections. Then there exists a partial basis projection \(F\) and a partial isometry \(u\), both commuting with \(P\) and \(P'\), such that \(F + \Gamma F \Gamma = 1 - E(0) - E(\pi/2)\), \(u^*u = F\) and \(uu^* = \Gamma F \Gamma\).

**Proof.** Use the notation in the proof of Theorem 11. The operator \(\varepsilon\) has at most three eigenvalues 0, \(i\) and \(-i\). The eigenprojection for 0 is \(1-E(0)-E(\pi/2)\). Let \(F\) be an eigenprojection for \(i\). Since \([\Gamma, \varepsilon] = 0, \Gamma F \Gamma\) must be an eigenprojection for \(-i\) and hence \(F\) is a partial basis projection commuting with \(\varepsilon\).

Next we modify the construction of \(K(e_v)\) as follows. We restrict our attention to \((1-E(0) - E(\pi/2))K\). Let \(K(e_v)\), \(\nu < \nu_0\) be given. Then choose a unit vector \(e'_{\nu_0}\) in \(F(\bigcup_{\nu < \nu_0} \{K(e_v) + iK(e_v)\})^1\). Let \(\sqrt{2} e_{\nu_0} = e'_{\nu_0} + \Gamma e'_{\nu_0}\) and \(\sqrt{2} e_{\nu_0+1} = i(e'_{\nu_0} - \Gamma e'_{\nu_0})\). Since \(\varepsilon e'_{\nu_0} = ie'_{\nu_0}, \varepsilon \Gamma e'_{\nu_0} = -i \Gamma e'_{\nu_0}\) and \([\varepsilon, \varepsilon] = 0, (\varepsilon \varepsilon', \varepsilon \Gamma e'_{\nu_0}) = 0\) and hence \(K(e_{\nu_0}) \perp K(e_{\nu_0+1})\). Note that \((\bigcup_{\nu < \nu_0} \{K(e_v) + iK(e_v)\})^2\) is invariant under \(F\) and \(F \neq 0\) on this subspace unless \(F + \Gamma F \Gamma = 1 - E\) is 0 on this subspace, which occurs only if this subspace is \(EK\).
We define \( u' \) to be 1 on \( K(e_{e_{1}}) \), \(-1\) on \( K(e_{e_{p+1}}) \) and 0 on \( EK \).
Then \( u = u'F \) commutes with \( \mathcal{F} \) and \( u^*u = F, \ uu^* = rFr \). Q.E.D.

References


