Symmetric Crystals for $\mathfrak{gl}_\infty$

Dedicated to Professor Heisuke Hironaka on the occasion of his seventy-seventh birthday

By

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Abstract

In the preceding paper, we formulated a conjecture on the relations between certain classes of irreducible representations of affine Hecke algebras of type B and symmetric crystals for $\mathfrak{gl}_\infty$. In the present paper, we prove the existence of the symmetric crystal and the global basis for $\mathfrak{gl}_\infty$.

§1. Introduction

Lascoux-Leclerc-Thibon ([LLT]) conjectured the relations between the representations of Hecke algebras of type A and the crystal bases of the affine Lie algebras of type A. Then, S. Ariki ([A]) observed that it should be understood in the setting of affine Hecke algebras and proved the LLT conjecture in a more general framework. Recently, we presented the notion of symmetric crystals and conjectured that certain classes of irreducible representations of the affine Hecke algebras of type B are described by symmetric crystals for $\mathfrak{gl}_\infty$ ([EK]).

The purpose of the present paper is to prove the existence of symmetric crystals in the case of $\mathfrak{gl}_\infty$.

Let us recall the Lascoux-Leclerc-Thibon-Ariki theory. Let $H^A_n$ be the affine Hecke algebra of type A of degree $n$. Let $K^A_n$ be the Grothendieck group $\mathcal{K}_n$ of $H^A_n$.
of the abelian category of finite-dimensional $H^A_n$-modules, and $K^A = \oplus_{n \geq 0} K^A_n$. Then it has a structure of Hopf algebra by the restriction and the induction. The set $I = \mathbb{C}^*$ may be regarded as a Dynkin diagram with $I$ as the set of vertices and with edges between $a \in I$ and $ap_1^2$. Here $p_1$ is the parameter of the affine Hecke algebra usually denoted by $q$. Let $g_f$ be the associated Lie algebra, and $g_f^\circ$ the unipotent Lie subalgebra. Let $U_I$ be the group associated to $g_f^\circ$. Hence $g_I$ is isomorphic to a direct sum of copies of $A^{(1)}_{\ell-1}$ if $p_1^2$ is a primitive $\ell$-th root of unity and to a direct sum of copies of $gl_\infty$ if $p_1$ has an infinite order. Then $C \otimes K^A$ is isomorphic to the algebra $O(U_I)$ of regular functions on $U_I$. Let $U_q^{\theta}(g_I)$ be the associated quantized enveloping algebra. Then $U_q^{\theta}(g_I)$ has an upper global basis $\{G^{\up}(b)\}_{b \in B(\infty)}$. By specializing $\bigoplus C[q, q^{-1}]G^{\up}(b)$ at $q = 1$, we obtain $O(U_I)$. Then the LLTA-theory says that the elements associated to irreducible $H^A$-modules corresponds to the image of the upper global basis.

In [EK], we gave analogous conjectures for affine Hecke algebras of type $B$. In the type $B$ case, we have to replace $U_q^{\theta}(g_I)$ and its upper global basis with symmetric crystals (see § 2.3). It is roughly stated as follows. Let $H^B_n$ be the affine Hecke algebra of type $B$ of degree $n$. Let $K^B_n$ be the Grothendieck group of the abelian category of finite-dimensional modules over $H^B_n$, and $K^B = \oplus_{n \geq 0} K^B_n$. Then $K^B$ has a structure of a Hopf bimodule over $K^A$. The group $U_I$ has the anti-involution $\theta$ induced by the involution $a \mapsto a^{-1}$ of $I = \mathbb{C}^*$. Let $U_I^{\theta}$ be the $\theta$-fixed point set of $U_I$. Then $O(U_I^{\theta})$ is a quotient ring of $O(U_I)$. The action of $O(U_I) \simeq C \otimes K^A$ on $C \otimes K^B$, in fact, descends to the action of $O(U_I^{\theta})$.

We introduce $V^{\theta}(\lambda)$ (see § 2.3), a kind of the $q$-analogue of $O(U_I^{\theta})$. The conjecture in [EK] is then:

(i) $V^{\theta}(\lambda)$ has a crystal basis and a global basis.

(ii) $K^B$ is isomorphic to a specialization of $V^{\theta}(\lambda)$ at $q = 1$ as an $O(U_I)$-module, and the irreducible representations correspond to the upper global basis of $V^{\theta}(\lambda)$ at $q = 1$.

Remark. In [KM], Miemietz and the second author gave an analogous conjecture for the affine Hecke algebras of type $D$. In the present paper, we prove that $V^{\theta}(\lambda)$ has a crystal basis and a global basis for $g = gl_\infty$ and $\lambda = 0$.

More precisely, let $I = \mathbb{Z}_{\text{odd}}$ be the set of odd integers. Let $\alpha_i (i \in I)$ be
the simple roots with

\[(\alpha_i, \alpha_j) = \begin{cases} 
2 & \text{if } i = j, \\
-1 & \text{if } i = j \pm 2, \\
0 & \text{otherwise.}
\end{cases} \]

Let \( \theta \) be the involution of \( I \) given by \( \theta(i) = -i \). Let \( \mathcal{B}_\theta(\mathfrak{gl}_\infty) \) be the algebra over \( K := \mathbb{Q}(q) \) generated by \( E_i, F_i \), and invertible elements \( T_i \) \((i \in I)\) satisfying the following defining relations:

(i) the \( T_i \)'s commute with each other,

(ii) \( T\theta(i) = T_i \) for any \( i \),

(iii) \( T_i E_j T_i^{-1} = q^{(\alpha_i + \alpha_{\theta(i)}), \alpha_j} E_j \) and \( T_i F_j T_i^{-1} = q^{(\alpha_i + \alpha_{\theta(i)}, -\alpha_j)} F_j \) for \( i, j \in I \),

(iv) \( E_i F_j = q^{-(\alpha_i, -\alpha_j)} F_j E_i + (\delta_{i,j} + \delta_{\theta(i),j}) T_i \) for \( i, j \in I \),

(v) the \( E_i \)'s and the \( F_i \)'s satisfy the Serre relations (see Definition 2.1 (4)).

Then there exists a unique irreducible \( \mathcal{B}_\theta(\mathfrak{gl}_\infty) \)-module \( V_\theta(0) \) with a generator \( \phi \) satisfying \( E_i \phi = 0 \) and \( T_i \phi = \phi \) (Proposition 2.11). We define the endomorphisms \( \tilde{E}_i \) and \( \tilde{F}_i \) of \( V_\theta(0) \) by

\[ \tilde{E}_i a = \sum_{n \geq 1} F_i^{(n-1)} a_n, \quad \tilde{F}_i a = \sum_{n \geq 0} F_i^{(n+1)} a_n, \]

when writing

\[ a = \sum_{n \geq 0} F_i^{(n)} a_n \quad \text{with} \quad E_i a_n = 0. \]

Here \( F_i^{(n)} = F_i^n/|n!| \) is the divided power. Let \( A_0 \) be the ring of functions \( a \in K \) which do not have a pole at \( q = 0 \). Let \( L_\theta(0) \) be the \( A_0 \)-submodule of \( V_\theta(0) \) generated by the elements \( \tilde{F}_{i_1} \cdots \tilde{F}_{i_\ell} \phi \) \((\ell \geq 0, i_1, \ldots, i_\ell \in I)\). Let \( B_\theta(0) \) be the subset of \( L_\theta(0)/qL_\theta(0) \) consisting of the \( \tilde{F}_{i_1} \cdots \tilde{F}_{i_\ell} \phi \)'s. In this paper, we prove the following theorem.

**Theorem** (Theorem 4.15).

(i) \( \tilde{F}_i L_\theta(0) \subset L_\theta(0) \) and \( \tilde{E}_i L_\theta(0) \subset L_\theta(0) \),

(ii) \( B_\theta(0) \) is a basis of \( L_\theta(0)/qL_\theta(0) \),

(iii) \( \tilde{F}_i B_\theta(0) \subset B_\theta(0) \), and \( \tilde{E}_i B_\theta(0) \subset B_\theta(0) \cup \{0\} \),
(iv) \( \tilde{F}_i \tilde{E}_i(b) = b \) for any \( b \in B_{\theta}(0) \) such that \( \tilde{E}_i b \neq 0 \), and \( \tilde{E}_i \tilde{F}_i(b) = b \) for any \( b \in B_{\theta}(0) \).

By this theorem, \( B_{\theta}(0) \) has a similar structure to the crystal structure. Namely, we have operators \( \tilde{F}_i : B_{\theta}(0) \to B_{\theta}(0) \) and \( \tilde{E}_i : B_{\theta}(0) \to B_{\theta}(0) \sqcup \{0\} \), which satisfy (iv). Moreover \( \varepsilon_i(b) := \max \{ n \in \mathbb{Z}_{\geq 0} \mid \tilde{E}_i^n b \in B_{\theta}(0) \} \) is finite. We call it the symmetric crystal associated with \((I, \theta)\). Contrary to the usual crystal case, \( \tilde{E}_i \) may coincide with \( \tilde{E}_i \) in the symmetric crystal case.

Let \( \bar{\phi} \) be the bar operator of \( V_{\theta}(0) \). Namely, \( \bar{\phi} \) is a unique endomorphism of \( V_{\theta}(0) \) such that \( \bar{\phi} = \phi, \bar{\omega} = \bar{a} \bar{\omega} \) and \( \bar{F}_i \bar{v} = F_i \bar{v} \) for \( a \in K \) and \( v \in V_{\theta}(0) \). Here \( a(q) = a(q^{-1}) \). Let \( V_{\theta}(0)A \) be the smallest submodule of \( V_{\theta}(0) \) over \( A := \mathbb{Q}[q, q^{-1}] \) such that it contains \( \phi \) and is stable by the \( F_i^{(n)} \)’s.

Then we prove the existence of global basis:

**Theorem** (Theorem 5.5).

(i) For any \( b \in B_{\theta}(0) \), there exists a unique \( G_{\theta}^{\text{low}}(b) \in V_{\theta}(0)A \cap L_{\theta}(0) \) such that \( G_{\theta}^{\text{low}}(b) = G_{\theta}^{\text{low}}(b) \) and \( b = G_{\theta}^{\text{low}}(b) \mod q L_{\theta}(0) \).

(ii) \( \{ G_{\theta}^{\text{low}}(b) \}_{b \in B_{\theta}(0)} \) is a basis of the \( A \)-module \( L_{\theta}(0) \), the \( A \)-module \( V_{\theta}(0)A \) and the \( K \)-vector space \( V_{\theta}(0) \).

We call \( G_{\theta}^{\text{low}}(b) \) the lower global basis. The \( B_{\theta}(\mathfrak{g}_{\text{reduced}}) \)-module \( V_{\theta}(0) \) has a unique symmetric bilinear form \((\cdot, \cdot)\) such that \((\phi, \phi) = 1 \) and \( E_i \) and \( F_i \) are transpose to each other. The dual basis to \( \{ G_{\theta}^{\text{low}}(b) \}_{b \in B_{\theta}(0)} \) with respect to \((\cdot, \cdot)\) is called an upper global basis.

Let us explain the strategy of our proof of these theorems. We first construct a PBW type basis \( \{ P_{\theta}(m) \phi \}_{m} \) of \( V_{\theta}(0) \) parametrized by the \( \theta \)-restricted multisegments \( m \). Then, we explicitly calculate the actions of \( E_i \) and \( F_i \) in terms of the PBW basis \( \{ P_{\theta}(m) \phi \}_{m} \). Then, we prove that the PBW basis gives a crystal basis by the estimation of the coefficients of these actions. For this we use a criterion for crystal bases (Theorem 4.1).

