L-invariant of the Symmetric Powers of Tate Curves

By

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In my earlier paper [H07] and in my talk at the workshop on “Arithmetic Algebraic Geometry” at RIMS in September 2006, we made explicit a conjectural formula of the L-invariant of symmetric powers of a Tate curve over a totally real field (generalizing the conjecture of Mazur-Tate-Teitelbaum, which is now a theorem of Greenberg-Stevens). In this paper, we prove the formula for Greenberg’s L-invariant when the symmetric power is of adjoint type, assuming a standard conjecture (see Conjecture 0.1) on the ring structure of a Galois deformation ring of the symmetric powers.

Let p be an odd prime and F be a totally real field of degree d<∞ with integer ring O. Order all the prime factors of p in O as p₁,...,pᵦ. Throughout this paper, we study an elliptic curve Eₓ over O with split multiplicative reduction at pⱼ|ⱼ for j = 1,2,...,b and ordinary good reduction at pⱼ|ⱼ for j > b. Write Fⱼ = Fₚⱼ for the pⱼ-adic completion of F and qⱼ ∈ Fₚⱼ with j ≤ b for the Tate period of Eₓ/Fⱼ. Put Qⱼ = Nₓ/Fⱼ/qⱼ.(qⱼ). When b = 0, as a convention, we assume that Eₓ/F has good ordinary reduction at every p-adic place of F. We assume throughout the paper that E does not have complex
multiplication, and for simplicity, we also assume that $E$ is semi-stable over $O$.

Some cases of complex multiplication are treated in [HMI] Section 5.3.3. Take an algebraic closure $\overline{F}$ of $F$. Writing $\rho_E : \text{Gal}(\overline{F}/F) \to GL_2(\mathbb{Q}_p)$ for the Galois representation on $T_pE \otimes_{F_p} \mathbb{Q}_p$ for the Tate module $T_pE = \lim_{\text{proj}} E[p^n]$, at each prime factor $\mathfrak{p}|p$, we have $\rho_E|_{\text{Gal}(\overline{F}_\mathfrak{p}/F_p)} \sim \left( \begin{smallmatrix} \beta_\mathfrak{p} & * \\ 0 & \alpha_\mathfrak{p} \end{smallmatrix} \right)$ for an unramified character $\alpha_\mathfrak{p}$. Since $\beta_\mathfrak{p}$ restricted to the inertia subgroup $I_\mathfrak{p} \subset \text{Gal}(\overline{F}_\mathfrak{p}/F_p)$ is equal to the $p$-adic cyclotomic character $\chi$, we have $\alpha_\mathfrak{p}^* \neq \beta_\mathfrak{p}^*$ for any pair of integers $(i,j)$ except for $i = j = 0$. Write $\rho_{n,0}$ for the symmetric $n$-th tensor power of $\rho_E$, which is an $(n+1)$-dimensional Galois representation semi-stable over $O$.

More generally, we write $\rho_{n,m}$ for $\rho_{n,0} \otimes \chi^m : \text{Gal}(\overline{F}/F) \to G_n(\mathbb{Q}_p)$, where $\chi$ is the $p$-adic cyclotomic character. By semi-stability, the sets of ramification primes for $\rho_\mathfrak{p}$ and $\rho_{n,m}$ are equal.

Consider $J_1 = \left( \begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix} \right)$. We then define $J_n = \text{Sym}^n(J_1)$. Since $\iota \alpha J_1 \alpha = \det(\alpha)J_1$ for $\alpha \in GL(2)$, we have $\iota \rho_{n,0}(\sigma)J_n \rho_{n,0}(\sigma) = N^n(\sigma)J_n$. Define an algebraic group $G_n$ over $\mathbb{Z}_p$ by

$$G_n(A) = \{ \alpha \in GL_{n+1}(A)|\iota \alpha J_n \alpha = \nu(\alpha)J_n \}$$

with the similitude homomorphism $\nu : G_n \to \mathbb{Z}_n$. Then $G_n$ is a quasi-split orthogonal or symplectic group according as $n$ is even or odd. The representation $\rho_{n,0}$ of $\text{Gal}(\overline{F}/F)$ actually factors through $G_n(\mathbb{Q}_p) \subset GL_{n+1}(\mathbb{Q}_p)$.

Two representations $\rho$ and $\rho' : G \to G_n(A)$ for a group $G$ are isomorphic if $\rho(g) = x\rho'(g)x^{-1}$ for $x \in G_n(A)$ independent of $g \in G$. If $\rho$ is isomorphic to $\rho'$, we write $\rho \cong \rho'$.

Let $S$ be the set of prime ideals of $O$ prime to $p$ where $E$ has bad reduction (and by semi-stability, $S \cup \{ p|p \} \cup \{ \infty \}$ gives the set of ramified primes for $\rho_{n,0}$). Let $K/\mathbb{Q}_p$ be a finite extension with $p$-adic integer ring $W$. We may take $K = \mathbb{Q}_p$, but it is useful to formulate the result allowing other choices of $K$. Start with $\rho_{n,0}$ and consider the deformation ring $(R_n, \rho_n)$ which is universal among the following deformations: Galois representations $\rho_A : \text{Gal}(\overline{F}/F) \to G_n(A)$ for Artinian local $K$-algebras $A$ with residue field $K = A/m_A$ such that

- (K$_n$1) unramified outside $S$, $\infty$ and $p$;
- (K$_n$2) $\rho_A|_{\text{Gal}(\overline{F}_\mathfrak{p})} \cong \left( \begin{smallmatrix} \alpha_0 & \cdots & \alpha_{n-1} & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \alpha_n \end{smallmatrix} \right)$ with $\alpha_j, \lambda \equiv (\beta_\mathfrak{p}^{n-j} \alpha_j^\mathfrak{p} \mod m_A$ with $\alpha_j, \lambda|_{\mathfrak{p}}$ factoring through $\text{Gal}(F_p^\text{unr}/F_p)(\mathfrak{p})$ for the maximal unramified extension $F_p^\text{unr}/F_p$ for all prime factors $\mathfrak{p}$ of $p$;
(Ku3) \( \nu \circ \rho_A = N^n \) for the \( p \)-adic cyclotomic character \( N \);

(Ku4) \( \rho_A \equiv \rho_{0,0} \mod m_A \).

Since \( \rho_{0,0} \) is absolutely irreducible as long as \( E \) does not have complex multiplication (because \( \text{Im}(\rho_E) \) is open in \( GL_2(\mathbb{Z}_p) \) by a result of Serre) and all \( \alpha_i \beta_i^{n-i} \) for \( i = 0, 1, \ldots, n \) are distinct, the deformation problem specified by (Ku1-1) is representable by a universal couple \((R_n, \rho_n)\) (see [Ti]). In other words, for any \( \rho_A \) as above, there exists a unique \( K \)-algebra homomorphism \( \varphi : R_n \to A \) such that \( \varphi \circ \rho_n \cong \rho_A \).

Write now

\[
\rho_i |_{\text{Gal}(F_p/F_p)} \mapsto \begin{pmatrix}
\delta_{0,p} & \cdots & 0 \\
\delta_{1,p} & \ddots & \vdots \\
0 & \cdots & \delta_{n,p}
\end{pmatrix}
\]

with \( \delta_{j,p} \equiv \beta_p^{n-j} \alpha_p^j \mod m_n \) (for \( m_n = m_{R_n} \)).

Let \( \Gamma_p \) be the maximal torsion-free quotient of \( \text{Gal}(F_{p^{n|p}}/F_p) \). Then the character \( \delta_{j,p} = \delta_{j,p}(\beta_p^{n-j} \alpha_p^j)^{-1} \) restricted to \( I_p \) factors through \( \Gamma_p \), giving rise to an algebra structure of \( R_n \) over \( W[\Gamma_p] \). Take the product \( \Gamma = \prod_{p \mid \mathfrak{p}} \Gamma_p^{n+1} \) of \( n+1 \) copies of \( \Gamma_p \) over all prime factors \( p \) of \( \mathfrak{p} \) in \( F \). We write general elements of \( \Gamma \) as \( x = (x_{j,p})_{j,p} \) with \( x_{j,p} \) in the \( j \)-th component \( \Gamma_p \) in \( \Gamma \) (\( j = 0, 1, \ldots, n \)). Consider the character \( \delta : \Gamma \to R_n^* \) given by \( \delta(x) = \prod_{j=0}^{n} \prod_{p} \delta_{j,p}(x_{j,p}) \). Choosing a generator \( \gamma_p = \gamma_p \) (for \( \mathfrak{p} = p \)) of the topologically cyclic group \( \Gamma_p \), we identify \( W[\Gamma] \) with a power series ring \( W[[X_{j,p}]]_{j,p} \) by associating the generator \( \gamma_{p} \) of the \( j \)-th component: \( \Gamma_p \) of \( \Gamma \) with \( 1 + X_{j,p} \). The character \( \delta : W[[\Gamma]] \to R_n \) extends uniquely to an algebra homomorphism \( \delta : W[[X_{j,p}]]_{j,p} \to R_n \) by the universality of the (continuous) group ring \( W[\Gamma] \). Thus \( R_n \) is naturally an algebra over \( K[[X_{j,p}]]_{j,p} \). This algebra structure of \( R_n \) over the local Iwasawa algebra \( W[\Gamma] \) is a standard one which has been studied for long (about 20 years) in many places (for example, [Ti] Chapter 8 and [MFG] 5.2.2). The \((n+1)\) variables \( X_{j,p} \) may not be independent in \( R_n \), and we expect that only a half of them survives. More precisely, we have the following conjectural statement:

**Conjecture 0.1.** Suppose that \( n \) is odd. Then \( R_n \) is isomorphic to the power series ring \( K[[X_{j,p}]]_{j,p,j,\text{odd}} \) of \( \frac{n+1}{2} \) variables.

When \( n = 1 \), we write \( \beta_i = \delta_{0,p,i} \), \( \alpha_i = \delta_{1,p,i} \), and \( T_i = X_{1,p,i} \). If \( n = 1 \) and \( F = Q \), via the solution of the Shimura-Taniyama conjecture, this conjecture follows from Kisin’s work (generalizing earlier works of Wiles, Taylor-Wiles
and Skinner-Wiles). Assuming potential modularity of $\rho_E$ (see [Tu]) with additional assumptions that $\text{Im}(\overline{\rho})$ is nonsoluble and that the semi-simplification of $\overline{\rho}|_{\text{Gal}(\overline{F}/\mathbb{F}_p)}$ is non-scalar for all prime $p|p$ in $F$, we will prove this conjecture for $n=1$ in this paper (see Proposition 2.1). Assuming Hilbert modularity over $F$ of $E$ and the following two conditions:

(ai) The $\mathbb{F}_p$-linear Galois representation $\overline{\rho} = (T_p E \mod p)$ is absolutely irreducible over $\text{Gal}(\overline{F}/\mathbb{F}[\mu_p])$.

