Corrections to “Finiteness of the Number of Discrete Eigenvalues of the Schrödinger Operator for a Three Particle System”

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Remark 4, p. 59, was an error due to the negligence of the fact that $R$ depends on $Z$. The correct assertion is the following:

There exists some constant $Z_0 \left( \frac{Z_3}{2} \geq Z_0 > 0 \right)$ depending only on $Z_3$ such that for any positive constants $Z_1, Z_2 (Z_0 \geq Z_1 \geq Z_2 > 0)$ the operator of the form

$$H = -\Delta_1 - \Delta_2 - \frac{Z_1}{r_1} - \frac{Z_2}{r_2} + \frac{Z_3}{|r_1 - r_2|}$$

has no discrete eigenvalues.

In fact let $\frac{Z_3}{2} \geq Z_1 \geq Z_2 > 0$. Then taking into consideration Remark 1 and the fact that $\mu$ given to (1) by (2.7) and (2.8) equals $-\frac{Z_3^2}{4}$, (3.1) and (3.2) are satisfied by $R = \frac{c_1}{Z_1}$, where $c_1$ is sufficiently large constant depending only on $Z_3$. On the other hand, we can show the following fact which is more precise than Lemma 5.

There exist an “extension operator” $\phi$, which maps $\mathcal{E}_{13}(\Omega_1)$ to $\mathcal{D}_{12}(\mathbb{R}^n)$ and some constant $c_2$ depending only on $Z_3$ such that for any $\varphi \in \mathcal{E}_{13}(\Omega_1)$
Indeed, for $\varphi \in C^1_b(\Omega_1)$ we define $f \in C^1_b(\Omega_0)$ by

$$f(x) = \varphi(Rx) \quad \text{for} \quad x \in \Omega_0,$$

where $\Omega_0 = \{x \in \mathbb{R}^n ; r_1 < 1, r_2 < 1\}$. Then we have

$$\left\{ \begin{array}{l}
\|f\|_{\infty}^2 = \frac{1}{R^2} \|\varphi\|_{\infty}^2,
\|f\|_2^2 = \frac{R^2}{R^2} \|\varphi\|_2^2.
\end{array} \right.$$

Let $\phi$ and $c_0 = c_0(\Omega_0, \phi)$ be the $\phi$ and $\tilde{c}$ satisfying the relations given in Lemma 5 with $\Omega_1$ replaced by $\Omega_0$. Now we define $\phi, \varphi \in \mathcal{D}^1_{\infty}(\mathbb{R}^n)$ by

$$\left( \phi, \varphi \right)(x) = \left( \phi f \right) \left( \frac{x}{R} \right) \quad \text{for} \quad x \in \mathbb{R}^n.$$

Then by Lemma 5 $\phi$ satisfies (2), and

$$\left( \begin{array}{l}
\frac{1}{R^2} \|\phi, \varphi\|_{\infty}^2 = \|\phi f\|_{\infty}^2 \leq c_0 \|f\|_2^2 = \frac{c_0}{R^2} \|\varphi\|_2^2,
\end{array} \right.$$

and

$$\frac{R^2}{R^2} \|\phi(\phi, \varphi)\|_{\infty}^2 = \|\phi f\|_{\infty}^2 \leq c_0 \left( \|\varphi f\|_2^2 + \|f\|_2^2 \right)$$

By (7) and (8) we have (3).