§2. General Definitions and Conjectures

§2.1. Quantized universal enveloping algebras and its reduced \( q \)-analogues

We shall recall the quantized universal enveloping algebra \( U_q(\mathfrak{g}) \). Let \( I \) be an index set (for simple roots), and \( Q \) the free \( \mathbb{Z} \)-module with a basis \( \{ \alpha_i \}_{i \in I} \).
Let \((\bullet, \bullet) : Q \times Q \to \mathbb{Z}\) be a symmetric bilinear form such that \((\alpha_i, \alpha_i)/2 \in \mathbb{Z}_{>0}\) for any \(i\) and \((\alpha'_i, \alpha_j) \in \mathbb{Z}_{\leq 0}\) for \(i \neq j\) where \(\alpha'_i := 2\alpha_i/(\alpha_i, \alpha_i)\). Let \(q\) be an indeterminate and set \(K := \mathbb{Q}(q)\). We define its subrings \(A_0, A_\infty\) and \(A\) as follows.

\[
A_0 = \{ f \in K \mid f \text{ is regular at } q = 0 \},
A_\infty = \{ f \in K \mid f \text{ is regular at } q = \infty \},
A = \mathbb{Q}[q, q^{-1}].
\]

**Definition 2.1.** The quantized universal enveloping algebra \(U_q(\mathfrak{g})\) is the \(K\)-algebra generated by elements \(e_i, f_i\) and invertible elements \(t_i\) \((i \in I)\) with the following defining relations.

1. The \(t_i\)'s commute with each other.
2. \(t_i e_i t_j^{-1} = q^{(\alpha_j, \alpha_i)} e_i\) and \(t_j f_i t_j^{-1} = q^{-\langle \alpha_j, \alpha_i \rangle} f_i\) for any \(i, j \in I\).
3. \([e_i, f_j] = \delta_{ij} \frac{t_i - q^{-1}_i}{q_i - q_i^{-1}}\) for \(i, j \in I\). Here \(q_i := q^{(\alpha_i, \alpha_i)/2}\).
4. (Serre relation) For \(i \neq j,\)

\[
\sum_{k=0}^{b} (-1)^k e_i^{(k)} e_j e_i^{(b-k)} = 0, \quad \sum_{k=0}^{b} (-1)^k f_i^{(k)} f_j f_i^{(b-k)} = 0.
\]

Here \(b = 1 - (\alpha'_i, \alpha_j)\) and

\[
e_i^{(k)} = e_i^k/[k]_i!, \quad f_i^{(k)} = f_i^k/[k]_i!, \quad [k]_i = (q_i^k - q_i^{-k})/(q_i - q_i^{-1}), \quad [k]_i! = [1]_i \cdots [k]_i.
\]

Let us denote by \(U_q^- (\mathfrak{g})\) (resp. \(U_q^+ (\mathfrak{g})\)) the \(K\)-subalgebra of \(U_q(\mathfrak{g})\) generated by the \(f_i\)'s (resp. the \(e_i\)'s).

Let \(e_i'\) and \(e_i''\) be the operators on \(U_q^- (\mathfrak{g})\) (see [K1, 3.4]) defined by

\[
[e_i, a] = \frac{(e_i'^a) t_i - t_i^{-1} e_i'^a}{q_i - q_i^{-1}} \quad (a \in U_q^- (\mathfrak{g})).
\]

These operators satisfy the following formulas similar to derivations:

\[
e_i' (ab) = e_i' (a) b + (\text{Ad}(t_i) a) e_i' b,
\]

\[
e_i'' (ab) = a e_i'' b + (e_i'' a)(\text{Ad}(t_i) b).
\]
Note that in [K1], the operator $e''_i$ was defined. It satisfies $e''_i = - \circ e'_i \circ -$, while $e^*_i$ satisfies $e^*_i = * \circ e'_i \circ *$. They are related by $e^*_i = \text{Ad}(t_i) \circ e''_i$.

The algebra $U_q^-(g)$ has a unique symmetric bilinear form $(\cdot, \cdot)$ such that $(1, 1) = 1$ and 
\[
(e'_i a, b) = (a, f_i b)
\]
for any $a, b \in U_q^-(g)$. It is non-degenerate and satisfies $(e^*_i a, b) = (a, f^*_i b)$. The left multiplication of $f_j, e'_i$ and $e^*_i$ have the commutation relations 
\[
e'_i f_j = q^{-(\alpha_i, \alpha_j)} f_j e'_i + \delta_{ij}, \quad e^*_i f_j = f_j e^*_i + \delta_{ij} \text{Ad}(t_i),
\]
and both the $e'_i$'s and the $e^*_i$'s satisfy the Serre relations.

**Definition 2.2.** The reduced $q$-analogue $B(g)$ of $g$ is the $K$-algebra generated by $e'_i$ and $f_i$.

§2.2. Review on crystal bases and global bases

Since $e'_i$ and $f_i$ satisfy the $q$-boson relation, any element $a \in U_q^-(g)$ can be uniquely written as 
\[
a = \sum_{n \geq 0} f_i^{(n)} a_n \quad \text{with} \quad e'_i a_n = 0.
\]

Here $f_i^{(n)} = \frac{f_i^n}{[n]_q!}$.

**Definition 2.3.** We define the modified root operators $\tilde{e}_i$ and $\tilde{f}_i$ on $U_q^-(g)$ by 
\[
\tilde{e}_i a = \sum_{n \geq 1} f_i^{(n-1)} a_n, \quad \tilde{f}_i a = \sum_{n \geq 0} f_i^{(n+1)} a_n.
\]

**Theorem 2.4 ([K1]).** We define 
\[
L(\infty) = \sum_{\ell \geq 0, i_1, \ldots, i_\ell \in I} A_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \subset U_q^-(g),
\]
\[
B(\infty) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \mod qL(\infty) \mid \ell \geq 0, i_1, \ldots, i_\ell \in I \right\} \subset L(\infty)/qL(\infty).
\]

Then we have 
(i) $\tilde{e}_i L(\infty) \subset L(\infty)$ and $\tilde{f}_i L(\infty) \subset L(\infty)$,
(ii) $B(\infty)$ is a basis of $L(\infty)/qL(\infty)$,
We call \((L(\infty), B(\infty))\) the crystal basis of \(U_q^- (\mathfrak{g})\).

Let \(-\) be the automorphism of \(K\) sending \(q\) to \(q^{-1}\). Then \(\overline{A}_0\) coincides with \(A_\infty\).

Let \(V\) be a vector space over \(K\), \(L_0\) an \(A_0\)-submodule of \(V\), \(L_\infty\) an \(A_\infty\)-submodule, and \(V_A\) an \(A\)-submodule. Set \(E := L_0 \cap L_\infty \cap V_A\).

\textbf{Definition 2.5} ([K1], [K2, 2.1]). We say that \((L_0, L_\infty, V_A)\) is balanced if each of \(L_0\), \(L_\infty\) and \(V_A\) generates \(V\) as a \(K\)-vector space, and if one of the following equivalent conditions is satisfied.

(i) \(E \to L_0/qL_0\) is an isomorphism,

(ii) \(E \to L_\infty/q^{-1}L_\infty\) is an isomorphism,

(iii) \((L_0 \cap V_A) \oplus (q^{-1}L_\infty \cap V_A) \to V_A\) is an isomorphism,

(iv) \(A_0 \otimes Q E \to L_0, A_\infty \otimes Q E \to L_\infty, A \otimes Q E \to V_A\) and \(K \otimes Q E \to V\) are isomorphisms.

Let \(-\) be the ring automorphism of \(U_q(\mathfrak{g})\) sending \(q, t_i, e_i, f_i\) to \(q^{-1}, t_i^{-1}, e_i, f_i\).

Let \(U_q(\mathfrak{g})_A\) be the \(A\)-subalgebra of \(U_q(\mathfrak{g})\) generated by \(e_i^{(n)}, f_i^{(n)}\) and \(t_i\). Similarly we define \(U_q^- (\mathfrak{g})_A\).

\textbf{Theorem 2.6.} \((L(\infty), L(\infty)^-, U_q^- (\mathfrak{g})_A)\) is balanced.

Let

\[ G^{\text{low}} : L(\infty)/qL(\infty) \to E := L(\infty) \cap L(\infty)^- \cap U_q^- (\mathfrak{g})_A \]

be the inverse of \(E \sim L(\infty)/qL(\infty)\). Then \(\{ G^{\text{low}}(b) \mid b \in B(\infty) \}\) forms a basis of \(U_q^- (\mathfrak{g})\). We call it a (lower) global basis. It is first introduced by G. Lusztig ([L]) under the name of “canonical basis” for the \(A, D, E\) cases.

\textbf{Definition 2.7.} Let

\[ \{ G^{\text{up}}(b) \mid b \in B(\infty) \} \]

be the dual basis of \(\{ G^{\text{low}}(b) \mid b \in B(\infty) \}\) with respect to the inner product \((\bullet, \bullet)\). We call it the upper global basis of \(U_q^- (\mathfrak{g})\).
§2.3. Symmetric crystals

Let $\theta$ be an automorphism of $I$ such that $\theta^2 = \text{id}$ and $(\alpha_{\theta(i)}, \alpha_{\theta(j)}) = (\alpha_i, \alpha_j)$. Hence it extends to an automorphism of the root lattice $Q$ by $\theta(\alpha_i) = \alpha_{\theta(i)}$, and induces an automorphism of $U_q(\mathfrak{g})$.

**Definition 2.8.** Let $B_{\theta}(\mathfrak{g})$ be the $K$-algebra generated by $E_i, F_i$, and invertible elements $T_i$ ($i \in I$) satisfying the following defining relations:

(i) the $T_i$'s commute with each other,

(ii) $T_i E_j T_i^{-1} = q^{(\alpha_i, \alpha_j)} E_j$ and $T_i F_j T_i^{-1} = q^{(\alpha_i + \alpha_{\theta(i)}, -\alpha_j)} F_j$ for $i, j \in I$,

(iii) $E_i F_j = q^{-(\alpha_i, \alpha_j)} F_j E_i + (\delta_{i,j} + \delta_{\theta(i),j} T_i)$ for $i, j \in I$,

(iv) the $E_i$'s and the $F_i$'s satisfy the Serre relations (Definition 2.1 (4)).

We set $E_i^{(n)} = E_i^n / [n]_i!$ and $F_i^{(n)} = F_i^n / [n]_i!$.

**Lemma 2.9.** Identifying $U_q^-(\mathfrak{g})$ with the subalgebra of $B_{\theta}(\mathfrak{g})$ by the morphism $f_i \mapsto F_i$, we have

\begin{align*}
T_i a &= \left(\text{Ad}(t_i t_{\theta(i)})a\right) T_i, \\
E_i a &= \left(\text{Ad}(t_i) a\right) E_i + c_i' a + \left(\text{Ad}(t_i)(c_{\theta(i)}^* a)\right) T_i
\end{align*}

for $a \in U_q^-(\mathfrak{g})$.

**Proof.** The first relation is obvious. In order to prove the second, it is enough to show that if $a$ satisfies (2.3), then $f_j a$ satisfies (2.3). We have

\begin{align*}
E_i(f_j a) &= (q^{-(\alpha_i, \alpha_j)} f_j E_i + \delta_{i,j} + \delta_{\theta(i),j} T_i) a \\
&= q^{-(\alpha_i, \alpha_j)} f_j \left(\left(\text{Ad}(t_i) a\right) E_i + c_i' a + \left(\text{Ad}(t_i)(c_{\theta(i)}^* a)\right) T_i\right) \\
&\quad + \delta_{i,j} a + \delta_{\theta(i),j} \left(\text{Ad}(t_i t_{\theta(i)})a\right) T_i \\
&= \left(\text{Ad}(t_i)(f_j a)\right) E_i + c_i' (f_j a) + \left(\text{Ad}(t_i)(c_{\theta(i)}^* (f_j a))\right) T_i.
\end{align*}

The following lemma can be proved in a standard manner and we omit the proof.
Lemma 2.10. Let $K[T_i^\pm; i \in I]$ be the commutative $K$-algebra generated by invertible elements $T_i$ $(i \in I)$ with the defining relations $T_{\theta(i)} = T_i$. Then the map $U_q^{-}(g) \otimes K[T_i^\pm; i \in I] \otimes U_q^{+}(g) \to B_\theta(g)$ induced by the multiplication is bijective.

Let $\lambda \in P_+ := \{\lambda \in \text{Hom}(Q, \mathbb{Q}) \mid \langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}_{\geq 0} \text{ for any } i \in I\}$ be a dominant integral weight such that $\theta(\lambda) = \lambda$.

Proposition 2.11.