(ds) $\overline{\rho}^{ss}$ has a non-scalar value over $\text{Gal}(\overline{F}_p/\mathbb{F}_p)$ for all prime factors $p | p$.

the conjecture for $n=1$ follows from a result of Fujiwara (see [F] and [F1]) and Skinner-Wiles [SW1] as described in [HMI] Theorem 3.65 and Proposition 3.78.

In the special case of rational elliptic curve $E/\mathbb{Q}$ with multiplicative reduction at $p$, the following conjecture (generalizing the one by Mazur-Tate-Teitelbaum in [MTT]) was proven by R. Greenberg for his $L$-invariant of symmetric powers of $E$. His proof is described in his remark in page 170 of [Gr]. Although his proof might also be generalized to our setting, our point of view is different from [Gr], relating the following conjecture to Conjecture 0.1, and indeed, if one can generalize Greenberg’s proof to cover the following conjecture, it might supply us with a proof of Conjecture 0.1 (we hope to discuss this point in our future work).

**Conjecture 0.2.** Let the notation and the assumption be as in Theorem 0.3. Suppose that the $n$-th symmetric power motive $\text{Sym}^\otimes n(H_1(E))(-m)$ with Tate twist by an integer $m$ is critical at 1. Then if $\text{Ind}_F^{Q}(\text{Sym}^\otimes n(\rho_E)(-m))$ has an exceptional zero at $s=1$, we have

$$L(\text{Ind}_F^{Q}(\rho_{n,m})) = \begin{cases} \left(\prod_{i=1}^{b} \frac{\log(Q_i)}{\text{ord}_p(Q_i)}\right) L(m) & \text{for a constant } L(m) \in \mathbb{Q}^\times \text{ if } n = 2m \text{ with odd } m, \\
\prod_{i=1}^{b} \frac{\log(Q_i)}{\text{ord}_p(Q_i)} & \text{if } n \neq 2m.
\end{cases}$$

We have $L(m) = 1$ if $b = e$, and the value $L(1)$ when $b < e$ is given by

$$L(1) = \det \left( \frac{\partial \delta_i([p,F_i])}{\partial X_j} \right)_{i>b, j>b} \bigg|_{X_1=X_2=\cdots=X_e=0} \prod_{i>b} \frac{\log_p(\gamma_i)}{[F_p:Q_p] \alpha_i([p,F_i])}$$

for the local Artin symbol $[p,F_i]$, where $\gamma_p$ is the generator of $\mathcal{N}(\text{Gal}(F_p[\mu_p^\infty]/F_p))$ by which we identify the group algebra $W[[\Gamma_p]]$ with $W[[X_p]]$. 

The analytic $L$-invariant of $p$-adic analytic $L$-functions (when $n = 1$) is studied by C.-P. Mok [M] following the method of [GS], and his result confirms the conjecture in some special cases (see a remark in [H07] after Conjecture 1.3).

The motive $\text{Sym}^n(H_1(E))(-m)$ is critical at 1 if and only if the following two conditions are satisfied:

- $0 \leq m < n$;
- either $n$ is odd or $n = 2m$ with odd $m$.

We will specify $L(m)$ in Definition 1.11 assuming Conjecture 0.1. There is a wild guess that $L(m)$ might be independent of $m$ only depending on $E$. We hope to discuss this matter in our future work.

We will prove in this paper (for Greenberg’s $L$-invariant of $\rho_{2n,n}$) that Conjecture 0.1 implies the above conjecture for $\rho_{2m,m}$. Here are some additional remarks about the conjecture:

1. When $n = 2m$ with even $m$, the motive associated to $\text{Sym}^n(\rho_E)(-m)$ is not critical at $s = 1$; so, the situation is drastically different (and in such a case, we do not make any conjecture; see [H00] Examples 2.7 and 2.8).

2. The above conjecture applies to arithmetic and analytic $p$-adic $L$-functions.

We let $\sigma \in \text{Gal}(\overline{F}/F)$ act on the Lie algebra of $G_{n/K}$

$$s_n(K) = \{ x \in M_{n+1}(K) | \text{Tr}(x) = 0 \text{ and } ^t xJ_n + J_n x = 0 \}$$

by conjugation: $x \mapsto \sigma x = \rho_{n,0}(\sigma)x\rho_{n,0}(\sigma)^{-1}$. This representation $\text{Ad}(\rho_{n,0})$ is isomorphic to $\bigoplus_{0 < j \leq n, j \text{ odd}} \rho_{2j,1}$ and is called the adjoint square representation of $\rho_{n,0}$. By using a canonical isomorphism between the tangent space of $\text{Spf}(R_n)$ and a certain Selmer group of $\text{Ad}(\rho_{n,0})$, we get

**Theorem 0.3.** Let $m$ be an odd positive integer. Assume Conjecture 0.1 for all odd integers $n$ with $0 < n \leq m$. Then Conjecture 0.2 holds for Greenberg’s $L$-invariant of $\rho_{2m,m}$.

All the assumptions in [Gr] (particularly, $\text{Sel}_F(\rho_{2m,m}) = 0$: Lemma 1.2) made to define the invariant can be verified under Conjecture 0.1 for $\rho_{2m,m}$. The assumption in the theorem that $E$ has split (multiplicative) reduction at $p_j$ with $j \leq b$ is inessential, because $\text{Ad}(\rho_{n,0}) \cong \text{Ad}(\text{Sym}^n(\rho_E \otimes \chi))$ (for a $K^\times$-values Galois character $\chi$) and we can bring any elliptic curve with multiplicative reduction at $p_j$ to an elliptic curve with split multiplicative reduction at $p_j$ by a quadratic twist. We will prove this theorem as Theorem 1.14 later.
Conjecture 0.1 and Conjecture 0.2 are logically close. Since $\rho_{2m,m}$ is self dual, the complex $L$-function $L(s, \rho_{2m,m})$ has functional equation of the form $s \mapsto 1 - s$, and the complex $L$-value $L(1, \rho_{2m,m})$ should not vanish at $s = 1$ (the abscissa of convergence). Conjecturally, this should imply $\text{Sel}_{F}(\rho_{2m,m}) = 0$, since $\rho_{2m,m}$ with odd $m$ is critical at 1. This vanishing is essential for Greenberg’s definition of his $L$-invariant to work (especially in his definition of the subspace $\mathbf{T} \subset H^{1}(\text{Gal}(\overline{F}/F), \rho_{2m,m})$ ($\mathbf{T}$ is written later as $\mathbf{H}_{F}$ in this paper; see [Gr] page 163–4). Conjecture 0.1 for an integer $n \geq m$ implies $\text{Sel}_{F}(\rho_{2m,m}) = 0$ for odd $m > 0$ (see Lemma 1.2). Indeed, at least in appearance, a much weaker infinitesimal version than Conjecture 0.1 asserting that $R_{n}$ shares the tangent space with $K[[X_{j}, p]|_{p, 0 < j \leq n, j \text{ odd}}$ (that is, $K[[X_{j}, p]]/(X_{j}, p)^{2} \cong R_{n}/m_{n}^{2}$) is sufficient for this vanishing $\text{Sel}_{F}(\rho_{2m,m}) = 0$ and to prove Conjecture 0.2. However, for example, if $m = 1$ and $n = 1$, any characteristic 0 $p$-adic (motivic) Galois deformation $\rho$ over $\mathbb{Z}_{p}$ (not over $\mathbb{Q}_{p}$ in Conjecture 0.1) of $\rho := (\rho_{E} \mod p)$ has its $p$-adic $L$-function $L_{p}(s, \rho_{2,1})$ with an exceptional zero at $s = 1$. Thus the weaker infinitesimal statement at each $p$ should actually imply the stronger statement as in Conjecture 0.1 (if we admit the “$T = T$” theorem as in [MFG] Theorem 5.29 for $F = \mathbb{Q}$ or [HMI] Theorem 3.50 for general $F$ for nearly ordinary deformations). In this sense, the two conjectures are almost equivalent if we include motivic deformations $\rho$ of $\overline{\rho}$ in the scope of Conjecture 0.2 not limiting ourselves to elliptic curves. This point will be discussed in more details in our future work.

§1. Symmetric Tensor $L$-Invariant

We recall briefly an $F$-version (given in [HMI] Definition 3.85) of Greenberg’s formula of the $L$-invariant for a general $p$-adic totally $p$-ordinary Galois representation $V$ (of $\text{Gal}(\overline{F}/F)$) with an exceptional zero. This definition is equivalent to the one in [Gr] if we apply it to $\text{Ind}_{F}^{Q}V$ as proved in [HMI] (in Definition 3.85). When $V = \rho_{2m,m}$ with odd $m$, the definition can be outlined as follows. Under some hypothesis, he found a unique subspace $H \subset H^{1}(Q, \text{Ind}_{F}^{Q} \rho_{2m,m})$ of dimension $e$. By Shapiro’s lemma, $H^{1}(Q, \text{Ind}_{F}^{Q} \rho_{2m,m}) \cong H^{1}(F, \rho_{2m,m})$, and one can give a definition of the image $H_{F}$ of $H$ in $H^{1}(F, \rho_{2m,m})$ without reference to the induction $\text{Ind}_{F}^{Q} \rho_{2m,m}$ ([HMI] Definition 3.85) as we recall the precise definition later (see Lemma 1.7). The space $H_{F}$ is represented by cocycles $c : \text{Gal}(\overline{F}/F) \to \rho_{2m,m}$ such that

(1) $c$ is unramified outside $p$;

(2) $c$ restricted to the decomposition subgroup $\text{Gal}(\overline{F}_{p}/F_{p}) \cong D_{p} \subset \text{Gal}(\overline{F}/F)$
at each $p|p$ has values in $\mathcal{F}_p^+\rho_{2m,m}$ and $c|D_p$ modulo $\mathcal{F}_p^+\rho_{2m,m}$ becomes unramified over $F_p[[\mu_{p^\infty}]]$ for all $p|p$.

Here $\mathcal{F}_p^+\rho_{2m,m} = F_p^0\rho_{2m,m}$, $\mathcal{F}_p^+\rho_{2m,m} = F_p^1\rho_{2m,m}$, and $\mathcal{F}_p^+\rho_{2m,m}$ is the decreasing filtration on $\rho_{2m,m}$ such that $I_p$ acts by $N^j$ on $F_p^j\rho_{2m,m}/F_p^{j+1}\rho_{2m,m}$.