Then using the well-known inequality

$$\int_{\mathbb{R}^n} \frac{|x|^2}{R^2} dx \leq 4 \int_{\mathbb{R}^n} |x|^2 dx \quad \text{for} \quad \varphi \in \mathcal{D}^1_{\infty}(\mathbb{R}^n),$$

and Schwartz's inequality, we have by (2) and (3) for any $\psi \in \mathcal{D}^1_{\infty}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \frac{1}{R^2} dx \leq \int_{\mathbb{R}^n} \frac{1}{R^2} dx \leq \left( \int_{\mathbb{R}^n} \frac{1}{R^2} dx \right)^{1/2} \|\phi, \varphi\|_{\infty}^2.$$
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\[ = \left( \int_{R^3} \int_{R^3} \left( \frac{\phi_1 \psi}{r_1^2} \right)^2 \, dr_1 \right)^{1/2} \cdot \| \phi_1 \psi \|_{R^3} \leq 2 \left( \int_{R^3} \int_{R^3} |P_1(\phi_1 \psi)|^2 \, dr_1 \right)^{1/2} \cdot \| \phi_1 \psi \|_{R^3} \]

\[ \leq \frac{c_0}{Z_1} \cdot \| P_1(\phi_1 \psi) \|_{R^3}^2 + \frac{Z_1}{c_0} \| \phi_1 \psi \|_{R^3}^2 \]

\[ \leq \frac{c_0 c_\eta}{Z_1} \cdot \| P \psi \|_{L^2}^2 + c_2 \left( \frac{\eta}{c_0} \right) Z_1 \| \psi \|_{L^2}^2 \]

and similarly

\[ \left( \int_{\Omega_1} \frac{|\psi|}{r_2} \, dx \leq \frac{c_0 c_\eta}{Z_1} \cdot \| P \psi \|_{L^2}^2 + c_2 \left( \frac{\eta}{c_0} \right) Z_1 \| \psi \|_{L^2}^2, \right. 

where \( \eta \) is an arbitrary positive constant. Let \( \eta = \frac{1}{2c_2} \). Then if we take into consideration \( \frac{Z_3}{r_1 - r_2} \rightarrow \frac{Z_3}{2R} = \frac{Z_3}{2c_1} \) in \( \Omega_1 \), we have by (10) and (11) for any \( \psi \in D_{L^2}(R^n) \)

\[ L_1[\psi] \geq \left( 1 - c_0 c_\eta - \frac{Z_3}{Z_1} c_0 c_\eta \right) \| P \psi \|_{L^2}^2 + 

\[ + \left\{ \frac{Z_3}{2c_1} - c_2 \left( \frac{\eta}{c_0} \right) Z_1 \right\} \| \psi \|_{L^2}^2 \]

\[ \geq N_1 \left\{ \frac{Z_3}{2c_1} - 2c_2 \left( \frac{\eta}{c_0} \right) Z_1 \right\} \| \psi \|_{L^2}^2. \]

Then there exists some constant \( Z_6 \) such that for any \( Z_1 \) and \( Z_2 (Z_6 \geq Z_6) \) we have

\[ L_1[\psi] \geq \mu \| \psi \|_{L^2}^2 = - \frac{Z_3^2}{4} \| \psi \|_{L^2}^2, \]

for any \( \psi \in D_{L^2}(R^n) \). By Lemma 1, Lemma 3 and (13), we have the assertion.

Remark. There exists some constant \( Z'_6 \) depending on \( Z_1 (Z'_6 \geq Z_6) \) such that for any \( Z_6 \geq Z'_6 \) the operator of the form (1) has no discrete eigenvalues.

In fact \( R \) satisfying (3.1) and (3.2) is independent of \( Z_3 \). Then by the same calculation as (3.16) and (3.17) we have

\[ L_1[\psi] \geq \left( 1 - 2c_\eta Z_3 \right) \| P \psi \|_{L^2}^2 + \left( \frac{Z_3^2}{2R} - 2Z_1(\eta + c(\eta)) \right) \| \psi \|_{L^2}^2. \]
Take $\gamma = \frac{1}{2\ell Z_1}$ and $Z_3$ sufficiently large. We have for any $\phi \in \mathcal{D}_+^1(R^n)$

\begin{equation}
L_1[\phi] \geq -\frac{Z_1^2}{4}\|\phi\|_2^2 = \mu\|\phi\|_2^2.
\end{equation}

Then by Lemma 1, Lemma 3 and (15), we have the assertion.