(i) There exists a $B_\theta(g)$-module $V_\theta(\lambda)$ generated by a non-zero vector $\phi_\lambda$ such that

(a) $E_i \phi_\lambda = 0$ for any $i \in I$,
(b) $T_i \phi_\lambda = q^{\langle \alpha_i, \lambda \rangle} \phi_\lambda$ for any $i \in I$,
(c) $\{ u \in V_\theta(\lambda) \mid E_i u = 0 \text{ for any } i \in I \} = K \phi_\lambda$.

Moreover such a $V_\theta(\lambda)$ is irreducible and unique up to an isomorphism.

(ii) there exists a unique symmetric bilinear form $(\cdot, \cdot)$ on $V_\theta(\lambda)$ such that $(\phi_\lambda, \phi_\lambda) = 1$ and $(E_i u, v) = (u, F_i v)$ for any $i \in I$ and $u, v \in V_\theta(\lambda)$, and it is non-degenerate.

Remark 2.12. Set $P_\theta = \{ \mu \in P \mid \theta(\mu) = \mu \}$. Then $V_\theta(\lambda)$ has a weight decomposition

$$V_\theta(\lambda) = \bigoplus_{\mu \in P_\theta} V_\theta(\lambda)_\mu,$$

where $V_\theta(\lambda)_\mu = \{ u \in V_\theta(\lambda) \mid T_i u = q^{\langle \alpha_i, \mu \rangle} u \}$. We say that an element $u$ of $V_\theta(\lambda)$ has a $\theta$-weight $\mu$ and write $\text{wt}_\theta(u) = \mu$ if $u \in V_\theta(\lambda)_\mu$. We have $\text{wt}_\theta(E_i u) = \text{wt}_\theta(u) + \langle \alpha_i, \theta(\lambda) \rangle$ and $\text{wt}_\theta(F_i u) = \text{wt}_\theta(u) - \langle \alpha_i, \theta(\lambda) \rangle$.

In order to prove Proposition 2.11, we shall construct two $B_\theta(g)$-modules, analogous to Verma modules and dual Verma modules.

Lemma 2.13. Let $U_q^{-}(g) \phi'_\lambda$ be a free $U_q^{-}(g)$-module with a generator $\phi'_\lambda$. Then the following action gives a structure of a $B_\theta(g)$-module on $U_q^{-}(g) \phi'_\lambda$:

$$
\begin{align*}
T_i(a \phi'_\lambda) &= q^{\langle \alpha_i, \lambda \rangle}(\text{Ad}(t_i t_{\theta(i)}) a) \phi'_\lambda, \\
E_i(a \phi'_\lambda) &= (e'_i a + q^{\langle \alpha_i, \lambda \rangle} \text{Ad}(t_i)(e'_{\theta(i)} a)) \phi'_\lambda, \\
F_i(a \phi'_\lambda) &= (f_i a) \phi'_\lambda
\end{align*}
$$

for any $i \in I$ and $a \in U_q^{-}(g)$.

Moreover $B_\theta(g)/ \sum_{i \in I} (B_\theta(g) E_i + B_\theta(g)(T_i - q^{\langle \alpha_i, \lambda \rangle})) \to U_q^{-}(g) \phi'_\lambda$ is an isomorphism.
Proof. We can easily check the defining relations in Definition 2.8 except the Serre relations for the $E_i$’s.

For $i \neq j \in I$, set $S = \sum_{n=0}^{b}(-1)^n E_i^{(n)} E_j E_i^{(b-n)}$ where $b = 1 - \langle h_i, \alpha_j \rangle$. It is enough to show that the action of $S$ on $U_q^{(-)}(g) \phi_\lambda$ is equal to 0. We can easily check that $SF_k = q^{-(\alpha_i, h_k + \alpha_j)} F_k S$. Since $S \phi_\lambda = 0$, we have $SU_q^{(-)}(g) \phi_\lambda = 0$.

Hence $U_q^{(-)}(g) \phi_\lambda$ has a $B_\theta(g)$-module structure.

The last statement is obvious.

Lemma 2.14. Let $U_q^{(-)}(g) \phi_\lambda''$ be a free $U_q^{(-)}(g)$-module with a generator $\phi_\lambda''$. Then the following action gives a structure of a $B_\theta(g)$-module on $U_q^{(-)}(g) \phi_\lambda''$:

\[
\begin{align*}
T_i(a \phi_\lambda'') &= q^{(\alpha_i, \lambda)}(\text{Ad}(t_i t_{\theta(i)}) a) \phi_\lambda'', \\
E_i(a \phi_\lambda'') &= (c'_{i} a) \phi_\lambda'', \\
F_i(a \phi_\lambda'') &= (f_i a + q^{(\alpha_i, \lambda)}(\text{Ad}(t_i) t_{\theta(i)}) a) \phi_\lambda''
\end{align*}
\]

for any $i \in I$ and $a \in U_q^{(-)}(g)$. Moreover, there exists a non-degenerate bilinear form $\langle \cdot, \cdot \rangle: U_q^{(-)}(g) \phi_\lambda' \times U_q^{(-)}(g) \phi_\lambda'' \to K$ such that $\langle F_i u, v \rangle = \langle u, E_i v \rangle$, $\langle E_i u, v \rangle = \langle u, F_i v \rangle$, $\langle T_i u, v \rangle = \langle u, T_i v \rangle$ for $u \in U_q^{(-)}(g) \phi_\lambda'$ and $v \in U_q^{(-)}(g) \phi_\lambda''$, and $\langle \phi_\lambda', \phi_\lambda'' \rangle = 1$.

Proof. There exists a unique symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $U_q^{(-)}(g)$ such that $(1, 1) = 1$ and $f_i$ and $c_i'$ are transpose to each other. Let us define $\langle \cdot, \cdot \rangle: U_q^{(-)}(g) \phi_\lambda' \times U_q^{(-)}(g) \phi_\lambda'' \to K$ by $\langle a \phi_\lambda', b \phi_\lambda'' \rangle = \langle a, b \rangle$ for $a \in U_q^{(-)}(g)$ and $b \in U_q^{(-)}(g)$. Then we can easily check $\langle F_i u, v \rangle = \langle u, E_i v \rangle$, $\langle T_i u, v \rangle = \langle u, T_i v \rangle$.

Hence the action of $E_i$, $F_i$, $T_i$ on $U_q^{(-)}(g) \phi_\lambda''$ satisfy the defining relations in Definition 2.8.

Proof of Proposition 2.11. Since $E_i \phi_\lambda' = 0$ and $\phi_\lambda''$ has a $\theta$-weight $\lambda$, there exists a unique $B_\theta(g)$-linear morphism $\psi: U_q^{(-)}(g) \phi_\lambda' \to U_q^{(-)}(g) \phi_\lambda''$ sending $\phi_\lambda'$ to $\phi_\lambda''$. Let $V_\psi(\lambda)$ be its image $\psi(U_q^{(-)}(g) \phi_\lambda')$.

(i) (c) follows from $\{ u \in U_q^{(-)}(g) \mid c'_i u = 0 \text{ for any } i \} = K$ and $U_q^{(-)}(g) \phi_\lambda'' \supset V_\psi(\lambda)$. The other properties (a), (b) are obvious. Let us show that $V_\psi(\lambda)$ is irreducible. Let $S$ be a non-zero $B_\theta(g)$-submodule. Then $S$ contains a non-zero vector $v$ such that $E_i v = 0$ for any $i$. Then (c) implies that $v$ is a constant multiple of $\phi_\lambda$. Hence $S = V_\psi(\lambda)$.

Let us prove (ii). For $u, u' \in U_q^{(-)}(g) \phi_\lambda'$, set $[(u, u')] = \langle u, \psi(u') \rangle$. Then it is a bilinear form on $U_q^{(-)}(g) \phi_\lambda'$ which satisfies

\[
\begin{align*}
\langle \phi_\lambda', \phi_\lambda'' \rangle &= 1, \quad \langle (F_i u, u') \rangle = \langle (u, E_i u') \rangle, \quad \langle (E_i u, u') \rangle = \langle (u, F_i u') \rangle, \quad \text{and} \\
\langle (T_i u, u') \rangle &= \langle (u, T_i u') \rangle.
\end{align*}
\]
It is easy to see that a bilinear form which satisfies (2.6) is unique. Since \((u', u)\) also satisfies (2.6), \((u, u')\) is a symmetric bilinear form on \(U_q^{-}(\mathfrak{g})\). Since \(\psi(u') = 0\) implies \((u, u') = 0\), \((u, u')\) induces a symmetric bilinear form on \(V_{\theta}(\lambda)\). Since \((\cdot, \cdot)\) is non-degenerate on \(U_q^{-}(\mathfrak{g})\), \((\cdot, \cdot)\) is a non-degenerate symmetric bilinear form on \(V_{\theta}(\lambda)\).

Lemma 2.15. There exists a unique endomorphism \(\tilde{\varphi}\) of \(V_{\theta}(\lambda)\) such that \(\varphi_{\lambda} = \varphi_{\lambda}\) and \(\varphi_{\lambda} = \tilde{\varphi}\) for any \(a \in K\) and \(v \in V_{\theta}(\lambda)\).

Proof. The uniqueness is obvious.

Let \(\xi\) be an anti-involution of \(U_q^{-}(\mathfrak{g})\) such that \(\xi(q) = q^{-1}\) and \(\xi(f_i) = f_{\theta(i)}\). Let \(\tilde{\rho}\) be an element of \(Q \otimes P\) such that \((\tilde{\rho}, \alpha_i) = (\alpha_i, \alpha_{\theta(i)})/2\). Define \(c(\mu) = ((\mu + \tilde{\rho}, \theta(\mu + \tilde{\rho})) - (\tilde{\rho}, \theta(\tilde{\rho}))/2 + (\lambda, \mu)\) for \(\mu \in P\). Then it satisfies
\[
c(\mu) - c(\mu - \alpha_i) = (\lambda + \mu, \alpha_{\theta(i)}).
\]
Hence \(c\) takes integral values on \(Q := \sum_i \mathbb{Z}\alpha_i\).

We define the endomorphism \(\Phi\) of \(U_q^{-}(\mathfrak{g})\) by \(\Phi(a\phi_{\lambda}^\mu) = q^{-c(\mu)}\xi(a)\phi_{\lambda}^\mu\) for \(a \in U_q^{-}(\mathfrak{g})_{\mu}\). Let us show that
\[
(2.7) \quad \Phi(F_i(a\phi_{\lambda}^\mu)) = F_i\Phi(a\phi_{\lambda}^\mu) \quad \text{for any } a \in U_q^{-}(\mathfrak{g}).
\]

For \(a \in U_q^{-}(\mathfrak{g})_{\mu}\), we have
\[
\Phi(F_i(a\phi_{\lambda}^\mu)) = \Phi(F_i(a + q^{(\alpha_i,\lambda + \mu)}a f_{\theta(i)}))\phi_{\lambda}^\mu = (q^{-c(\mu - \alpha_i)}\xi(a)f_{\theta(i)} + q^{-(\alpha_i, \lambda + \mu) - c(\mu - \alpha_{\theta(i)})}f_i\xi(a))\phi_{\lambda}^\mu.
\]
On the other hand, we have
\[
F_i\Phi(a\phi_{\lambda}^\mu) = F_i(q^{-c(\mu)}\xi(a)\phi_{\lambda}^\mu) = q^{-c(\mu)}(f_i\xi(a) + q^{(\alpha_i, \lambda + \theta(\mu))}\xi(a)f_{\theta(i)})\phi_{\lambda}^\mu.
\]
Therefore we obtain (2.7).

Hence \(\Phi\) induces the desired endomorphism of \(V_{\theta}(\lambda) \subset U_q^{-}(\mathfrak{g})\).

Hereafter we assume further that
\[\text{there is no } i \in I \text{ such that } \theta(i) = i.\]

We conjecture that \(V_{\theta}(\lambda)\) has a crystal basis under this assumption. This means the following. Since \(E_i\) and \(F_i\) satisfy the \(q\)-boson relation, any \(u \in V_{\theta}(\lambda)\) can be
uniquely written as $u = \sum_{n \geq 0} F_i^{(n)} u_n$ with $E_i u_n = 0$. We define the modified root operators $\tilde{E}_i$ and $\tilde{F}_i$ by:

$$\tilde{E}_i(u) = \sum_{n \geq 1} F_i^{(n-1)} u_n \quad \text{and} \quad \tilde{F}_i(u) = \sum_{n \geq 0} F_i^{(n+1)} u_n.$$ 

Let $L_\theta(\lambda)$ be the $A_0$-submodule of $V_\theta(\lambda)$ generated by $\tilde{F}_{i_1} \cdots \tilde{F}_{i_\ell} \phi_\lambda$ ($\ell \geq 0$ and $i_1, \ldots, i_\ell \in I$), and let $B_\theta(\lambda)$ be the subset

$$\{ \tilde{F}_{i_1} \cdots \tilde{F}_{i_\ell} \phi_\lambda \mod qL_\theta(\lambda) \mid \ell \geq 0, i_1, \ldots, i_\ell \in I \}$$

of $L_\theta(\lambda)/qL_\theta(\lambda)$.