Let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$-extension, and put $F_\infty/F$ for the composite of $F$ and $\mathbb{Q}_\infty$. By the condition (2), $(c|D_p, \bmod F_p^+\rho_{2m,m})$ with a prime $p'|p$ may be regarded as a homomorphism $a : D_p^+ \rightarrow K$ because $\mathcal{F}_p^+\rho_{2m,m}/\mathcal{F}_p^{+}\rho_{2m,m}$ is isomorphic to the trivial $D_p$-module $K$. Hence $a$ becomes unramified everywhere over the cyclotomic $\mathbb{Z}_p$-extension $F_\infty/F$. In other words, the cohomology class $[c]$ is in $\text{Sel}_{F_\infty}(\rho_{2m,m})$ but not in $\text{Sel}_F(\rho_{2m,m})$. In other words, we have

$$H_F \cong \text{Sel}_F^{\text{cyc}}(\rho_{2m,m}) := \text{Res}^{-1}(\text{Sel}_{F_\infty}(\rho_{2m,m}))$$

for the restriction map $\text{Res} : H^1(F,\rho_{2m,m}) \rightarrow H^1(F_\infty,\rho_{2m,m})$ (see the definition of various Selmer groups given in the following section).

Take a basis $\{e_p\}_{p|p}$ of $H_F$ over $K$. Write $a_p : D_p^+ \rightarrow K$ for $e_p$ mod $\mathcal{F}_p^+\rho_{2m,m}$ regarded as a homomorphism (identifying $\mathcal{F}_p^+\rho_{2m,m}/\mathcal{F}_p^+\rho_{2m,m}$ with $K$). We now have two $e \times e$ matrices with coefficients in $K$: $A = (a_p([p,F_p]))_{p,p'|p}$ and $B = (\log_p(\gamma_p)^{-1}a_p([\gamma_p,F_p]))_{p,p'|p}$. Under Conjecture 0.1 for $\rho_{n,0}$ for all odd $n \leq m$, we can show that $B$ is invertible. Then Greenberg’s $\mathcal{L}$-invariant is defined by

$$(1.1) \quad L(\text{Ind}_F^G \rho_{2m,m}) = \det(AB^{-1}).$$

The determinant $\det(AB^{-1})$ is independent of the choice of the basis $\{e_p\}_{p|p}$. Though $L(s,\text{Ind}_F^G \rho) = L(s,\rho)$ for a Galois representation $\rho : \text{Gal}(\mathbb{Q}/F) \rightarrow GL_n(K)$ in a compatible system, the (nonvanishing) modification Euler $p$-factors $E^+(\rho)$ and $E^+(\text{Ind}_F^G \rho)$ (cf. [Gr]) to define the corresponding $p$-adic $L$-functions could be different (see [H07] (1.1)). Thus the $\mathcal{L}(\rho)$ and $L(\text{Ind}_F^G \rho)$ could be slightly different. As in [H07] (1.1), we have the following relation

$$(1.2) \quad \mathcal{L}(\rho_{2m,m}) = \prod_{p|p} f_p \mathcal{L}(\text{Ind}_F^G \rho_{2m,m}),$$


Choose a generator $\gamma$ of $\mathcal{N}(\text{Gal}(F_\infty/F)) \subset \mathbb{Z}_p^\times$ for the $p$-adic cyclotomic character $\mathcal{N}$, and identify $\Lambda = W[[\text{Gal}(F_\infty/F)]]$ with $W[[T]]$ by $\gamma \mapsto 1 + T$. The Selmer group $\text{Sel}_{F_\infty}(\rho_{2m,m}) := \text{Sel}_{F_\infty}(\text{Sym}_2(T,E)(-m) \otimes (\mathbb{Q}_p/\mathbb{Z}_p))$ has its Pontryagin dual which is a $\Lambda$-module of finite type. Choose a characteristic power series $\Phi^{\text{arith}}(T) \in \Lambda$ of the Pontryagin dual. Put $L_p^{\text{arith}}(s,\rho_{2m,m}) = \Phi^{\text{arith}}(\gamma^{1-s}-1)$. We consider the following condition stronger than (ds):
(ds\_m) \overline{\eta}_{m,0} (for \overline{\eta}_{m,0} = \text{Sym}^\otimes_m(\overline{\rho})) is a direct sum of m+1 distinct characters of D\_p for all prime factors p|p.

For the known cases of the following conjecture, see [Gr] Proposition 4 and [H07] Theorem 5.3.

**Conjecture 1.1** (Greenberg). Suppose (ds\_m) and that \overline{\eta}_{m,0} is absolutely irreducible. Then \( L^\text{arith}_p(s, \rho_{2m,m}) \) has zero of order equal to \( e = |\{p|p\}| \) and for the constant \( \mathcal{L}(\rho_{2m,m}) \in K^\times \) given in (1.1) and (1.2), we have

\[
\lim_{s \to 1} \frac{L^\text{arith}_p(s, \rho_{2m,m})}{(s-1)^d} = \mathcal{L}(\rho_{2m,m}) |\text{Sel}_F(\rho_{2m,m}^*)|^{-1/[K:Q_p]}_p
\]

up to units.

This conjecture has been proven by Greenberg (see [Gr] Proposition 4) for more general ordinary Galois representation than \( \rho_{2m,m} \) under some (mild, believable but possibly restrictive) assumptions. Especially the assumption (5) in [Gr] proposition 4 is difficult to verify just by assuming (ds\_m) and absolute irreducibility of \( \overline{\eta}_{m,0} \) and could be far deeper (even for those of adjoint type like \( \rho_{2m,m} \)) than the modularity statement like Conjecture 0.1; so, unfortunately, the above statement remains to be a conjecture.

In the above conjecture, the modifying Euler factor at the \( p \)-adic places \( p_j \) of good reduction \( (j > b) \):

\[
\mathcal{E}^+(\rho_{2m,m}) = \prod_{j > b} \left( \prod_{i=1}^m (1 - \alpha_j^{-2i} N(p_j)^{i-1})(1 - \alpha_j^{-2i} N(p_j)\bar{i}) \right)
\]

does not appear, where \( \alpha_j = \alpha_j(\text{Frob}_{p_j}) \). However, if we replace Greenberg’s Selmer group \( \text{Sel}_F(\rho_{2m,m}^*) \) by the Bloch-Kato Selmer group \( \mathcal{S}_F(\rho_{2m,m}^*) \) over \( F \) (crystalline at \( p_j \) for \( j > b \)), we expect to have the relation

\[
|\text{Sel}_F(\rho_{2m,m}^*)|^{-1/[K:Q_p]}_p = \mathcal{E}^+(\rho_{2m,m}) |\mathcal{S}_F(\rho_{2m,m}^*)|^{-1/[K:Q_p]}_p
\]

up to \( p \)-adic units (as described in [MFG] page 284 for \( \rho_{2,1} \)). Thus if one uses the formulation of Bloch-Kato, we should have the modifying Euler factor in the formula, and the size of the Bloch-Kato Selmer group is expected to be equal to the primitive archimedean \( L \)-values (divided by a suitable period; see Greenberg’s Conjecture 0.1 in [H06]).
§1.1. Selmer groups

First we recall Greenberg’s definition of Selmer groups. Write $F^{(S)}/F$ for the maximal extension unramified outside $S$, $p$ and $\infty$. Put $\mathfrak{S} = \text{Gal}(F^{(S)}/F)$ and $\mathfrak{S}_M = \text{Gal}(F^{(S)}/M)$. Let $V$ be a potentially ordinary representation of $\mathfrak{S}$ on a $K$-vector space $V$. Thus $V$ has decreasing filtration $F_p^i V$ such that an open subgroup of $I_p$ (for each prime factor $p|p$) acts on $F_p^i V/F_p^{i+1} V$ by the $i$-th power $N^i$ of the $p$-adic cyclotomic character $N$. We fix a $W$-lattice $T$ in $V$ stable under $\mathfrak{S}$.

Put $F_p^i V = F_p^1 V$ and $F_p^{-i} V = F_p^0 V$. Writing $F_p^i T = T \cap F_p^i V$ and $F_p^i V/T = F_p^i V/F_p^i T$, we have a 3-step filtration for $A = V$, $T$ or $V/T$:

$$A \supseteq F_p^- A \supseteq F_p^0 A \supseteq \{0\}.$$ Its dual $V^*(1) = \text{Hom}_K(V, K) \otimes N$ again satisfies $(\text{ord})$.

Let $M/F$ be a subfield of $F^{(S)}$, and put $\mathfrak{S}_M = \text{Gal}(F^{(S)}/M)$. We write $p$ for a prime of $M$ over $p$ and $q$ for general primes outside $p$ of $M$. We write $I_p$ and $I_q$ for the inertia subgroup in $\mathfrak{S}_M$ at $p$ and $q$, respectively. We put

$$L_p(A) = \text{Ker} \left( \text{Res} : H^1(M_p, A) \to H^1 \left( I_p, \frac{A}{F_p^i (A)} \right) \right),$$

and

$$L_q(A) = \text{Ker} (\text{Res} : H^1(M_q, A) \to H^1(I_q, A)).$$

Then we define the Selmer submodule in $H^1(M, A)$ by

$$\text{Sel}_M(A) = \text{Ker} \left( H^1(\mathfrak{S}_M, A) \to \prod_q \frac{H^1(M_q, A)}{L_q(A)} \times \prod_p \frac{H^1(M_p, A)}{L_p(A)} \right)$$

for $A = V, V/T$. The classical Selmer group of $V$ is given by $\text{Sel}_M(V/T)$, equipped with discrete topology. We define the “minus”, the “locally cyclotomic” and the “strict” Selmer groups $\text{Sel}^-_M(A)$, $\text{Sel}^{loc}_M(A)$ and $\text{Sel}^{st}_M(A)$, respectively, replacing $L_p(A)$ by

$$L_p^-(A) = \text{Ker} \left( \text{Res} : H^1(M_p, V) \to H^1 \left( I_p, \frac{V}{F_p^i (A)} \right) \right) \supset L_p(A)$$

$$L_p^{loc}(A) = \text{Ker} \left( \text{Res} : L_p^-(A) \to H^1 \left( I_p, \frac{V}{F_p^i (A)} \right) \right) \subset L_p^-(A)$$

$$L_p^{st}(A) = \text{Ker} \left( \text{Res} : L_p^-(A) \to H^1 \left( M_p, \frac{V}{F_p^i (A)} \right) \right) \subset L_p(A),$$
where $I_{p, \infty}$ is the inertia group of $\text{Gal}(\overline{M}_p/M_p[\mu_p])$. Then we have

$$\text{Sol}_F^{\text{cyc}}(A) = \text{Res}_{F_{\infty}/F}^{-1}(\text{Sel}_{F_{\infty}}(A)).$$

**Lemma 1.2.** We have

$$\text{Sel}_F^{\text{cyc}}(\text{Ad}(\rho_{n,0})) \cong \bigoplus_{0 < m \leq n, m\text{ odd}} \text{Sel}_F^{\text{cyc}}(\rho_{2m,m}) \cong \text{Hom}_K(m_n/m_n^2, K),$$

where $m_n$ is the maximal ideal of $R_n$. If we suppose Conjecture 0.1 for odd $n > 0$, we have $\text{Sel}_F(\rho_{2m,m}) = 0$ for all odd $m$ with $0 < m \leq n$.