**Conjecture 2.16.** For a dominant integral weight $\lambda$ such that $\theta(\lambda) = \lambda$, we have

(1) $\tilde{F}_i L_\theta(\lambda) \subset L_\theta(\lambda)$ and $\tilde{E}_i L_\theta(\lambda) \subset L_\theta(\lambda)$,

(2) $B_\theta(\lambda)$ is a basis of $L_\theta(\lambda)/qL_\theta(\lambda)$,

(3) $\tilde{F}_i B_\theta(\lambda) \subset B_\theta(\lambda)$, and $\tilde{E}_i B_\theta(\lambda) \subset B_\theta(\lambda) \cup \{0\}$,

(4) $\tilde{F}_i \tilde{E}_i (b) = b$ for any $b \in B_\theta(\lambda)$ such that $\tilde{E}_i b \neq 0$, and $\tilde{E}_i \tilde{F}_i (b) = b$ for any $b \in B_\theta(\lambda)$.

As in [K1], we have

**Lemma 2.17.** Assume Conjecture 2.16. Then we have

(i) $L_\theta(\lambda) = \{ v \in V_\theta(\lambda) \mid (L_\theta(\lambda), v) \subset A_0 \}$,

(ii) Let $(\cdot, \cdot)_0$ be the $Q$-valued symmetric bilinear form on $L_\theta(\lambda)/qL_\theta(\lambda)$ induced by $(\cdot, \cdot)$. Then $B_\theta(\lambda)$ is an orthonormal basis with respect to $(\cdot, \cdot)_0$.

Moreover we conjecture that $V_\theta(\lambda)$ has a global crystal basis. Namely we have

**Conjecture 2.18.** The triplet $(L_\theta(\lambda), L_\theta(\lambda)^-, V_\theta(\lambda)^{\text{low}}_\lambda)$ is balanced. Here $V_\theta(\lambda)^{\text{low}}_\lambda := U^-_\theta (\mathfrak{g}) A_0 \phi_\lambda$.

Its dual version is as follows.

Let us denote by $V_\theta(\lambda)^{\text{up}}_\lambda$ the dual space $\{ v \in V_\theta(\lambda) \mid (V_\theta(\lambda)^{\text{low}}_\lambda, v) \subset A_0 \}$.

Then Conjecture 2.18 is equivalent to the following conjecture.
Conjecture 2.19. \((L_\theta(\lambda), c(L_\theta(\lambda)), V_\theta(\lambda)^{up})\) is balanced.

Here \(c\) is a unique endomorphism of \(V_\theta(\lambda)\) such that \(c(\phi_\lambda) = \phi_\lambda\) and \(c(\alpha v) = \alpha c(v), \ c(E_i v) = E_i c(v)\) for any \(a \in K\) and \(v \in V_\theta(\lambda)\). We have \((c(v'), v) = (v', v)\) for any \(v, v' \in V_\theta(\lambda)\).

Note that \(V_\theta(\lambda)^{up}\) is the largest \(A\)-submodule \(M\) of \(V_\theta(\lambda)\) such that \(M\) is invariant by the \(E_i\)'s and \(M \cap K\phi_\lambda = A\phi_\lambda\).

By Conjecture 2.19, \(L_\theta(\lambda) \cap c(L_\theta(\lambda)) \cap V_\theta(\lambda)^{up} \rightarrow L_\theta(\lambda)/qL_\theta(\lambda)\) is an isomorphism. Let \(G_\theta^{up}\) be its inverse. Then \(\{G_\theta^{up}(b)\}_{b \in B_\theta(\lambda)}\) is a basis of \(V_\theta(\lambda)\), which we call the upper global basis of \(V_\theta(\lambda)\). Note that \(\{G_\theta^{up}(b)\}_{b \in B_\theta(\lambda)}\) is the dual basis to \(\{G_\theta^{low}(b)\}_{b \in B_\theta(\lambda)}\) with respect to the inner product of \(V_\theta(\lambda)\).

We shall prove these conjectures in the case \(\mathfrak{g} = \mathfrak{gl}_\infty\) and \(\lambda = 0\).

§3. PBW Basis of \(V_\theta(0)\) for \(\mathfrak{g} = \mathfrak{gl}_\infty\)

§3.1. Review on the PBW basis

In the sequel, we set \(I = \mathbb{Z}_{odd}\) and

\[
(\alpha_i, \alpha_j) = \begin{cases} 
2 & \text{for } i = j, \\
-1 & \text{for } j = i \pm 2, \\
0 & \text{otherwise},
\end{cases}
\]

and we consider the corresponding quantum group \(U_q(\mathfrak{gl}_\infty)\). In this case, we have \(q_i = q\). We write \([n]\) and \([n]!\) for \([n]_{odd}\) and \([n]_{odd}!\) for short.

We can parametrize the crystal basis \(B(\infty)\) by the multisegments. We shall recall this parametrization and the PBW basis.

Definition 3.1. For \(i, j \in I\) such that \(i \leq j\), we define a segment \(\langle i, j \rangle\) as the interval \([i, j] \subset I := \mathbb{Z}_{odd}\). A multisegment is a formal finite sum of segments:

\[
m = \sum_{i < j} m_{ij} \langle i, j \rangle
\]

with \(m_{i,j} \in \mathbb{Z}_{\geq 0}\). We call \(m_{i,j}\) the multiplicity of a segment \(\langle i, j \rangle\). If \(m_{i,j} > 0\), we sometimes say that \(\langle i, j \rangle\) appears in \(m\). We sometimes write \(m_{i,j}(m)\) for \(m_{i,j}\).

We sometimes write \(\langle i \rangle\) for \(\langle i, i \rangle\). We denote by \(\mathcal{M}\) the set of multisegments. We denote by \(\emptyset\) the zero element (or the empty multisegment) of \(\mathcal{M}\).

Definition 3.2. For two segments \(\langle i_1, j_1 \rangle\) and \(\langle i_2, j_2 \rangle\), we define the ordering \(\succ_{PBW}\) by the following:

\[
\langle i_1, j_1 \rangle \succ_{PBW} \langle i_2, j_2 \rangle \iff \begin{cases} 
j_1 > j_2 \\
or \\
j_1 = j_2 \text{ and } i_1 \geq i_2.
\end{cases}
\]
We call this ordering the PBW-ordering.

**Definition 3.3.** For a multisegment $m$, we define the element $P(m) \in U_q^{-}(\mathfrak{gl}_\infty)$ as follows.

1. For a segment $\langle i, j \rangle$, we define the element $\langle i, j \rangle \in U_q^{-}(\mathfrak{gl}_\infty)$ inductively by
   
   $\langle i, i \rangle = f_i,$
   
   $\langle i, j \rangle = \langle i, j-2 \rangle \langle j, j \rangle - q \langle j, j \rangle \langle i, j-2 \rangle$ for $i < j$.

2. For a multisegment $m = \sum_{i \leq j} m_{ij} \langle i, j \rangle$, we define
   
   $P(m) = \prod \langle i, j \rangle^{(m_{ij})}$.

Here the product $\prod$ is taken over segments appearing in $m$ from large to small with respect to the PBW-ordering. The element $\langle i, j \rangle^{(n)}$ is the divided power of $\langle i, j \rangle$, i.e.

$$\langle i, j \rangle^{(n)} = \begin{cases} 
\frac{1}{[n]!} \langle i, j \rangle^n & \text{for } n > 0, \\
1 & \text{for } n = 0, \\
0 & \text{for } n < 0.
\end{cases}$$

Hence the weight of $P(m)$ is equal to $\text{wt}(m) := - \sum_{i \leq k \leq j} m_{i,j} \alpha_k$: $t_i P(m) t_i^{-1} = q^{(\alpha_i, \text{wt}(m))} P(m)$.

**Theorem 3.4 ([L]).** The set of elements $\{P(m) \mid m \in \mathcal{M}\}$ is a $K$-basis of $U_q^{-}(\mathfrak{gl}_\infty)$. Moreover this is an $A$-basis of $U_q^{-}(\mathfrak{gl}_\infty)$. We call this basis the PBW basis of $U_q^{-}(\mathfrak{gl}_\infty)$.

**Definition 3.5.** For two segments $\langle i_1, j_1 \rangle$ and $\langle i_2, j_2 \rangle$, we define the ordering $\geq_{\text{cry}}$ by the following:

$$\langle i_1, j_1 \rangle \geq_{\text{cry}} \langle i_2, j_2 \rangle \iff \begin{cases} j_1 > j_2 \\ \text{or} \\ j_1 = j_2 \text{ and } i_1 \leq i_2. \end{cases}$$

We call this ordering the crystal ordering.

**Example 3.6.** The crystal ordering is different from the PBW-ordering. For example, we have $\langle -1, 1 \rangle >_{\text{cry}} \langle 1, 1 \rangle >_{\text{cry}} \langle -1 \rangle$, while we have $\langle 1, 1 \rangle >_{\text{PBW}} \langle -1, 1 \rangle >_{\text{PBW}} \langle -1 \rangle$. 
**Definition 3.7.** We define the crystal structure on $\mathcal{M}$ as follows: for $m = \sum_{i,j} m_{ij}(i,j) \in \mathcal{M}$ and $i \in I$, set $A^{(i)}_{k}(m) = \sum_{k' \geq k} (m_{i,k'} - m_{i+2,k'+2})$ for $k \geq i$. Define $\varepsilon_{i}(m)$ as $\max \left\{ A^{(i)}_{k}(m) \mid k \geq i \right\} \geq 0$.

(i) If $\varepsilon_{i}(m) = 0$, then define $\tilde{e}_{i}(m) = 0$. If $\varepsilon_{i}(m) > 0$, let $k_{e}$ be the largest $k \geq i$ such that $\varepsilon_{i}(m) = A^{(i)}_{k}(m)$ and define $\tilde{e}_{i}(m) = m - \langle i, k_{e} \rangle + \delta_{k_{e} \neq i} \langle i + 2, k_{e} \rangle$.

(ii) Let $k_{f}$ be the smallest $k \geq i$ such that $\varepsilon_{i}(m) = A^{(i)}_{k}(m)$ and define $\tilde{f}_{i}(m) = m - \delta_{k_{f} \neq i} \langle i + 2, k_{f} \rangle + \langle i, k_{f} \rangle$.

**Remark 3.8.** For $i \in I$, the actions of the operators $\tilde{e}_{i}$ and $\tilde{f}_{i}$ on $m \in \mathcal{M}$ are also described by the following algorithm:

Step 1. Arrange the segments in $m$ in the crystal ordering.

Step 2. For each segment $\langle i, j \rangle$, write $-$, and for each segment $\langle i + 2, j \rangle$, write $+$. Step 3. In the resulting sequence of $+$ and $-$, delete a subsequence of the form $+-$ and keep on deleting until no such subsequence remains.

Then we obtain a sequence of the form $-\cdots-++\cdots+$.

(1) $\varepsilon_{i}(m)$ is the total number of $-$ in the resulting sequence.

(2) $\tilde{f}_{i}(m)$ is given as follows:

(a) if the leftmost $+$ corresponds to a segment $\langle i + 2, j \rangle$, then replace it with $\langle i, j \rangle$,

(b) if no $+$ exists, add a segment $\langle i, i \rangle$ to $m$.

(3) $\tilde{e}_{i}(m)$ is given as follows:

(a) if the rightmost $-$ corresponds to a segment $\langle i, j \rangle$ with $i < j$, then replace it with $\langle i + 2, j \rangle$.

(b) if the rightmost $-$ corresponds to a segment $\langle i, i \rangle$, then remove it,

(c) if no $-$ exists, then $\tilde{e}_{i}(m) = 0$.