**Proof.** Let $V = \text{Ad}(\rho_{n,0})$. Then we have the filtration:

$$V \supset F_p^- V \supset F_p^+ V \supset \{0\},$$

where taking a basis so that the semi-simplification of $\rho_{n,0}|_{D_p}$ is diagonal with diagonal character $\beta_p^n, \beta_p^{n-1} \alpha_p, \ldots, \alpha_p$ in this order from top to bottom, $F_p^- V$ is made up of upper triangular matrices and $F_p^+ V$ is made up of upper nilpotent matrices, and on $F_p^- V/F_p^+ V$, $D_p$ acts trivially (getting eigenvalue 1 for $\text{Frob}_p$). We consider the space $\text{Der}_K(R_n, K)$ of continuous $K$-derivations of $R_n$. Let $K[\varepsilon] = K[t]/(t^2)$ for the dual number $\varepsilon = (t \mod t^2)$. Then writing each $K$-algebra homomorphism $\phi : R_n \rightarrow K[\varepsilon]$ as $\phi(r) = \phi_0(r) + \varepsilon \phi_1(r)$ and sending $\phi$ to $\partial\phi \in \text{Der}_K(R_n, K)$, we have $\text{Hom}_{K\text{-alg}}(R_n, K[\varepsilon]) \cong \text{Der}_K(R_n, K) = \text{Hom}_K(m_n/m_n^2, K)$. By the universality of $(R_n, \rho_{n,0})$, we have

$$\text{Hom}_{K\text{-alg}}(R_n, K[\varepsilon]) \cong \{\rho : \text{Gal}(\overline{F}/F) \rightarrow G_n(K[\varepsilon])|\rho \text{ satisfies } (K_n1-4)\} \cong$$

by $\text{Hom}_{K\text{-alg}}(R_n, K[\varepsilon]) \ni \phi \mapsto \rho_\phi = \phi \circ \rho_{n,0} = \rho_{n,0} + \varepsilon \partial\phi \rho_{n,0}$. Pick $\rho = \rho_\phi$ as above. Write $\rho(\sigma) = \rho_0(\sigma) + \rho_1(\sigma)\varepsilon$ with $\rho_1(\sigma) = \partial\phi \rho_{n,0}(\varepsilon)$. Then $c_\rho = (\partial\phi \rho_{n,0})\rho_{n,0}^{-1}$ can be easily checked to be an inhomogeneous 1-cocycle having values in $M_{n+1}(K) \supset V$. Here $\sigma \in \text{Gal}(\overline{F}/F)$ acts on $x \in M_{n+1}(K)$ by $x \mapsto \rho_{n,0}(\sigma)x\rho_{n,0}(\sigma)^{-1}$.

Since $\nu \circ \rho = \nu \circ \rho_{n,0}$ by (K_n, 3), we have $\det(\rho) = \det(\rho_{n,0})$, which implies $\text{Tr}(c_\rho) = 0$; so, $c_\rho$ has values in $\mathfrak{sl}_{n+1}(K)$. For $\partial \in \text{Der}_K(R_n, K)$ and $X \in GL_{n+1}(R_n)$ with $^tXJ_nX = J_n$, writing $\overline{X} = (X \mod m_n) \in GL_{n+1}(K)$

$$0 = \partial(\overline{X}^{-1}X) = \overline{X}^{-1}\partial X + (\overline{X}^{-1})\overline{X}.$$

Since $^t\rho_nJ_n\rho_n = N^nJ_n = ^t\rho_{n,0}J_n\rho_{n,0}$, we have $^t\rho_{n,0}^{-1}\rho_nJ_n\rho_n\rho_{n,0}^{-1} = J_n$. Let $X = \rho_n\rho_{n,0}^{-1}$. Differentiating the identity: $^tXJ_nX = J_n$, by $\partial$, we have
\((\partial X J_n)X + (J_n \partial X) = 0\), which is equivalent to \(c_p(\sigma) \in \mathfrak{a}_n(K) = V\). By the reducibility condition \((K_n 2), [c_p]\) vanishes in \(H^1(M_n, V)\). By the local cyclotony condition in \((K_n 2), [c_p]\) vanishes in \(H^1(M_n, V)\). If \(E\) has multiplicative reduction at \(q\) (so, \(q \in S\), the unramifiedness of \(c_p\) follows from the following lemma. Thus the cohomology class \([c_p]\) of \(c_p\) is in \(\text{Sel}^{cyc}_E(V)\). We see easily that \(\rho \cong \rho' \iff [c_p] = [c_{p'}]\). We can reverse the above argument starting with a cocycle \(c\) giving an element of \(\text{Sel}^{cyc}_E(V)\) to construct a deformation \(\rho_c = \rho_{n,0} + \varepsilon(c)\) with values in \(G_n(K[\varepsilon])\). Thus we have

\[
\{ \rho : \text{Gal}(F/\mathbb{F}) \to G_n(K[\varepsilon]) | \rho \text{ satisfies the conditions } (K_n 1–4) \} \cong \text{Sel}^{cyc}_E(V).
\]

Recall that the isomorphism \(\text{Der}_K(R_n, K) \cong \text{Sel}^{cyc}_E(V)\) is given by

\[
\text{Der}_K(R_n, K) \ni \partial \mapsto [c_\partial] \in \text{Sel}^{cyc}_E(V)
\]

for the cocycle \(c_\partial = c_\partial = (\partial \rho_{n,0})\rho_{n,0}^{-1}\), where \(\rho = \rho_{n,0} + \varepsilon(\partial \rho_{n,0})\).

Suppose Conjecture 0.1. Since the algebra structure of \(R_n\) over \(W[\{X_j, p\}]_{p|p}\) is given by \(\delta_{j,p}(\beta_p^{m-1})\alpha_p^{j})^{-1}\) and \(\delta_{n-j,p} = N^n\), the \(K\)-derivation \(\partial = \partial_0 : R_n \to K\) corresponding to a \(K[\varepsilon]\)-deformation \(\rho\) is a \(W[\{X_j, p\}]\)-derivation for odd \(j\) if and only if \(\partial \rho_{n,0}|_{I_q}\) is upper nilpotent, which is equivalent to \([c_\partial] \in \text{Sel}_F(V)\). Thus we have \(\text{Sel}_F(V) \cong \text{Der}_W[\{X_j\}](R_n, K) = 0\). Since \(V \cong \bigoplus_{0 < m \leq n, m \text{ odd}} \rho_{2m,m}\) as global Galois modules, we have \(\text{Sel}_F(V) \cong \bigoplus_{0 < m \leq n, m \text{ odd}} \text{Sel}_E(\rho_{2m,m})\), and we conclude \(\text{Sel}_F(\rho_{2m,m}) = 0\).

**Lemma 1.3.** Let \(q\) be a prime outside \(p\) at which \(E\) has potentially multiplicative reduction. Then for a deformation \(\rho\) of \(\rho_{n,0}\) satisfying \((K_n 1–4), the cocycle \(c_\partial\) (defined in the above proof) is unramified at \(q\).

**Proof.** Since \(\text{Ad}(\rho_E \otimes \eta)|_{n,0}) \cong \text{Ad}(\rho_{n,0})\) twisting by a character \(\eta\), we may assume that the restriction of \(\rho_E\) to the inertia group \(I_q\) has values in the upper unipotent subgroup having the form \(\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \xi_1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}\) for \(\sigma \in I_q\) up to conjugation. Thus we may assume

\[
\rho_{n,0}|_{I_q} = \begin{pmatrix}
1 & \xi_1 & \cdots & \xi_{n-1} \\
0 & 1 & \cdots & \xi_{n-1} \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

Since \(I_q \ni \sigma \mapsto \log(\rho_{n,0}(\sigma))\) is a homomorphism of \(I_q\) into the Lie algebra \(\mathfrak{u}_n\) of the unipotent radical of the Borel subgroup of \(G_n\) containing the image...
of \( I_q \), it factors through the tame inertia group \( \cong \mathbb{Z}^{(q)}(1) \). By the theory of Tate curves, \( \rho_{n,0} \) ramifies at \( q \) and hence \( \xi_q \) is nontrivial. The \( p \)-factor of \( \mathbb{Z}^{(q)} \) is of rank 1 isomorphic to \( \mathbb{Z}_p(1) \). Then \( \rho(I_q) \) is cyclic, and therefore \( \dim_K \rho(I_q) = 1 = \dim_K \rho_{n,0}(I_q) \). Thus the deformation \( \rho \) is constant over the inertia subgroup, and hence \( e_p \) restricted to \( I_q \) is trivial.

**Corollary 1.4.** Let \( n \) be an odd positive integer. Suppose Conjecture 0.1 for all odd integers \( m \) with \( 0 < m \leq n \). Then we have \( \dim_K \text{Sel}_{E,F}^{\text{ge}}(\rho_{2n,n}) = e \).

**Proof.** Let \( V = \rho_{2n,n} \). By Lemma 1.2, we have \( \dim_K \text{Sel}_{E,F}^{\text{ge}}(\text{Ad}(\rho_{n,0})) = e \cdot \frac{m+1}{2} \). Since

\[
\text{Sel}_{E,F}^{\text{ge}}(\text{Ad}(\rho_{n,0})) = \text{Sel}_{E,F}^{\text{ge}}(\text{Ad}(\rho_{n-2,0})) \oplus \text{Sel}_{E,F}^{\text{ge}}(V),
\]

we find that \( \dim_K \text{Sel}_{E,F}^{\text{ge}}(V) = e \).\( \square \)

Let \( \rho_{n,m} = \text{Sym}^{\otimes n}(\rho_E)(-m) \), and write \( V \) for either the representation space of \( \rho_{n,m} \) or that of \( \text{Ad}(\rho_{n,0}) \). For each prime \( q \in S \cup \{ p |p \} \), we put

\[
(1.4) \quad \mathcal{T}_q(V) = \begin{cases} \text{Ker}(H^1(F_j, V) \to H^1(F_j, \frac{V_{q_j}}{\tau_{q_j}(V_{q_j})})) \subset L_{p_j}(V) & \text{if } q = p_j \text{ with } j \leq b, \\ L_q(V) & \text{otherwise} \end{cases}
\]

Once \( \mathcal{T}_q(V) \) is defined, we define \( \mathcal{T}_q(V^*(1)) = \mathcal{T}_q(V)^{\perp} \) under the local Tate duality between \( H^1(F_q, V) \) and \( H^1(F_q, V^*(1)) \), where \( V^*(1) = \text{Hom}_K(V, \mathbb{Q}_p(1)) \) as Galois modules. Then we define the balanced Selmer group \( \text{Sel}_{E,F}^{\text{ge}}(V) \) (resp. \( \text{Sel}_{E,F}(V^*(1)) \)) by the same formula as in (1.3) replacing \( L_q(V) \) (resp. \( L_q(V^*(1)) \)) by \( \mathcal{T}_p(V) \) (resp. \( \mathcal{T}_p(V^*(1)) \)). By definition, \( \text{Sel}_{E,F}(V) \subset \text{Sel}_{E,F}(V) \). We will show in Lemma 1.6, \( \mathcal{T}_p(V) = L_p(V) \) for \( V = \text{Ad}(\rho_{n,0}) \) and \( \rho_{2n,n} \) for odd \( n \), and we actually have \( \text{Sel}_{E,F}(V) = \text{Sel}_{E,F}(V) \).