Let us introduce a linear ordering on the set $\mathcal{M}$ of multisegments, lexicographic with respect to the crystal ordering on the set of segments.
Definition 3.9. For \( m = \sum_{i \leq j} m_{i,j} \langle i, j \rangle \in \mathcal{M} \) and \( m' = \sum_{i \leq j} m'_{i,j} \langle i, j \rangle \in \mathcal{M} \), we define \( m' < m \) if there exist \( i_0 \leq j_0 \) such that \( m'_{i_0,j_0} < m_{i_0,j_0} \), \( m'_{i,j_0} = m_{i,j_0} \) for \( i < i_0 \), and \( m'_{i,j} = m_{i,j} \) for \( j > j_0 \) and \( i \leq j \).

Theorem 3.10.
(i) \( L(\infty) = \bigoplus_{m \in \mathcal{M}} A_0 P(m) \).

(ii) \( B(\infty) = \{ P(m) \mod qL(\infty) \mid m \in \mathcal{M} \} \).

(iii) We have
\[
\tilde{e}_i P(m) \equiv P(\tilde{e}_i(m)) \mod qL(\infty),
\tilde{f}_i P(m) \equiv P(\tilde{f}_i(m)) \mod qL(\infty).
\]
Note that \( \tilde{e}_i \) and \( \tilde{f}_i \) in the left-hand-side is the modified root operators.

(iv) We have
\[
\overline{P(m)} \in P(m) + \sum_{m' < m} A P(m').
\]

Therefore we can index the crystal basis by multisegments. By this theorem we can easily see by a standard argument that \( (L(\infty), L(\infty)^-, U_q^- (\mathfrak{g})_A) \) is balanced, and there exists a unique \( G^{\text{low}}(m) \in L(\infty) \cap U_q^- (\mathfrak{g})_A \) such that \( G^{\text{low}}(m)^- = G^{\text{low}}(m) \) and \( G^{\text{low}}(m) \equiv P(m) \mod qL(\infty) \). Then \( \{ G^{\text{low}}(m) \}_{m \in \mathcal{M}} \) is a lower global basis.

§3.2. \( \theta \)-restricted multisegments

We consider the Dynkin diagram involution \( \theta \) of \( I := \mathbb{Z}_{\text{odd}} \) defined by \( \theta(i) = -i \) for \( i \in I \).

We shall prove in this case Conjectures 2.16 and 2.18 for \( \lambda = 0 \) (Theorems 4.15 and 5.5).

We set
\[
\tilde{V}_\theta(0) := B_\theta(\mathfrak{gl}_\infty)/ \sum_{i \in I} (B_\theta(\mathfrak{gl}_\infty) E_i + B_\theta(\mathfrak{gl}_\infty) (T_i - 1) + B_\theta(\mathfrak{gl}_\infty) (F_i - F_{\theta(i)}))
\sim U_q^- (\mathfrak{gl}_\infty)/ \sum_i U_q^- (\mathfrak{gl}_\infty) (f_i - f_{\theta(i)}).
Let $\tilde{\phi}$ be the generator of $\tilde{V}_\theta(0)$ corresponding to $1 \in B_\theta(\mathfrak{gl}_\infty)$. Since $F_i \phi''_0 = (f_i + f_{\theta(i)}) \phi''_0 = F_{\theta(i)} \phi''_0$, we have an epimorphism of $B_\theta(\mathfrak{gl}_\infty)$-modules

\[(3.1) \quad \tilde{V}_\theta(0) \twoheadrightarrow V_\theta(0).\]

We shall see later that it is in fact an isomorphism (see Theorem 4.15).

**Definition 3.11.** If a multisegment $m$ has the form

$$m = \sum_{-j \leq i \leq j} m_{ij} \langle i, j \rangle,$$

we call $m$ a $\theta$-restricted multisegment. We denote by $M_\theta$ the set of $\theta$-restricted multisegments.

**Definition 3.12.** For a $\theta$-restricted segment $\langle i, j \rangle$, we define its modified divided power by

$$\langle i, j \rangle^{[m]} = \begin{cases} 
\langle i, j \rangle^{(m)} = \frac{1}{[m]!} \langle i, j \rangle^m & (i \neq -j), \\
\frac{1}{\prod_{\nu=1}^{\lfloor m/2 \rfloor} \langle -j, j \rangle^m} & (i = -j).
\end{cases}$$

We understand that $\langle i, j \rangle^{[m]}$ is equal to 1 for $m = 0$ and vanishes for $m < 0$.

**Definition 3.13.** For $m \in M_\theta$, we define $P_\theta(m) \in U_q^- \mathfrak{gl}_\infty \subset B_\theta(\mathfrak{gl}_\infty)$ by

$$P_\theta(m) = \prod_{\langle i, j \rangle \in m} \langle i, j \rangle^{[m_{ij}]}.$$ 

Here the product $\prod$ is taken over the segments appearing in $m$ from large to small with respect to the PBW-ordering.

If an element $m$ of the free abelian group generated by $\langle i, j \rangle$ does not belong to $M_\theta$, we understand $P_\theta(m) = 0$.

We will prove later that $\{P_\theta(m) \phi \}_{m \in M_\theta}$ is a basis of $V_\theta(0)$ (see Theorem 4.15). Here and hereafter, we write $\phi$ instead of $\phi_0 \in V_\theta(0)$.

**§3.3. Commutation relations of $\langle i, j \rangle$**

In the sequel, we regard $U^-_q \mathfrak{gl}_\infty$ as a subalgebra of $B_\theta(\mathfrak{gl}_\infty)$ by $f_i \mapsto F_i$.

We shall give formulas to express products of segments by a PBW basis.

**Proposition 3.14.** For $i, j, k, l \in I$, we have
(1) \( \langle i, j \rangle(k, \ell) = \langle k, \ell \rangle(i, j) \) for \( i \leq j, k \leq \ell \) and \( j < k - 2 \),

(2) \( \langle i, j \rangle(j + 2, k) = \langle i, k \rangle + q\langle j + 2, k \rangle\langle i, j \rangle \) for \( i \leq j < k \),

(3) \( \langle j, k \rangle(i, \ell) = \langle i, \ell \rangle(j, k) \) for \( i < j \leq k \leq \ell \),

(4) \( \langle i, k \rangle(j, k) = q^{-1}\langle j, k \rangle\langle i, k \rangle \) for \( i < j \leq k \),

(5) \( \langle i, j \rangle(i, k) = q^{-1}\langle i, k \rangle\langle i, j \rangle \) for \( i \leq j < k \),

(6) \( \langle i, k \rangle(j, \ell) = \langle j, \ell \rangle\langle i, k \rangle + (q^{-1} - q)\langle i, \ell \rangle\langle i, k \rangle \) for \( i < j \leq k \leq \ell \).

Proof. (1) is obvious. We prove (2) by the induction on \( k - j \). If \( k - j = 2 \),

It is trivial by the definition. If \( j < k - 2 \), then \( \langle k \rangle \) and \( \langle i, j \rangle \) commute. Thus, we have

\[
\langle i, j \rangle(j + 2, k) = \langle i, j \rangle(\langle j + 2, k - 2 \rangle(\langle k \rangle - q(\langle j + 2, k - 2 \rangle)) = (\langle i, k - 2 \rangle + q(\langle j + 2, k - 2 \rangle)\langle i, j \rangle - q(\langle i, k - 2 \rangle + q(\langle j + 2, k - 2 \rangle)\langle i, j \rangle)) = \langle i, k \rangle + (j + 2, k)\langle i, j \rangle.
\]

In order to prove the other relations, we first show the following special cases.

Lemma 3.15. We have for any \( j \in I \)

(a) \( \langle j - 2, j \rangle\langle j \rangle = q^{-1}\langle j \rangle\langle j - 2, j \rangle \) and \( \langle j \rangle\langle j, j + 2 \rangle = q^{-1}\langle j, j + 2 \rangle\langle j \rangle \),

(b) \( \langle j \rangle\langle j - 2, j + 2 \rangle = \langle j - 2, j + 2 \rangle\langle j \rangle \),

(c) \( \langle j - 2, j \rangle\langle j, j + 2 \rangle = \langle j, j + 2 \rangle\langle j - 2, j \rangle + (q^{-1} - q)\langle j - 2, j + 2 \rangle\langle j \rangle \).

Proof. The first equality in (a) follows from

\[
\langle j - 2, j \rangle\langle j \rangle - q^{-1}\langle j \rangle\langle j - 2, j \rangle = (f_{j - 2}f_j - qf_jf_{j - 2})f_j - q^{-1}f_j(f_{j - 2}f_j - qf_jf_{j - 2}) = f_{j - 2}f_j^2 - (q + q^{-1})f_jf_{j - 2} + f_j^2f_{j - 2} - 0.
\]

We can similarly prove the second.
Let us show (b) and (c). We have, by (a)

\[
\langle j - 2, j \rangle (j, j + 2) = \langle j - 2, j \rangle (\langle j \rangle (j + 2) - q(j + 2) \langle j \rangle)
\]

\[
= q^{-1}(j) (\langle j - 2, j \rangle (j + 2) - q((j - 2, j + 2) + q(j + 2) \langle j - 2, j \rangle) \langle j \rangle
\]

\[
= q^{-1}(j) (\langle j - 2, j + 2 \rangle + q(j + 2) \langle j - 2, j \rangle)
\]

(3.2)

\[
= (\langle j \rangle (j + 2) - q(j + 2) \langle j \rangle) (j - 2, j)
\]

\[
+ q^{-1}(j) (j - 2, j + 2) - q(j - 2, j + 2) \langle j \rangle
\]

\[
= (j, j + 2)(j - 2, j) + q^{-1}(j) (j - 2, j + 2) - q(j - 2, j + 2) \langle j \rangle.
\]

Similarly, we have

\[
\langle j - 2, j \rangle (j, j + 2) = (\langle j - 2 \rangle (j) - q(j) \langle j - 2 \rangle) (j, j + 2)
\]

\[
= q^{-1}(j - 2) (\langle j, j + 2 \rangle(j) - q(j) (\langle j - 2, j + 2 \rangle(j) + q(j, j + 2) (j - 2)\rangle)
\]

\[
= q^{-1}(j - 2, j + 2) + q(j, j + 2)(j - 2)\rangle(j)
\]

(3.3)

\[
= (j, j + 2)(\langle j - 2 \rangle (j) - q(j) \langle j - 2 \rangle)\rangle(j)
\]

\[
+ q^{-1}(j - 2, j + 2) \langle j \rangle - q(j \langle j - 2, j + 2 \rangle \rangle(j)
\]

\[
= (j, j + 2)(j - 2, j) + q^{-1}(j - 2, j + 2) \langle j \rangle - q(j \langle j - 2, j + 2 \rangle \langle j \rangle).
\]

Then, (3.2) and (3.3) imply (b) and (c).

We shall resume the proof of Proposition 3.14. By Lemma 3.15 (b), \((i, k)\) commutes with \((j)\) for \(i < j < k\). Thus we obtain (3).

We shall show (4) by the induction on \(k - j\). Suppose \(k - j = 0\). The case \(i = k - 2\) is nothing but Lemma 3.15 (a).

If \(i < k - 2\), then

\[
\langle i, k \rangle \langle k \rangle = \langle i, k - 4 \rangle \langle k - 2, k \rangle \langle k \rangle - q \langle k - 2, k \rangle \langle i, k - 4 \rangle \langle k \rangle
\]

\[
= q^{-1}(k) \langle i, k - 4 \rangle \langle k - 2, k \rangle - \langle k \rangle \langle k - 2, k \rangle \langle i, k - 4 \rangle = q^{-1}(k) \langle i, k \rangle.
\]

Suppose \(k - j > 0\). By using the induction hypothesis and (3), we have

\[
\langle i, k \rangle \langle j, k \rangle = \langle i, k \rangle (j, j + 2) - q(i, k) (j + 2, k) \langle j \rangle
\]

\[
= (j) \langle i, k \rangle \langle j + 2, k \rangle - (j + 2, k) \langle i, k \rangle \langle j \rangle
\]

\[
= q^{-1}(j) (j + 2, k) \langle i, k \rangle - (j + 2, k) \langle j, k \rangle = q^{-1}(j, k) \langle i, k \rangle.
\]

Similarly we can prove (5).
Let us prove (6). We have
\[
\langle i, k \rangle \langle j, \ell \rangle = (\langle i, j \rangle - 2 \langle j, k \rangle \langle i, j - 2 \rangle \langle j, \ell \rangle)
\]
\[
= q^{-1} \langle i, j \rangle \langle j, \ell \rangle (j, k) - q(j, k) ((i, \ell) + q(j, \ell) \langle i, j - 2 \rangle)
\]
\[
= q^{-1} ((i, \ell) + q(j, \ell) \langle i, j - 2 \rangle) (j, k)
\]
\[
- q(i, \ell) (j, k) - q(j, \ell) (j, k) \langle i, j - 2 \rangle
\]
\[
= \langle j, \ell \rangle \langle i, k \rangle + (q^{-1} - q) \langle i, \ell \rangle \langle j, k \rangle.
\]

Lemma 3.16.