**Lemma 1.5.** Let \( V \) be \( \text{Ad}(\rho_{n,0}) \) or \( \rho_{n,m} \). If \( V \) is critical at \( s = 1 \),

\[
(V) \quad \text{Sel}_{E,F}(V) = 0 \Rightarrow H^1(\mathcal{Q}, V) \cong \prod_{q \in S} \frac{H^1(F_q, V)}{L_q(V)} \times \prod_{p \not| p} \frac{H^1(F_p, V)}{\mathcal{T}_p(V)}.
\]

**Proof.** Since \( \text{Sel}_{E,F}(V) \subset \text{Sel}_{E,F}(V) \), the assumption implies \( \text{Sel}_{E,F}(V) = 0 \). Then the Poitou-Tate exact sequence tells us the exactness of the following sequence:

\[
\text{Sel}_{E,F}(V) \to H^1(\mathcal{Q}, V) \to \prod_{l \in \mathbb{Q} \cup \{ p \}} \frac{H^1(F_l, V)}{L_l(V)} \to \text{Sel}_{E,F}(V^*(1))^*.
\]
It is an old theorem of Greenberg (which assumes criticality at $s = 1$) that
\[
\dim \Sel_F(V) = \dim \Sel_F(V^*(1))^* 
\]
(see [Gr] Proposition 2 or [HMI] Proposition 3.82); so, we have the assertion (V). In [HMI], Proposition 3.82 is formulated in terms of $\Sel_Q(\Ind_F^G V)$ and $\Sel_Q(\Ind_F^G V^*(1))$ defined in [HMI] (3.4.11), but this does not matter because we can easily verify $\Sel_Q(\Ind_F^G V^*) \cong \Sel_F(V^*)$ (similarly to [HMI] Corollary 3.81).

§1.2. Greenberg’s $L$-invariant

In this subsection, we let $V = \rho_{2n,n}$ or $Ad(\rho_{n,0})$ for odd $n$ (so, $V$ is critical at $s = 1$). Write $t(p)$ for $\dim \mathcal{F}^p F / \mathcal{F}^0 V$ (thus, $t(p) = 1$ or $\frac{p+1}{2}$ according as $V = \rho_{2n,n}$ or $Ad(\rho_{n,0})$). We recall a little more detail of the $F$-version of Greenberg’s definition of $\mathcal{L}(\Ind_F^G V)$ (which is equivalent to the one given in [Gr] if we apply Greenberg’s definition to $\Ind_F^G V$ as explained in [HMI] 3.4.4 without assuming the simplifying condition). Let $\mathcal{F}^\text{gal}_p$ be the Galois closure of $F_p/Q_p$ in $\overline{Q}_p$. Write $D_p = \text{Gal}(\overline{Q}_p/Q_p)$, $D_p = \text{Gal}(\overline{Q}_p/F_p)$ and $D_p^{\text{gal}} = \text{Gal}(\overline{Q}_p/F_p^{\text{gal}})$. Write $D_L = \text{Gal}(\overline{Q}_p/L)$ for an intermediate field $L$ of $F_p^{\text{gal}}/Q_p$. For a $D_L$-module $M$ (which is a $K$-vector space), the group $D_L$ acts on $H^*(F_p^{\text{gal}}, M)$ naturally through the finite quotient $\text{Gal}(F_p^{\text{gal}}/L)$. Since, for $q > 0$,
\[
H^q(\text{Gal}(F_p^{\text{gal}}/L), H^0(D_p^{\text{gal}}, M)) = 0,
\]
by the inflation-restriction sequence, taking $L = Q_p$ and $L = F_p$, we verify that $H^1(F_p^{\text{gal}}, M)^{D_p}$ is canonically isomorphic to a subspace of $H^1(F_p, M)$ even if $F_p/Q_p$ is not a normal extension. We regard $H^1(F_p^{\text{gal}}, M)^{D_p}$ as a subspace of $H^1(F_p, M)$.

The long exact sequence associated to the short one $\mathcal{F}^p V / \mathcal{F}^0 V \hookrightarrow V / \mathcal{F}^p V \twoheadrightarrow V / \mathcal{F}^p V$ gives a homomorphism
\[
H^1 \left( F_p^{\text{gal}}, \frac{\mathcal{F}^p V}{\mathcal{F}^0 V} \right) \overset{D_p}{\to} \text{Hom} \left( \left( D_p^{\text{gal}} \right)^{ab}, \frac{\mathcal{F}^p V}{\mathcal{F}^0 V} \right) \overset{\iota}_{D_p} \to H^1(F_p^{\text{gal}}, V) / \mathcal{F}_p(V),
\]
where $D_p$ acts on $H^1(F_p^{\text{gal}}, \frac{\mathcal{F}^p V}{\mathcal{F}^0 V})$ regarding $\frac{\mathcal{F}^p V}{\mathcal{F}^0 V}$ as the trivial $D_p$-module; so, its action on $\phi \in \text{Hom}(\left( D_p^{\text{gal}} \right)^{ab}, \frac{\mathcal{F}^p V}{\mathcal{F}^0 V})$ is given by $\phi \mapsto \tau \cdot \phi(\sigma) = \phi(\tau \sigma \tau^{-1})$. Note that canonically
\[
H^1 \left( F_p^{\text{gal}}, \frac{\mathcal{F}^p V}{\mathcal{F}^0 V} \right) \overset{\iota}_{\text{Res}} \leftarrow \text{Hom} \left( D_p^{ab}, \frac{\mathcal{F}^p V}{\mathcal{F}^0 V} \right) \cong \text{Hom} \left( Q_p^{\text{ab}}, \frac{\mathcal{F}^p V}{\mathcal{F}^0 V} \right) \cong (\mathcal{F}^p V / \mathcal{F}^0 V)^2 \cong K^{2t(p)}
\]
by \( \phi \mapsto (\frac{a(\gamma[p])}{\log(p)}, \phi([p,F_p])) \). Here, as before, \([x,F_p]\) is the local Artin symbol. Identifying \( H^1(F_p^{ab}, \frac{\mathbb{F}_p}{\mathbb{F}_p^+}) \) with \( \text{Hom}(D_p^{ab}, \frac{\mathbb{F}_p}{\mathbb{F}_p^+}) \), a homomorphism \( \phi : D_p^{ab} \to \frac{\mathbb{F}_p}{\mathbb{F}_p^+} \) in \( \text{Ker}(\phi) \) is unramified if \( p = p_i \) with \( i > b \); so, the image of \( \phi \) is one-dimensional (those ramified classes modulo unramified ones).

In other words, the image of \( \phi \) is isomorphic to \( \mathbb{F}_p \) by \( \phi \). We may assume that \( \phi(\gamma) = 1 \). Writing \( Y \hookrightarrow V \) and \( Y \hookrightarrow V/\mathbb{F}_p^+V \), we need to show that \( \text{Im}(\gamma) \cap \text{Ker}(\text{Res}) = 0 \).

Similarly \( \text{Ker}(\text{Res}_Y : H^1(Y) \to H^1(I_p, Y)) = H^1(D_p/I_p, Y/I_p) \) and \( \text{Ker}((\gamma, F): H^1(Y) \to H^1(I_p, Y)) \) are injective. Identify \( H^1(Y) \) with its image in \( H^1(V) \). We have

\[
\text{Im}(\gamma) = \text{Im}(\gamma, F : H^1(Y) \to H^1(Y/I_p)) \subset H^1(V).
\]

By the inflation-restriction sequence,

\[
\text{Ker}(\text{Res}) = H^1(D_p/I_p, Y/I_p) = \mathbb{F}_p^+/\text{Frob}_p - 1)\mathbb{F}_p^+ = \mathbb{F}_p^+V/\mathbb{F}_p^+V.
\]

Similarly

\[
\text{Ker}(\text{Res}_Y : H^1(Y) \to H^1(I_p, Y)) = H^1(D_p/I_p, Y/I_p) = \mathbb{F}_p^+/\text{Frob}_p - 1)\mathbb{F}_p^+ Y = \mathbb{F}_p^+V/\mathbb{F}_p^+V.
\]

Thus inside \( H^1(V) \), \( \text{Ker}(\text{Res}) = \text{Ker}((\gamma, F) \), and we may replace \( V \) by \( Y \) in our argument. We therefore need to show that

\[
\text{Im}(\gamma, F : H^1(Y) \to H^1(Y/I_p)) \cap \text{Ker}(\text{Res}_Y : H^1(Y) \to H^1(I_p, Y)) = 0.
\]
We have the long exact sequence attached to the short one $F_p^Y \hookrightarrow Y \rightarrow \overline{Y}$:

$$0 \to \overline{Y} = H^0(\overline{Y}) \to H^1(F_p^Y) \to H^1(Y) \to H^2(F_p^Y) \to H^2(Y) = 0.$$ 

By the non-splitting of the short sequence, $H^0(\overline{Y})$ injects into $H^1(F_p^Y)$. By the local Tate duality,

$$\dim_K H^2(Y) = \dim_K H^0((\text{Hom}_K(Y, K(1)))) = 0 \text{ and } \dim_K H^2(F_p^Y) = t(p).$$

This shows that $\dim_K H^1(Y) = 2t(p)d$ and $\dim_K \text{Im}(\overline{Y}) = t(p)d$, because by Kummer’s theory

$$H^1(K(1)) = K \otimes_{\mathbb{Z}_p} \lim_n F_p^Y/(F_p^Y)^p = K^{d+1}$$

and $H^1(K) \cong \text{Hom}((F_p)^*, K) \cong K^{d+1}$ for $d = [F_p, \mathbb{Q}_p]$. By the inflation-restriction sequence, we have

$$L_p(\overline{Y}) := \text{Ker}(H^1(\overline{Y}) \to H^1(I_p, \overline{Y})) \cong H^1(D_p/I_p, \overline{Y}^I) \cong \overline{Y}.$$ 