(i) For $1 \leq i \leq j$, we have $\langle -j, -i \rangle \tilde{\phi} = \langle i, j \rangle \tilde{\phi}$.

(ii) For $1 \leq i < j$, we have $\langle -j, i \rangle \tilde{\phi} = q^{-1} \langle -i, j \rangle \tilde{\phi}$.

Proof. (i) If $i = j$, it is obvious. By the induction on $j - i$, we have
\[
\langle -j, -i \rangle \tilde{\phi} = ((-j, -i - 2) (-i) - q(-i) (-j, -i - 2)) \tilde{\phi}
\]
\[
= ((-j, -i - 2) (i) - q(-i) (i + 2, j)) \tilde{\phi}
\]
\[
= ((i) (-j, -i - 2) - q(i + 2, j) (-i)) \tilde{\phi}
\]
\[
= ((i) (i + 2, j) - q(i + 2, j) (i)) \tilde{\phi} = \langle i, j \rangle \tilde{\phi}.
\]

(ii) By (i), we have
\[
\langle -j, i \rangle \tilde{\phi} = \langle (-j, -i) \langle 1, i \rangle - q(1, i) \langle -i, -1 \rangle \rangle \tilde{\phi}
\]
\[
= \langle (-j, -i) \langle -i, -1 \rangle - q(1, i) \langle 1, j \rangle \tilde{\phi}
\]
\[
= (q^{-1} \langle -i, -1 \rangle \langle -j, -1 \rangle - \langle 1, j \rangle \langle 1, i \rangle) \tilde{\phi}
\]
\[
= (q^{-1} \langle -i, -1 \rangle \langle 1, j \rangle - \langle 1, j \rangle \langle -i, -1 \rangle) \tilde{\phi} = q^{-1} \langle -i, j \rangle \tilde{\phi}.
\]

Proposition 3.17.

(i) For a multisegment $m = \sum_{i,j} m_{i,j} \langle i, j \rangle$, we have
\[
\text{Ad}(t_k) P(m) = q^{\sum_{i,j}(m_{i,k-2} - m_{i,k}) + \sum_{j}(m_{k+2,j} - m_{k,j})} P(m).
\]
we have Finally we show

\[
\begin{align*}
\text{induction on} & \quad n \\
\text{For} & \quad n \\
\text{The case} & \quad k \\
\text{We shall prove} & \quad (ii) \\
\text{Proof.} & \quad (i) \text{ is obvious. Let us show (ii). It is obvious that } e_k'(i, j)^{(n)} = 0 \\
& \quad \text{unless } i \leq k \leq j. \text{ It is known ([K1]) that we have } e_k'(k)^{(n)} = q^{1-n}(k)^{(n-1)}. \\
& \quad \text{We shall prove } e_k'(k, j)^{(n)} = (1 - q^2)q^{1-n}(k + 2, j)^{(k, j)^{(n-1)}} \text{ for } k < j \text{ by the} \\
& \quad \text{ induction on } n. \text{ By (2.1), we have} \\
& \quad e_k'(k, j) = e_k'(k, j)^{(n)} = (1) = q(k + 2, j)^{(k, j)}(k) \\
& \quad = (k + 2, j) - q^2(k + 2, j) = (1 - q^2)(k + 2, j). \\
& \quad \text{For } n \geq 1, \text{ by the induction hypothesis and Proposition 3.14 (4), we get} \\
& \quad [n]e_k'(k, j)^{(n)} = e_k'(k, j)^{(n-1)} \\
& \quad = (1 - q^2)(k + 2, j)^{(n-1)} + q^{-1}(k, j)^{(n-2)}(1 - q^2)q^{2-n}(k + 2, j)^{(k, j)^{(n-2)}} \\
& \quad = (1 - q^2)\left\{ (k + 2, j)^{(n-1)} + q^{1-n}(k + 2, j)^{(k, j)^{(n-2)}} \right\} \\
& \quad = (1 - q^2)(1 + q^{-n}\{n - 1\}) (k + 2, j)^{(k, j)^{(n-1)}} \\
& \quad = (1 - q^2)q^{-n}\{k + 2, j\} (k, j)^{(n-1)}. \\
& \quad \text{Finally we show } e_k'(i, j) = 0 \text{ if } k \neq i. \text{ We may assume } i < k \leq j. \text{ If } i < k < j, \text{ we have} \\
& \quad e_k'(i, j) = e_k'(i, k - 2)q(k, j) - q(k, j)q(k, j) \\
& \quad = q(i, k - 2)e_k'(i, j) - q(e_k'(i, j))q(k, j) \\
& \quad = q(1 - q^2)(i, k - 2)q(k + 2, j) - q(1 - q^2)q(k + 2, j)q(i, k - 2) \\
& \quad = 0. \\
& \quad \text{The case } k = j \text{ is similarly proved.} \\
& \quad \text{The proof for } e_k^* \text{ is similar.} \quad \Box
\end{align*}
\]
§3.4. Actions of divided powers

Lemma 3.18. Let $a, b$ be non-negative integers, and let $k \in I > 0 := \{k \in I \mid k > 0\}$.

1. For $\ell > k$, we have
   \[
   \langle -k \rangle \langle -k + 2, \ell \rangle^{(a)} \langle -k, \ell \rangle^{(b)} = \left[ b + 1 \right] \langle -k + 2, \ell \rangle^{(a-1)} \langle -k, \ell \rangle^{(b+1)}
   + q^{a-b} \langle -k + 2, \ell \rangle^{(a)} \langle -k, \ell \rangle^{(b)} \langle -k \rangle.
   \]

2. We have
   \[
   \langle -k \rangle \langle -k + 2, k \rangle^{(a)} \langle -k, k \rangle^{(b)} = \left[ 2b + 2 \right] \langle -k + 2, k \rangle^{(a-1)} \langle -k, k \rangle^{(b+1)}
   + q^{a-b} \langle -k + 2, k \rangle^{(a)} \langle -k, k \rangle^{(b)} \langle -k \rangle.
   \]

3. For $k > 1$, we have
   \[
   \langle -k \rangle \langle -k + 2, k - 2 \rangle^{(a)} \langle -k, k - 2 \rangle^{(a)} = (q^a + q^{-a})^{-1} \langle -k + 2, k - 2 \rangle^{(a-1)} \langle -k, k - 2 \rangle
   + q^a \langle -k + 2, k - 2 \rangle^{(a)} \langle -k \rangle.
   \]

4. If $\ell \leq k - 2$, we have
   \[
   \langle \ell, k \rangle^{(a)} \langle \ell, k - 2 \rangle^{(a-1)} + q^a \langle k \rangle \langle \ell, k - 2 \rangle^{(a)}.
   \]

5. For $k > 1$, we have
   \[
   \langle -k + 2, k - 2 \rangle^{[a]} \langle k \rangle = (q^a + q^{-a})^{-1} \langle -k + 2, k \rangle \langle -k + 2, k - 2 \rangle^{[a-1]} \langle -k \rangle
   + q^a \langle k \rangle \langle -k + 2, k - 2 \rangle^{[a]}.
   \]

Proof. We show (1) by the induction on $a$. If $a = 0$, it is trivial. For $a > 0$, we have

\[
[a] \langle -k \rangle \langle -k + 2, \ell \rangle^{(a)} \langle -k, \ell \rangle^{(b)}
= \left( \langle -k, \ell \rangle + q \langle -k + 2, \ell \rangle \langle -k \rangle \right) \langle -k + 2, \ell \rangle^{(a-1)} \langle -k, \ell \rangle^{(b)}
= [b + 1] q^{1-a} \langle -k + 2, \ell \rangle^{(a-1)} \langle -k, \ell \rangle^{(b+1)}
+ q \langle -k + 2, \ell \rangle \left[ [b + 1] \langle -k + 2, \ell \rangle^{(a-2)} \langle -k, \ell \rangle^{(b+1)}
+ q^{a-b-1} \langle -k + 2, \ell \rangle^{(a-1)} \langle -k, \ell \rangle^{(b)} \langle -k \rangle \right]
= [b + 1] \left( q^{1-a} + q[a - 1] \right) \langle -k + 2, \ell \rangle^{(a-1)} \langle -k, \ell \rangle^{(b+1)}
+ q^{a-b} [a] \langle -k + 2, \ell \rangle^{(a)} \langle -k, \ell \rangle^{(b)} \langle -k \rangle.
\]
Since $q^{1-a} + q[a-1] = [a]$, the induction proceeds.

The proof of (2) is similar by using $(-k, k)^{[b]} = [2b](\langle -k, k \rangle^{b-1}) \langle -k, k \rangle$.

We prove (3) by the induction on $a$. The case $a = 0$ is trivial. For $a > 0$, we have

$$[2a]\langle -k \rangle \langle -k + 2, k - 2 \rangle^{[a]}$$

$$= ((-k, k - 2) + q(-k + 2, k - 2) \langle -k \rangle) \langle -k + 2, k - 2 \rangle^{[a-1]}$$

$$+ q(-k + 2, k - 2) \{(q^{a-1} + q^{1-a})^{-1}(-k + 2, k - 2)^{[a-2]}(-k, k - 2) + q^{a-1}(-k + 2, k - 2)^{[a-1]}(-k)\}$$

$$= (q^{1-a} + \frac{q[2a - 2]}{q^{a-1} + q^{1-a}})(-k + 2, k - 2)^{[a-1]}(-k, k - 2)$$

$$+ q^a[2a](\langle -k + 2, k - 2 \rangle^a(-k)$$

$$= (q^a + q^{-a})^{-1}[2a]\langle -k + 2, k - 2 \rangle^{[a-1]}(-k, k - 2)$$

$$+ q^a[2a](\langle -k + 2, k - 2 \rangle^a(-k).$$

Similarly, we can prove (4) and (5) by the induction on $a$. □

**Lemma 3.19.** For $k > 1$ and $a, b, c, d > 0$, set

$$(a, b, c, d) = \langle k \rangle^{(a)} \langle -k + 2, k \rangle^{(b)} \langle -k, k \rangle^{(c)} \langle -k + 2, k - 2 \rangle^{[d]} \phi.$$

Then, we have

$$\langle -k \rangle(a, b, c, d) = [2c + 2](a, b - 1, c + 1, d)$$

$$+ [b + 1]q^{a-2c}(a, b + 1, c, d - 1)$$

$$+ [a + 1]q^{2d-2c}(a + 1, b, c, d).$$

**Proof.** We shall show first

$$\langle -k \rangle(-k + 2, k - 2)^{[d]} \phi$$

$$= (\langle -k + 2, k \rangle(-k + 2, k - 2)^{[d-1]} + q^{2d}\langle -k + 2, k - 2 \rangle^{[d]} \phi).$$

By Lemma 3.18 (3), we have

$$\langle -k \rangle(-k + 2, k - 2)^{[d]} \phi$$

$$= ((q^d + q^{-d})^{-1}\langle -k + 2, k - 2 \rangle^{[d-1]}(-k, k - 2)$$

$$+ q^d\langle -k + 2, k - 2 \rangle^{[d]}(-k)\phi.$$
By Lemma 3.16 and Lemma 3.18 (5), it is equal to

\[
((q^d + q^{-d})^{-1}q^{-1}(-k + 2, k - 2)^{[d-1]}(-k + 2, k) + q^d(-k + 2, k - 2)^{[d]}(k))\tilde{\phi}
\]

\[
= \left((q^d + q^{-d})^{-1}q^{-1}q^{1-d}(-k + 2, k)(-k + 2, k - 2)^{[d-1]}
+ q^d((q^d + q^{-d})^{-1}(-k + 2, k)(-k + 2, k - 2)^{[d-1]}
+ q^d(k)(-k + 2, k - 2)^{[d]})\right)\tilde{\phi}.
\]