Thus $\dim L_p(\overline{Y}) + \dim \text{Im}(\overline{Y}) = \dim H^1(I_p, \overline{Y})$. Thus we need to show $L_p(\overline{Y}) + \text{Im}(\overline{Y}) = H^1(I_p, \overline{Y})$. By the local Tate duality, noting $Y^*(1) \cong Y$, this statement is equivalent to

$$\text{Ker}(\delta : H^1(F_p^Y) \to H^1(Y)) \cap L_p(\overline{Y}) = 0.$$ 

Here $L_p(\overline{Y}) = H^1(I_p, F_p^Y) = \overline{Y} \otimes_{\mathbb{Z}_p} \lim_n (O_p^*/(O_p^*)^p) \subset H^1(\overline{Y}(1))$, because $\overline{Y}(1) = \overline{Y}(1) = K(1)^{t(p)}$. Since $\text{Ker}(\delta)$ gives rise to the subspace spanned by extension class of $K(1)^{t(p)} = F_p^Y \hookrightarrow Y \rightarrow \overline{Y} \cong K^{t(p)}$, it is given by the cocycles in $\xi_q \otimes \overline{Y}$ for the Tate period $q$ of $E$ at $p = p_j$ (where $\xi_q$ is as in the proof of Lemma 1.3). Defining $\xi_q : D_p \to \mu_{p^n}$ by $\xi_q(\sigma) = (q^{1/p^n})^{q-1}$, the map $\xi_q = \lim_n \xi_q$, having values in $\mathbb{Z}_p(1) \subset K(1)$ is an explicit form of the cocycle $\xi_q$ (see [H07] Section 4). In particular, $(\overline{Y} \otimes \xi_q) \cap H^1(I_p, F_p^Y)$ is given by

$$(q \otimes \overline{Y}) \cap (\overline{Y} \otimes_{\mathbb{Z}_p} \lim_n (O_p^*/(O_p^*)^p)^n)$$

inside $\overline{Y} \otimes_{\mathbb{Z}_p} \lim_n (F_p^Y/(F_p^Y)^p)^n$, which is trivial (because $q$ is a nonunit). \(\square\)

Suppose $R_n \cong K[[X_p]]_{\sigma/p}$. Then by (V) in Lemma 1.5 (and Lemma 1.2), we have a unique subspace $H_F$ of $H^1(\mathfrak{C}, V)$ projecting down onto

$$\prod_p \text{Im}(\iota_p) \hookrightarrow \prod_p \frac{H^1(F_p, V)}{L_p(V)}.$$
Then by the restriction, $H_F$ gives rise to a subspace $L = L_V$ of

$$\prod_p \Hom((D_p^{gal})|_{H_v}, F_p V/F_p^+ V)^{D_p}\simeq \prod_p \Hom(D_p^{ab}, F_p V/F_p^+ V) \simeq \prod_p (F_p^+ V/F_p^+ V)^2$$

isomorphic to $\prod_p (F_p^+ V/F_p^+ V)$. If a cocycle $c$ representing an element in $H_F$ is unramified, it gives rise to an element in $\Sel_F(V)$. By the vanishing of $\Sel_F(V)$ (Lemma 1.2), this implies $c = 0$; so, the projection of $L$ to the first factor $\prod_p F_p^+ V/F_p^+ V$ (via $\phi \mapsto (\phi(\gamma, F_p^{gal}))|_{\log_p(\gamma)}$) is surjective. Thus this subspace $L$ is a graph of a $K$–linear map

$$(1.5) \quad \mathcal{L}: \prod_p F_p^+ V/F_p^+ V \rightarrow \prod_p F_p^+ V/F_p^+ V.$$

We then define $\mathcal{L}(\text{Ind}_{F}^{V}) = \det(\mathcal{L}) \in K$. This is a description of the direct construction of $H_F$. In the following lemma, we verify the equivalence between the earlier definition and this direct one:

**Lemma 1.7.** Let $V = \text{Ad}(\rho_{n,0})$ or $\rho_{2m,m}$ for an odd $m > 0$, and assume that $\Sel_p(V) = 0$. The space $H_F$ defined above consists of cohomology classes of 1-cocycles $c : \text{Gal}(\overline{\mathbb{F}}/F) \rightarrow V$ such that

1. $c$ is unramified outside $p$;
2. $c$ restricted to the decomposition subgroup $\text{Gal}(\overline{F}_p/F_p) \simeq D_p \subset \text{Gal}(\overline{F}/F)$ at each $p | p$ has values in $F_p^+ V$ and $c|_{D_p}$ modulo $F_p^+ V$ becomes unramified over $F_p^{\mu_{p^{\infty}}}$ for all $p | p$.

We here give a sketch of the proof, assuming $F_p = F_p^{gal}$ (leaving the general case to the attentive reader).

**Proof.** Since $\text{Ad}(\rho_{n,0}) \cong \bigoplus_{0<j \leq n, j \text{ odd}} \rho_{2j, j}$, we may assume that $V = \text{Ad}(\rho_{n,0})$. Recall the decomposition groups $D_p \supset D_p$ in $\text{Gal}(\overline{F}/\mathbb{Q})$ at $p$, and write $I_p \supset I_p$ for the corresponding inertia groups. Let $H_F \subset H^1(\text{Gal}(\overline{F}/F), V)$ be the subspace spanned by the cohomology classes satisfying (1) and (2). Take a cocycle $c$ satisfying (1) and (2). Note that for any $\sigma \in D_p$, $\sigma(F_p) = F_p$ by our simplifying assumption. Since $Q_\nu[\mu_{p^{\infty}}]/Q_p$ is abelian, we have $\sigma\gamma\sigma^{-1} = \gamma$ for any $\gamma \in \text{Gal}(F_p^{\mu_{p^{\infty}}}/F_p)$. Since $(c|_{I_p}) \mod F_p^+ V: I_p \rightarrow F_p^+ V/F_p^+ V$ factors through $\text{Gal}(F_p^{\mu_{p^{\infty}}}/F_p)$, for any $\sigma \in D_p$, $c(\sigma\gamma\sigma^{-1}) = c(\gamma)$ for any $\gamma \in I_p$. Since $D_p = \phi^{\sigma} \times I_p$ for a Frobenius element $\phi = \text{Frob}_p$, the cocycle $(c|_{D_p}) \mod F_p^+ V$ is actually $D_p$-invariant. Thus $c|_{D_p} \mod F_p^+ V$ is in
Groups $G \triangleleft H$ of $X$hausted by $\Gamma = \text{Gal}(\mathbb{Q}(\sqrt[n]{s})/\mathbb{Q})$ for all odd $n$.

Further we have the restriction map in Lemma 1.5.

Then it extends to a homomorphism $\overline{G} \to \overline{H}$ and hence the above map $\text{Res}$ is the map in Lemma 1.5. Thus we conclude $H_2^\prime \subset \text{Res}^{-1}(\prod_{p \mid \ell} \text{Im}(i_p))$.

Conversely, we suppose that the class $[c|_{D_p} \mod \mathcal{F}_p] \in \text{Im}(i_p)$. Thus the homomorphism $(c|_{D_p} \mod \mathcal{F}_p) : D_p \to \mathcal{F}_p / \mathcal{F}_p$ is $D_p$-invariant. Then it extends to a homomorphism $\tilde{c}_p : V \to \mathcal{F}_p / \mathcal{F}_p$. Indeed, for any two groups $G \supset H$ with finite index and a torsion-free divisible abelian group $X$, every $G$-invariant homomorphism $\phi : X \to X$ extends to a homomorphism $\phi : G \to X$ by Schur’s theory of multipliers (e.g. [MFG] 4.3.5), because the obstruction lies in $H^2(G/H, X)$ which vanishes by the finiteness of $G/H$ and divisibility of $X$. Then $\tilde{c}_p$ has to factor through $\text{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ for the maximal abelian extension $\mathbb{Q}_p^{ab}/\mathbb{Q}_p$, which is equal to $\mathbb{Q}_p^{ur}[\mu_{p \infty}]$ for the maximal unramified extension $\mathbb{Q}_p^{ur}/\mathbb{Q}_p$ (by local class field theory); so, $(c|_{I_p}) \mod \mathcal{F}_p$ factors through $\text{Gal}(\mathbb{Q}_p^{ur}/\mathbb{Q}_p)$ and $c$ satisfies (2). The condition (1) for $c|_{D_q} (q \nmid p)$ is equivalent to the vanishing of $i_q(c|_{D_q})$ in $H^1(V, \mathbb{Q}_q(V))$. Then we get the reverse inclusion. Since $\text{Res}$ is an isomorphism if $\text{Sel}_F(V) = 0$ by Lemma 1.5, $H_2^\prime \subset \text{Res}^{-1}(\prod_{p \mid \ell} \text{Im}(i_p))$ is a surjective isomorphism, and hence $H_2^\prime = H_2$.

If one restricts $c \in H_2$ to $\frak{S}_\infty = \text{Gal}(\mathbb{F}(S)/\mathbb{F}_\infty)$, its ramification is exhausted by $\Gamma = \text{Gal}(\mathbb{F}(S)/\mathbb{F})$ (because of the definition of $\text{Sel}_F^{\text{reg}}(\rho_{2n,m})$ and $H_2$) giving rise to a class $[c] \in \text{Sel}_{\frak{S}_\infty}(V)$. The kernel of the restriction map: $H^1(\frak{S}_\infty(V)) \to H^1(\frak{S}_\infty, V)$ is given by $H^1(\Gamma, H^0(\frak{S}_\infty(V))) = 0$ because $H^0(\frak{S}_\infty, V) = 0$. Thus the image of $H_2$ in $\text{Sel}_{\frak{S}_\infty}(V/T)$ gives rise to the order $e$ exceptional zero of $L^{\text{ar}}(s, \rho_{2n,m})$ at $s = 1$. We have replaced the first half of the following result in [Gr] Proposition 1.

Proposition 1.8. Let $n$ be an odd positive integer. Suppose Conjecture 0.1 for all odd $m \leq n$. Then for the number $e$ of prime factors of $p$ in $F$, we have

$$\text{ord}_{s=1} L^{\text{ar}}(s, \rho_{2n,m}) \geq e.$$ 

Further we have $L(p_{2n,m}) = 0 \iff \text{ord}_{s=1} L^{\text{ar}}(s, \rho_{2n,m}) > e$.

The last assertion follows from [Gr] Proposition 3. In [Gr] Proposition 3, Conjecture 0.1 is not assumed. However, in the very definition of Greenberg’s $\mathcal{L}$-invariant, the condition (V) in Lemma 1.5 is necessary as explicitly pointed out.
in pages 163–4 of [Gr]. As is clear from Lemma 1.2, Conjecture 0.1 supplies us
the vanishing Sel$_{F}(V) = 0$ (which is equivalent to the finiteness of Greenberg’s
Selmer group $S_{A}(Q)$ in [Gr]).

\section{Factorization of $\mathcal{L}$-invariants}

In this section, we factorize $\mathcal{L}(\text{Ind}_{F}^{G_{P}} \rho_{2n,n})$ and $\mathcal{L}(\text{Ind}_{F}^{G_{P}} \text{Ad}(\rho_{n,0}))$ for odd $n$ into the product over multiplicative places and the contribution of the good
reduction part. This good reduction part gives $\mathcal{L}(n)$ for $\mathcal{L}(\text{Ind}_{F}^{G_{P}} \rho_{2n,n})$ in Conjecture 0.2. We keep notation introduced in the previous section; so, $V$ is either
$\rho_{2n,n}$ or $\text{Ad}(\rho_{n,0})$.

\textbf{Proposition 1.9.} Let $V$ be either $\rho_{2n,n}$ or $\text{Ad}(\rho_{n,0})$. Suppose $b > 0$, and fix an index $k$ with $1 \leq k \leq b$. Let $a \in \prod_{i=1}^{b} \text{Hom}(D_{p_{i}}^{\text{red}}, \mathcal{F}_{p_{i}} V/\mathcal{F}_{p_{i}}^{+} V)^{D_{p_{i}}}$ be
induced by $c \in H_{F}$ such that $c \in H_{F}$ restricts down trivially to $H^{1}(F_{p_{i}}, V)$ for all $i \neq k$. Then we have $a([\gamma_{i}, F_{i}]) = 0$ for all $i \neq k$ and $a([p, F_{k}]) = 0$ for all $k' \neq k$ with $k' \leq b$.

\textbf{Proof.} For the index $k \leq b$, $\mathcal{L}_{p_{i}}(V)$ is exactly $\mathcal{F}_{p_{i}}^{+} H^{1}(F_{k}, V)$. Take a cocycle $c \in H_{F}$ restricting down to $H^{1}(F_{p_{i}}, V)$ trivially to $H^{1}(F_{p_{i}}, V)$ for all $i \neq k$. Since $H_{F} \cong \prod_{i=1}^{b} \text{Im}(\tau_{p_{i}})$ by the restriction map (Lemmas 1.2 and 1.5), such cocycles $c$ form a direct summand of $H_{F}$ isomorphic to $\text{Im}(\tau_{p_{i}})$.

If $i > b$, $L_{p_{i}}(V)$ is made of classes of cocycles becoming unramified modulo
those with values in $\mathcal{F}_{p_{i}}^{+} V$; so, even if $c|_{D_{p_{i}}}$ vanishes in $H^{1}(F_{p_{i}}, V)$ (that is, $c|_{D_{p_{i}}} \in L_{p_{i}}(V)$), we cannot pull out much information on the value $a([p, F_{i}])$ because of the ambiguity modulo unramified cocycles with values in $\mathcal{F}_{p_{i}} V/\mathcal{F}_{p_{i}}^{+} V$. Anyway, $a([\gamma_{i}, F_{i}]) = 0$ because $[\gamma_{i}, F_{i}] \in \tau_{p_{i}}$.

For $i \leq b$ with $i \neq k$, $\mathcal{L}_{p_{i}}(V)$ is made of cocycles of $D_{p_{i}}$, with values in $\mathcal{F}_{p_{i}}^{+} V$, and the condition that $c|_{D_{p_{i}}} \in \tau_{p_{i}}(V)$ implies the vanishing of $a(\sigma) = c(\sigma)$ mod $\mathcal{F}_{p_{i}}^{+} V$ for all $\sigma \in D_{p_{i}}$. This shows the last assertion: $a([p, F_{k}]) = 0$.

By the above lemma, we get immediately the following fact.

\textbf{Corollary 1.10.} Let the notation be as in Proposition 1.9. Then the
linear operator $\mathcal{L}$ acting on $\prod_{p} \mathcal{F}_{p} V/\mathcal{F}_{p}^{+} V$ preserves the following exact sequence:

$$0 \to \prod_{i>b} \mathcal{F}_{p_{i}} V/\mathcal{F}_{p_{i}}^{+} V \to \prod_{p} \mathcal{F}_{p} V/\mathcal{F}_{p}^{+} V \to \prod_{k \leq b} \mathcal{F}_{p_{k}} V/\mathcal{F}_{p_{k}}^{+} V \to 0,$$

and $\mathcal{L}$ acting on the quotient $\prod_{k \leq b} \mathcal{F}_{p_{k}} V/\mathcal{F}_{p_{k}}^{+} V$ sends $\mathcal{F}_{p_{k}} V/\mathcal{F}_{p_{k}}^{+} V$ into itself for each $k \leq b$. 

**Definition 1.11.** Define $\mathcal{L}(n)$ (resp. $\mathcal{L}_k(V)$) by

$$\det \left( \mathcal{L} \big|_{\prod_p \mathcal{F}_p^{-1} V/\mathcal{F}_p^{+} V} \right) \in \mathbb{Q}_p$$

for $V = \rho_{2n,n}$ (resp. the determinant of the linear operator induced by $\mathcal{L}$ on $\prod_p \mathcal{F}_p^{-1} V/\mathcal{F}_p^{+} V/\prod_{i \neq k} \mathcal{F}_p, V/\mathcal{F}_p^{+} V$ for $V = \rho_{2n,n}$ and $V = \text{Ad}(\rho_{n,0})$).

**Corollary 1.12.** Let the notation be as above. Then we have

$$\mathcal{L}(\text{Ind}_{\mathbb{Q}}^{\mathbb{F}} \rho_{2n,n}) = \mathcal{L}(n) \prod_{k=1}^{b} \mathcal{L}_k(\rho_{2n,n})$$

for odd $n \geq 1$.

**Proposition 1.13.** Suppose $n = 1$. Then for $k \leq b$, we have $\mathcal{L}_k(\rho_{2,1}) = \frac{\log_p(Q_k)}{\text{ord}_p(Q_k)}$, where $Q_k = N_{F_k/\mathbb{Q}_p}(q_k)$ for the Tate period $q_k$ of $E_{/F_k}$.

This follows from [H07] Theorem 5.3. In [H07], the above corollary is proved by automorphic means in Section 3 of [H07], but replacing the result of [H07] Section 3 by the above factorization result, the same argument proving Theorem 5.3 there proves the above proposition.

We now generalize Proposition 1.13 to arbitrary odd $n > 1$.

**Theorem 1.14.** Let $n$ be an odd positive integer, and assume $V = \rho_{2n,n}$. Suppose Conjecture 0.1 for all odd positive $m \leq n$. Then $\mathcal{L}_k(V) = \frac{\log_p(Q_k)}{\text{ord}_p(Q_k)}$ for $k \leq b$, where $Q_k = N_{F_k/\mathbb{Q}_p}(q_k)$ for the Tate period $q_k$ of $E$.

**Proof.** Fix $k \leq b$, and write $p = p_k$. Write $X_i = X_{i,p}$ if $i$ is odd. Define $\mathfrak{M}_\ell$ be the ideal generated by $X_i$ for odd $i \neq \ell$ and $X_\ell^2$. We fix an odd $\ell$ with $0 < \ell \leq n$, and write $\mathfrak{M}$ for $\mathfrak{M}_\ell$ and $\bar{K} = R_n/\mathfrak{M} \cong K[\varepsilon]$ with $\varepsilon^2 = 0$ by $X_\ell \mapsto \varepsilon$. Let $\bar{p} = (\rho_n \mod \mathfrak{M})$, and write $\bar{d}_i$ for $\delta_{i,p} \mod \mathfrak{M}$. We consider the exact sequence of $\bar{K}[D_p]$-modules:

$$0 \to \mathcal{F}_p^{i+1} \bar{p} \to \mathcal{F}_p^{i+2} \bar{p} \to \mathcal{F}_p^{i+3} \bar{p} \to 0.$$

Writing $\bar{K}(\psi)$ for the rank one free $\bar{K}$-module on which $D_p$ acts by a character $\psi : D_p \to \bar{K}^\times$, this exact sequence gives the following exact sequence

$$0 \to \bar{K}(\bar{d}_{i+1}) \to \mathcal{F}_p^{i+1} \bar{p} \to \bar{K}(\bar{d}_i) \to 0.$$
Twisting by $\mathfrak{d}_{i+1}^{-1}N$, we get another exact sequence of $\overline{K}[D_p]$-modules:

$$0 \to \overline{K}(N) \to M \to \overline{K}(\mathfrak{d}_{i+1}^{-1}N') \to 0.$$ 

By [H07] Lemma 5.1, this sequence gives the top row of the following commutative diagram of $D_p$-modules with exact rows:

$$\begin{array}{cccccc}
\overline{K}(N) & \longrightarrow & M & \longrightarrow & \overline{K}(\mathfrak{d}_{i+1}^{-1}N') \\
\mod X_t & \downarrow & \mod X_t & \downarrow & \mod X_t \\
K(N) & \longrightarrow & T_pE \otimes_{Z_p} K & \longrightarrow & K.
\end{array}$$

Then by taking the induction from $\text{Gal}(\mathcal{F}_p/F_p)$ to $\text{Gal}(\overline{\mathcal{F}_p}/\mathbb{Q}_p)$, we get the following new commutative diagram with exact rows:

$$\begin{array}{cccccc}
\text{Ind}_{\mathcal{F}_p}^{\mathbb{Q}_p} \overline{K}(N) & \longrightarrow & \text{Ind}_{\mathcal{F}_p}^{\mathbb{Q}_p} M & \longrightarrow & \text{Ind}_{\mathcal{F}_p}^{\mathbb{Q}_p} \overline{K}(\mathfrak{d}_{i+1}^{-1}N') \\
\mod X_t & \downarrow & \mod X_t & \downarrow & \mod X_t \\
\text{Ind}_{\mathcal{F}_p}^{\mathbb{Q}_p} K(N) & \longrightarrow & \text{Ind}_{\mathcal{F}_p}^{\mathbb{Q}_p} T_pE \otimes_{Z_p} K & \longrightarrow & \text{Ind}_{\mathcal{F}_p}^{\mathbb{Q}_p} K.
\end{array}$$