Thus we obtain (3.5). Applying Lemma 3.18 (2), we have

\[
\langle -k \rangle(a, b, c, d) = \langle k \rangle^{(a)}\left([2c + 2][-k + 2, k)^{(b-1)}(-k, k]^{[c+1]}
+ q^{b-c}(-k + 2, k)^{(b)}(-k, k]^{[c]}\langle -k, k - 2\rangle^{[d]}\tilde{\phi}
\right)
\]

\[
= [2c + 2](a, b - 1, c + 1, d) + q^{b-c}\langle -k + 2, k \rangle^{(b)}\langle -k, k \rangle^{[c]}
\times \left((-k + 2, k)(-k + 2, k - 2)^{[d-1]} + q^{2d}(k)(-k + 2, k - 2)^{[d]}\right)\tilde{\phi}
\]

\[
= [2c + 2](a, b - 1, c + 1, d) + q^{b-2c}[b + 1](a, b + 1, c, d - 1)
+ q^{b-c+2d-c-b}[a + 1](a + 1, b, c, d).
\]

Hence we have (3.4).

\[\square\]

**Proposition 3.20.**

1. We have

\[
\langle -1 \rangle^{(a)}\langle -1, 1 \rangle^{[m]}\tilde{\phi} = \sum_{s=0}^{[a/2]} \left(\prod_{\nu=1}^{s} \frac{[2m + 2\nu]}{[2\nu]}\right) q^{2-2(s-2)[a-2s+1]} \times \langle 1 \rangle^{(a-2s)}\langle -1, 1 \rangle^{[m+s]}\tilde{\phi}.
\]

2. For \(k > 1\), we have

\[
\langle -k \rangle^{(a)}\langle -k + 2, k - 2 \rangle^{[a]}\tilde{\phi} = \sum_{i+j+2l=n, j=t=u} q^{2ai + i(l-1) - i(t+u)}
\times \langle k \rangle^{(i)}\langle -k + 2, k \rangle^{(j)}\langle -k, k \rangle^{[l]}(-k + 2, k - 2)^{[a-u]}\tilde{\phi}.
\]

3. If \(\ell > k\), we have

\[
\langle k \rangle^{(n)}\langle k + 2, \ell \rangle^{(a)} = \sum_{s=0}^{n} q^{(n-s)(a-s)}\langle k + 2, \ell \rangle^{(n-s)}\langle k, \ell \rangle^{(s)}\langle k \rangle^{(n-s)}.
\]
Proof. We prove (1) by the induction on $a$. The case $a = 0$ is trivial. Assume $a > 0$. Then, Lemma 3.18 (2) implies

$$\langle -1 \rangle (1)^{(a)} \langle -1, 1 \rangle^{[m]} \overline{\phi}$$

$$= \left( [2m + 2] (1)^{(n-1)} \langle -1, 1 \rangle^{[m+1]} + q^{n-m} (1)^{(n)} \langle -1, 1 \rangle^{[m]} \right) \overline{\phi}$$

$$= \left( [2m + 2] (1)^{(n-1)} \langle -1, 1 \rangle^{[m+1]} + q^{n-m} (1)^{(n)} \langle -1, 1 \rangle^{[m]} \right) \overline{\phi}$$

$$= \left( [2m + 2] (1)^{(n-1)} \langle -1, 1 \rangle^{[m+1]} + q^{n-m} (1)^{(n+1)} \langle -1, 1 \rangle^{[m]} \right) \overline{\phi}.$$ 

Put

$$c_s = \left( \prod_{\nu=1}^{s} \frac{[2m + 2 \nu]}{[2\nu]} \right) q^{-2(a-s)m+\frac{(a-2s)(a-2s-1)}{2}}.$$ 

Then we have

$$[a + 1] \langle -1 \rangle (a+1) \langle -1, 1 \rangle^{[m]} \overline{\phi} = \langle -1 \rangle (a) \langle -1, 1 \rangle^{[m]} \overline{\phi}$$

$$= \langle -1 \rangle \sum_{s=0}^{[a/2]} c_s \langle 1 \rangle (a-2s) \langle -1, 1 \rangle^{[m+s]} \overline{\phi}$$

$$= \sum_{s=0}^{[a/2]} c_s \left\{ [2(m + s + 1)] \langle 1 \rangle (a-2s-1) \langle -1, 1 \rangle^{[m+s+1]} \right.$$

$$+ q^{a-2s-2(m+s)[a - 2s + 1]} \langle 1 \rangle (a-2s+1) \langle -1, 1 \rangle^{[m+s]} \} \overline{\phi}.$$ 

In the right-hand-side, the coefficients of $\langle 1 \rangle^{a+1-2r} \langle -1, 1 \rangle^{[m+r]} \overline{\phi}$ are

$$[2(m + r)] c_{r-1} + q^{a-2m-4r}[a - 2r + 1] c_r$$

$$= \prod_{\nu=1}^{r} \frac{[2m + 2 \nu]}{[2\nu]} q^{-2(a-r+1)m+\frac{m-2r(a-2r+1)}{2}} \left( [2r] q^{a-2r+1} + [a - 2r + 1] q^{-2r} \right)$$

$$= [a + 1] \prod_{\nu=1}^{r} \frac{[2m + 2 \nu]}{[2\nu]} q^{-2(a-r+1)m+\frac{a-2r(a-2r+1)}{2}}.$$ 

Hence we obtain (1).

We prove (2) by the induction on $n$. We use the following notation for short:

$$(i, j, t, a) := \langle k \rangle^{(i)} \langle -k + 2, k \rangle^{(j)} \langle -k, k \rangle^{[t]} \langle -k + 2, k - 2 \rangle^{[a]} \overline{\phi}.$$ 

Then Lemma 3.19 implies that

$$\langle -k \rangle (i, j, t, a) = [2t + 2] (i, j - 1, t + 1, a)$$

$$+ [j + 1] q^{-2t} (i, j + 1, t, a - 1)$$

$$+ [i + 1] q^{2a-2t} (i + 1, j, t, a).$$
Hence, by assuming (2) for \( n \), we have
\[
[n + 1](-k)^{(n+1)}(-k + 2, k - 2)^{[a]}\widetilde{\phi} = (-k)^{(n)}(-k + 2, k - 2)^{[a]}\widetilde{\phi}
\]
\[
= \sum_{i+j+2t=n,j+t=u} \left\{ [2t + 2]q^{2ai + \frac{i(i-1)}{2} + i(t+u)(i, j - 1, t + 1, a - u)} + \left[ j + 1 \right]q^{2ai + \frac{j(j-1)}{2} + j(t+u)+2t(i, j + 1, t, a - u - 1)} \right\}.
\]

Then in the right hand side, the coefficients of \((i', j', t', a - u')\) satisfying \( i' + j' + 2t' = n + 1, j' + t' = u' \) are
\[
[2t']q^{2ai' + \frac{(i'-1)(i'-2)}{2} - i'(t' - 1 + u')} + [j']q^{2ai' + \frac{j'(j'-1)}{2} - i'(t' + u' - 1) + j' - 1 - 2t'}
\]
\[
+ [i']q^{2a(i' - 1) + L(t'-1) - (i' - 1)(t' + u') + 2a - 2u' - 2t'}
\]
\[
= q^{2a(i' + L(t'-1) - i'(t' + u')) (2t') + [j']q^{i' - 2t'} + [i']q^{j' - (t' + u')}}
\]
\[
= q^{2a(i' + L(t'-1) - i'(t' + u')) [n + 1]}. \]

We can prove (3) similarly as above. ∎

§3.5. Actions of \( E_k, F_k \) on the PBW basis

For a \( \theta \)-restricted multisegment \( m \), we set
\[
\widetilde{P}_b(m) = P_b(m)\widetilde{\phi}.
\]

We understand \( \widetilde{P}_b(m) = 0 \) if \( m \) is not a multisegment.

**Theorem 3.21.** For \( k \in I > 0 \) and a \( \theta \)-restricted multisegment \( m = \sum_{-j \leq i \leq j} m_{i,j} (i, j) \), we have
\[
F_{-k}\widetilde{P}_b(m)
\]
\[
= \sum_{m_{-k,t}} [m_{-k,t} + 1]q^{m_{-k+2,t} - m_{-k,t}} \widetilde{P}_b(m - (-k + 2, t))
\]
\[
+ q^{m_{-k+2,t} - m_{-k,t}} \sum_{m_{-k+2,t} - m_{-k,t}} [2m_{-k,k} + 2] \widetilde{P}_b(m - (-k + 2, k) + (-k, k))
\]
\[
+ q^{m_{-k+2,t} - m_{-k,t}} \sum_{m_{-k+2,t} - m_{-k,t}} [m_{-k+2,k} - 2m_{-k,k} - 2m_{-k,k}]
\]
\[
\times [m_{-k+2,k} + 1] \widetilde{P}_b(m - \delta_{k\neq1}(-k + 2, k - 2) + (-k + 2, k))
\]
\[
+ \sum_{-k+2 < i < k} q^{m_{i,k}} \sum_{m_{i,k}} [m_{i,k} + 1] \widetilde{P}_b(m - \delta_{i<k} (i, k - 2) + (i, k)).
\]
Hence the last term in (3.6) is equal to

\[ k \]

If \( k = 1 \), we understand \((-k + 2, k - 2)^{[m]} = 1\). By Lemma 3.18 (1), we have

\[ \langle -k \rangle P_\theta(m_1) = \sum_{\ell > k} q^{\ell' + \ell}(m_{-k+2,\ell - m_{-k,\ell} + 1}) P_\theta(m_1 = (-k + 2, \ell) + \langle -k, \ell \rangle) \]

and Lemma 3.18 (2) implies

\[ \langle -k \rangle P_\theta(m_2) = [2m_{-k,k} + 2] P_\theta(m_2 = (-k + 2, k) + \langle -k, k \rangle) + q^{m_{-k+2,k} - m_{-k,k}} P_\theta(m_2)(-k) \]

Since we have \( \langle -k \rangle P_\theta(m_3) = P_\theta(m_3)(-k) \), we obtain

\[ (3.6) \] \[ \langle -k \rangle \tilde{P}_\theta(m) = \sum_{\ell > k} q^{\ell' + \ell}(m_{-k+2,\ell - m_{-k,\ell}}) \tilde{P}_\theta(m = (-k + 2, \ell) + \langle -k, \ell \rangle) \]

\[ + q^{\sum_{\ell > k}(m_{-k+2,\ell - m_{-k,\ell}})}[2m_{-k,k} + 2] \tilde{P}_\theta(m = (-k + 2, k) + \langle -k, k \rangle) \]

\[ + q^{\sum_{\ell > k}(m_{-k+2,\ell - m_{-k,\ell}}) P_\theta(m_1 + m_2 + m_3)} \times \langle -k \rangle (m_{-k+2,k} - 2) \tilde{\phi} \]

By (3.5), we have

\[ \langle -k \rangle (-k + 2, k - 2)^{[m_{-k+2,k-2}]} \tilde{\phi} = \langle -k + 2, k \rangle (-k + 2, k - 2)^{[m_{-k+2,k-2} - 1]} \tilde{\phi} \]

\[ + \delta_{k \neq 1} q^{2m_{-k+2,k-2}} \langle k \rangle (-k + 2, k - 2)^{[m_{-k+2,k-2}]} \tilde{\phi} \]