By [H07] Lemma 4.8, we have a unique extension $\tilde{\delta}_j$ of $\overline{\mathfrak{d}}_j$ to $\text{Gal}(\mathcal{F}_p/\mathbb{Q}_p)$ with $\tilde{\delta}_j \equiv N^{m-j} \mod m_n$. We write this extension as $\overline{\mathfrak{d}}_j$. For any potentially ordinary $\text{Gal}(\mathcal{F}_p/\mathbb{Q}_p)$-module $X$, write the maximal quotient of $\mathcal{F}^+X$ on which $\text{Gal}(\mathcal{F}_p/\mathbb{Q}_p)$ acts by $\mathcal{N}$ as $\mathcal{F}^+X/\mathcal{F}^{11}X$. Similarly, we define $\mathcal{F}^+X \subset \mathcal{F}^{10}X \subset X$ by $\mathcal{F}^{10}X/\mathcal{F}^+X = H^1(\text{Gal}(\mathcal{F}_p/\mathbb{Q}_p), X/\mathcal{F}^+X)$. Then the above commutative diagram yields another commutative diagram with exact rows:

$$\begin{array}{cccccc}
\overline{K}(N) & \longrightarrow & \mathcal{F}^{10} \text{Ind}_{\mathcal{F}_p}^{\mathbb{Q}_p} M/\mathcal{F}^{11} \text{Ind}_{\mathcal{F}_p}^{\mathbb{Q}_p} M & \longrightarrow & \overline{K}(\mathfrak{d}_{i+1}^{-1}N') \\
\mod X_t & \downarrow & \mod X_t & \downarrow & \mod X_t \\
K(N) & \longrightarrow & \mathcal{F}^{10} \text{Ind}_{\mathcal{F}_p}^{\mathbb{Q}_p} T_pE \otimes_{Z_p} K & \longrightarrow & \mathcal{F}^{11} \text{Ind}_{\mathcal{F}_p}^{\mathbb{Q}_p} T_pE \otimes_{Z_p} K & \longrightarrow \ K.
\end{array}$$

By Theorem 4.7 of [H07], this implies

$$\frac{\partial \tilde{\delta}_j \mathfrak{d}_{i+1}^{-1}N}{\partial X_t}([Q_k, \mathbb{Q}_p]) = 0.$$

Since $\mathcal{N}([Q_k, \mathbb{Q}_p])$ is constant in $\mathbb{Q}_p^\times$, we get

$$\frac{\partial \tilde{\delta}_j \mathfrak{d}_{i+1}^{-1}}{\partial X_t}([Q_k, \mathbb{Q}_p]) = 0.$$
which yields by the Leibnitz formula
\[
\left( \delta_{i-1} \frac{\partial \tilde{\delta}_i}{\partial X_{\ell}} - \delta_{i+1} \frac{\partial \tilde{\delta}_i}{\partial X_{\ell}} \right) ([Q_k, Q_p]) = 0, 
\]
where \( \tilde{\delta}_i = (\tilde{\delta}_i \mod m_n) = N^{n-i} \). Since this holds for \( i = 0, 1, \ldots, n \), we get
\[
\left( \delta_{0-1} \frac{\partial \tilde{\delta}_0}{\partial X_{\ell}} - \delta_{n+1} \frac{\partial \tilde{\delta}_n}{\partial X_{\ell}} \right) ([Q_k, Q_p]) = 0. 
\]
Since \( \tilde{\delta}_0 \tilde{\delta}_n = N \) which is the unique extension of \( \tilde{\delta}_0 \tilde{\delta}_n = N \) to \( \text{Gal}(\overline{F}_\ell/Q_p) \) congruent to \( N \) modulo \( m_n \) (see Lemma 4.8 of [H07]), we have
\[
\delta_{0-1} \frac{\partial \tilde{\delta}_0}{\partial X_{\ell}} = -\delta_{n+1} \frac{\partial \tilde{\delta}_n}{\partial X_{\ell}}, 
\]
and hence
\[
\delta_{n+1} \frac{\partial \tilde{\delta}_n}{\partial X_{\ell}} ([Q_k, Q_p]) = 0. 
\]
This in turn yields
\[
\delta_{n+1} \frac{\partial \tilde{\delta}_n}{\partial X_{\ell}} ([Q_k, Q_p]) = 0 
\]
for all \( i = 0, 1, \ldots, n \).

Write \( Q_k = p^a u \) for \( a = \text{ord}_p(Q_k) \) and \( u \in Z_p^\times \). Then \( \log_p(u) = \log_p(Q_k) \).

Write \( d_k = [F_k : Q_p] \) and \( N_k = N_{F_k/Q_p} : F_k^\times \rightarrow Q_p^\times \) for the norm map. Since \([p, Q_p]^{d_k} = [N_k(p), Q_p] = [p, F_k][Q_p^\times u] \) and \([u, Q_p]^{d_k} = [N_k(u), Q_p] = [u, F_k][Q_p^\times u] \), for odd \( i \), we have
\[
\tilde{\delta}_i([N(q_k), Q_p]^{d_k}) = \delta_i([p, F_k])^{d_k} \delta_i([u, F_k]) 
\]
with respect to \( X_{\ell} \), we get from \( \delta_i([p, F_k]) = N^{n-i}([p, F_k]) = 1 \)
\[
a \frac{\partial \delta_i}{\partial X_{\ell}} \bigg|_{X=0} ([p, F_k]) = \frac{d_k \log_p(u)}{\log_p(\gamma_k)} = 0 
\]
and
\[
a \frac{\partial \delta_i}{\partial X_{\ell}} \bigg|_{X=0} ([p, F_k]) = 0 \text{ if odd } i \neq \ell. 
\]
Since \( a \neq 0 \), we have
\[
\frac{\partial \delta_i}{\partial X_{\ell}} \bigg|_{X=0} ([p, F_k]) = 0 \text{ if odd } i \neq \ell, 
\]
and
\[ \frac{\partial \delta_k}{\partial X^\ell} \bigg|_{X=0} ([p, F_k]) d_k^{-1} \log p(\gamma_k) = \frac{\log p(Q_k)}{\text{ord}_p(Q_k)}. \]

Since \( \frac{\partial \rho}{\partial X^\ell} \bigg|_{X=0} \rho_{n,0}^{-1} \) for odd \( \ell \) with \( 0 < \ell \leq n \) gives a basis of the \( p \)-part of \( H_F \) isomorphic to \( \text{Im}(\iota_p) \), we find that \( L_k(\text{Ad}(\rho_{n,0})) = \frac{\log p(Q_k)}{\text{ord}_p(Q_k)} \). Since \( \text{Sel}^{\text{cyc}}_F(\text{Ad}(\rho_{n,0})) \cong \bigoplus_{0 < m \leq n, \text{odd}} \text{Sel}^{\text{cyc}}_F(\rho_{2m,m}) \), we find
\[ L_k(\text{Ad}(\rho_{n,0})) = \prod_{0 < m \leq n, \text{odd}} L_k(\rho_{2m,m}) = \frac{\log p(Q_k)}{\text{ord}_p(Q_k)}^{(n+1)/2}. \]

By induction on \( m \) starting with the case \( m = 1 \) treated in Proposition 1.13, we find \( L_k(\rho_{2n,n}) = \frac{\log p(Q_k)}{\text{ord}_p(Q_k)} \) as desired. \( \square \)

§2. Proof of Conjecture 0.1 under Potential Modularity

When \( n=1 \)

We suppose that
(\( \text{NS} \)) \( \overline{\rho} = E[p] \) has non-soluble image in \( GL_2(\mathbb{F}_p) \);
(\( \text{DS} \)) the semi-simplification of \( \overline{\rho} \) restricted to \( D_p \) is non-scalar.

We now give a sketch of a proof of Conjecture 0.1 under these two conditions:

**Proposition 2.1.** Suppose \( \text{NS} \) and \( \text{DS} \). If there exists a totally real Galois extension \( L/F \) totally split at \( p \) such that \( \overline{\rho}_L = \overline{\rho}_{| \text{Gal}(L/F)} \) is associated to a Hilbert modular form, then we have \( R_1 \cong K[[X_1,p]]/p^n \).

By the result of [Ta] and [Ta1], the Galois representation \( \overline{\rho} \) is potentially modular in the sense that there exists a totally real Galois extension \( L/F \) in which \( p \) totally split and \( \overline{\rho}_L \) is associated to a Hilbert cusp form of weight 2. Actually, in the above paper of Taylor, details of the proof is given for \( F = \mathbb{Q} \), but we should be able to adjust his argument to prove the result for general \( F \) (see [V] Theorem 1.1).

**Proof.** To indicate the dependence of \( R_n \) on the base-field \( L \), we write \( (R_{n/L}, \rho_{n/L}) \) if we consider the universal couple of \( \rho_{F|_{\text{Gal}(L/F)}} \) (under \( (K_n, 1-4) \)). By the potential modularity assumption, \( \overline{\rho}_L \) is modular. By further making a soluble base-change, by the potential level-lowering done by [SW], we may assume that \( \overline{\rho}_L \) is associated to a Hilbert modular cusp form of weight 2 of level
\(\Gamma_0(\mathfrak{N}p)\) satisfying the conditions (h1–4) of [HMI] page 185 for the prime-to-\(p\) Artin conductor \(\mathfrak{N}\) of \(T_L\). Then by [HMI] Corollary 3.77 and Proposition 3.78, we have \(R_{1/L} \cong K[[X_1,\mathfrak{P}]][\mathfrak{P}]/\mathfrak{P}\), where \(\mathfrak{P}\) runs over all prime factors of \(p\) in \(L\). For \(\sigma \in \text{Gal}(L/F)\), we take a lift \(\tilde{\sigma} \in \text{Gal}(\overline{F}/L)\) inducing \(\sigma\) on \(L\), for any deformation \(\rho\) of \(\rho_E\) over \(L\), we can define \(\rho^\sigma(g) = \rho(\tilde{\sigma}g\tilde{\sigma}^{-1})\). The isomorphism class of \(\rho^\sigma\) is determined independently of the choice of the lift \(\tilde{\sigma}\) and depends only on \(\sigma\). Since \(E\) is defined over \(F\), \(\rho_{E,F}^\sigma \cong \rho_{E,F}\) is another deformation of \(\rho_E\) over \(L\) satisfying (K\(_n\),1–4). Thus we have a unique ring automorphism \(\sigma \in \text{Aut}(R_{n/L})\) such that \(\rho_{n/L}^\sigma \cong [\sigma] \circ \rho_{n/L}\). In this way, \(\Delta := \text{Gal}(L/F)\) acts on \(R_{n/L}\). Since \(\delta_1,\mathfrak{P}(\tilde{\sigma}g\tilde{\sigma}^{-1})\) coincides with \(\delta_1,\mathfrak{P}\), we have \(\mathcal{A}(X_1,\mathfrak{P}) = \mathcal{A}_1,\mathfrak{P}\). By the \(K\)-deformation version of Theorem 5.42 in [MFG], we have \(R_{1/F} \cong R_{1/L}/\sum_{\sigma \in \Delta} R_{1/L}([\sigma] - 1)R_{1/L}\), where \(\sum_{\sigma \in \Delta} R_{1/L}([\sigma] - 1)R_{1/L}\) is the ideal of \(R_{1/L}\) generated by \([\sigma](r) - r\) for all \(r \in R_{1/L}\). Then it is clear that \(R_{1/F} \cong K[[X_1,\mathfrak{P}]]/\mathfrak{P}\).

\[\square\]

Remark 2.1. Since the potential modularity for \(\rho_{n,0}\) is proven in [Ta2] under mild assumptions, we expect that the above argument (or a modified version) would prove Conjecture 0.1 for general \(n\) in near future.

\[\begin{align*}
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