Hence the last term in (3.6) is equal to

\[ q^{\sum_{\ell > k}(m_{-k+2,\ell + m_{-k,\ell}})} \tilde{P}_\theta(m = \delta_{k \neq 1} (-k + 2, k - 2) + \langle -k + 2, k \rangle) \]

\[ + \delta_{k \neq 1} q^{\sum_{\ell > k}(m_{-k+2,\ell - m_{-k,\ell}}) + 2m_{-k+2,k-2}} \times P_\theta(m_1 + m_2 + m_3) \langle k \rangle (-k + 2, k - 2)^{[m_{-k+2,k-2}]} \tilde{\phi} \].
For $k \neq 1$, Lemma 3.18 (4) implies
\[ P_\theta(m_3)(k) = \sum_{-k+2<j<i\leq k} q^{\sum_{-k+2<j<i} m_j} m_{i,k-2} (i,k) P_\theta(m_3 - \delta_{i<k}(i,k-2)), \]
and Proposition 3.14 implies
\[ P_\theta(m_2)(i,k) = q^{-\sum_{j<i} m_{j,k}} [m_{i,k} + 1] P_\theta(m_2 + (i,k)). \]
Hence we obtain
\[ P_\theta(m_1) P_\theta(m_2) P_\theta(m_3)(k)(-k+2,k-2)^{[m_{k+k+2,k-2}]_\tilde{\phi}} = \sum_{-k+2<j<i\leq k} q^{\sum_{-k+2<j<i} m_j} m_{i,k-2}^2 \sum_{-k+2<j<i} m_{j,k} \times [m_{i,k} + 1] P_\theta(m - \delta_{i<k}(i,k-2) + (i,k)). \]
Thus we obtain the desired result. \hfill \Box

**Theorem 3.22.** For $k \in \mathbb{Z}_{>0}$ and a $\theta$-restricted multisegment $m = \sum_{j < i \leq k} m_{i,j}(i,j)$, we have
\[ E_{-k} \tilde{P}_\theta(m) \]
\[ = (1 - q^2) \sum_{\ell \geq k} q^{1+\sum_{\ell' \leq \ell} (m_{-k+2,\ell'} - m_{-k,\ell'})} \times [m_{-k+2,\ell+1} + 1] \tilde{P}_\theta(m - (k,\ell) + (-k+2,\ell)) \]
\[ + (1 - q^2) q^{1+\sum_{\ell \geq k} (m_{-k+2,\ell} - m_{-k,\ell}) + m_{-k+2,k-2} - 2m_{-k,k}} \times [m_{-k+2,k} + 1] \tilde{P}_\theta(m - (-k,k) + (-k+2,k)) \]
\[ + (1 - q^2) q^{1+\sum_{\ell \geq k} (m_{-k+2,\ell} - m_{-k,\ell}) + 2m_{-k+2,k-2} - 2m_{-k,k} + \sum_{-k+2<j<i\leq k} (m_{i,k-2} - m_{i,k})} \times [m_{i,k-2} + 1] \tilde{P}_\theta(m - (i,k) + (i,k-2)) \]
\[ + \delta_{k \neq 1} (1 - q^2) q^{1+\sum_{\ell \geq k} (m_{-k+2,\ell} - m_{-k,\ell}) + 2m_{-k+2,k-2} - 2m_{-k,k}} \times [2(m_{-k+2,k-2} + 1)] \tilde{P}_\theta(m - (-k+2,k) + (-k+2,k-2)) \]
\[ + q^{(m_{-k+2,\ell} - m_{-k,\ell}) - 2m_{-k,k} + \delta_{k \neq 1} (1 - m_{k,k} + 2m_{-k+2,k-2} + \sum_{-k+2<j<i\leq k} (m_{i,k-2} - m_{i,k}))} \times \tilde{P}_\theta(m - (k)). \]

**Proof.** We shall divide $m$ into \[ m = m_1 + m_2 + m_3 \]
where $m_1 = \sum_{i\leq j < k} m_{i,j}(i,j)$ and $m_2 = \sum_{i \leq k} m_{i,k}(i,k)$ and $m_3 = \sum_{i \leq j < k} m_{i,j}(i,j)$.

By (2.3) and Proposition 3.17, we have

$$E_{-k} \tilde{P}_\theta(m) = \left( (e'_{-k} P_\theta(m_1)) P_\theta(m_2) + m_3 \right)
+ \left( \text{Ad}(t_{-k}) P_\theta(m_1) (e'_{-k} P_\theta(m_2) + m_3) \right)
+ \text{Ad}(t_{-k}) \{ P_\theta(m_1) (e'_{k} P_\theta(m_2)) \text{Ad}(t_k) P_\theta(m_3) \} \tilde{\phi}.$$  

By Proposition 3.17, the first term is

$$\left( (e'_{-k} P_\theta(m_1)) P_\theta(m_2) + m_3 \right)
= (1 - q^2) \sum_{t > k} q^t \sum_{m_{k+2,i} \leq m_{k,t}}^{m_{k+2,i} = m_{k,t}} \times [m_{k+2,i} + 1] P_\theta (m - \langle -k, t \rangle + \langle -k, 2, k \rangle).$$

The second term is

$$\left( \text{Ad}(t_{-k}) P_\theta(m_1) (e'_{k} P_\theta(m_2)) \text{Ad}(t_k) P_\theta(m_3) \right)
= q^t \sum_{m_{k+2,i} \leq m_{k,t} + \delta_{i=1}^{m_{i,k} - 2}}^{m_{k+2,i} = m_{k,t} + \delta_{i=1}^{m_{i,k} - 2}} P_\theta(m_1) (e'_{k} P_\theta(m_2)) P_\theta(m_3).$$

Let us calculate the last part of (3.7). We have

$$\text{Ad}(t_{-k}) \left( P_\theta(m_1) (e'_{k} P_\theta(m_2)) \text{Ad}(t_k) P_\theta(m_3) \right)
= q^t \sum_{m_{k+2,i} \leq m_{k,t} + \delta_{i=1}^{m_{i,k} - 2}}^{m_{k+2,i} = m_{k,t} + \delta_{i=1}^{m_{i,k} - 2}} P_\theta(m_1) (e'_{k} P_\theta(m_2)) P_\theta(m_3).$$

We have

$$e'_{k} P_\theta(m_2) = q^{1-m_{k} \sum_{i < k} m_{i,k}} P_\theta(m_2 - \langle k \rangle)
+ (1 - q^2) \sum_{-k < i < k} q^{1-m_{k} \sum_{i < k} m_{i,k}} P_\theta(m_2 - \langle i, k \rangle)(i, k - 2)
+ \frac{[m_{k,k}]^{-1}}{[2m_{k,k}]} (1 - q^2) q^{1-m_{k,k}} P(m_2 - \langle -k, k \rangle)(-k, k - 2).$$

For $-k < i < k$, we have

$$\langle i, k - 2 \rangle P_\theta(m_3)
= q^{1 \sum_{i > k, j < k} m_{i,j} = k - 2} \sum_{i > k, j < k} \left( [1 + \delta_{i=1}^{m_{i,k} - 2} + 1] P_\theta(m_3 + \langle i, k - 2 \rangle).$$
By Lemma 3.16, we have
\[
\langle -k, k - 2 \rangle P_\theta(m_3) = \sum_{-k+2 \leq i \leq k-2} m_{i,k-2} P_\theta(m_3) \langle -k, k - 2 \rangle = q^{-k+2} \sum_{-k+2 \leq i \leq k-2} m_{i,k-2} P_\theta(m_3) \langle -k, k - 2 \rangle.
\]

Hence we obtain
\[
P_\theta(m_3)(\langle -k, k - 2 \rangle) P_\theta(m_3) = q^{-k+2} \sum_{-k+2 \leq i \leq k-2} m_{i,k-2} P_\theta(m_3) \langle -k, k - 2 \rangle.
\]

Hence the coefficient of \( \tilde{\phi} \) in \( E_{-k} \hat{P}_\theta(m) \) is
\[
\sum_{q^t} \left( \sum_{i \leq k} m_{i,k} - \sum_{i > k} m_{i,k} \right) \left( \sum_{i \leq k} m_{i,k} \right) = q^{\sum_{i \leq k} (m_{i,k} - m_{i,k})} \left( \sum_{i \leq k} m_{i,k} \right)
\]

The coefficient of \( \hat{P}_\theta(m) \) in \( E_{-k} \hat{P}_\theta(m) \) is
\[
(1 - q^2) q^{1+\sum_{i \leq k} m_{i,k} \left( m_{i,k} + 1 \right)} \left[ \sum_{i \leq k} m_{i,k} \right] \left( 1 + q^{2m-k} \right)
\]

\[
= (1 - q^2) q^{1+\sum_{i \leq k} m_{i,k} \left( m_{i,k} + 1 \right)} \left[ \sum_{i \leq k} m_{i,k} \right] \left( 1 + q^{2m-k} \right)
\]

\[
= (1 - q^2) q^{1+\sum_{i \leq k} m_{i,k} \left( m_{i,k} + 1 \right)} \left[ \sum_{i \leq k} m_{i,k} \right] \left( 1 + q^{2m-k} \right)
\]

\[
= (1 - q^2) q^{1+\sum_{i \leq k} m_{i,k} \left( m_{i,k} + 1 \right)} \left[ \sum_{i \leq k} m_{i,k} \right] \left( 1 + q^{2m-k} \right)
\]

\[
= (1 - q^2) q^{1+\sum_{i \leq k} m_{i,k} \left( m_{i,k} + 1 \right)} \left[ \sum_{i \leq k} m_{i,k} \right] \left( 1 + q^{2m-k} \right)
\]

\[
= (1 - q^2) q^{1+\sum_{i \leq k} m_{i,k} \left( m_{i,k} + 1 \right)} \left[ \sum_{i \leq k} m_{i,k} \right] \left( 1 + q^{2m-k} \right)
\]

\[
= (1 - q^2) q^{1+\sum_{i \leq k} m_{i,k} \left( m_{i,k} + 1 \right)} \left[ \sum_{i \leq k} m_{i,k} \right] \left( 1 + q^{2m-k} \right)
\]

\[
= (1 - q^2) q^{1+\sum_{i \leq k} m_{i,k} \left( m_{i,k} + 1 \right)} \left[ \sum_{i \leq k} m_{i,k} \right] \left( 1 + q^{2m-k} \right)
\]
For \(-k+2 < i \leq k-2\), the coefficient of \(\tilde{P}_\theta(m - \langle i, k \rangle + \langle i, k-2 \rangle)\) in \(E_{-k}\tilde{P}_\theta(m)\) is
\[
(1-q^2) \sum \frac{m_{i',k-2} - \delta_{k=1} + 1 - \sum m_{i',k} - \sum m_{i',k-2}}{i' \leq i} [m_{i,k-2} + 1]
\]
Finally, for \(k \neq 1\), the coefficient of \(\tilde{P}_\theta(m - \langle -k+2, k \rangle + \langle -k+2, k-2 \rangle)\) in \(E_{-k}\tilde{P}_\theta(m)\) is
\[
(1-q^2) \sum \frac{m_{i',k-2} - \delta_{k=1} + 1 - m_{i,k-2} - m_{i+k-2}}{i' \leq i} [m_{i,k-2} + 1]
\]

**Theorem 3.23.** For \(k > 0\) and \(m \in M_\theta\), we have
\[
E_k\tilde{P}_\theta(m) = \sum_{\ell \geq k} (1-q^2) q^{1+\sum_{i' > i}(m_{i',\ell} - m_{i,k})} \times \sum \frac{m_{i,k-2} - \delta_{k=1} + 1 - m_{i,k-2} - m_{i+k-2}}{i' \leq i} [2(m_{i+k-2,k-2} + 1)]
\]
\[
F_k\tilde{P}_\theta(m) = \sum_{\ell \neq k} q^{1+\sum_{i' > i}(m_{i',\ell} - m_{i,k})} [m_{i,k} + 1] \tilde{P}_\theta(m - \delta_{\ell \neq k}(k+2, \ell) + \langle k, \ell \rangle).
\]

**Proof.** The first follows from \(e^*_k P_b(m) = 0\) and Proposition 3.17, and the second follows from Proposition 3.20. \(\square\)

**§4. Crystal Basis of \(V_\theta(0)\)**

**§4.1. A criterion for crystals**

We shall give a criterion for a basis to be a crystal basis. Although we treat the case for modules over \(\mathcal{B}(\mathfrak{g})\) in this paper, similar results hold also for \(U_q(\mathfrak{g})\).

Let \(K[e,f]\) be the ring generated by \(e\) and \(f\) with the defining relation \(ef = q^{-2}fe + 1\). We define the divided power by \(f^{(n)} = f^n/[n]!\).

Let \(P\) be a free \(\mathbb{Z}\)-module, and let \(\alpha\) be a non-zero element of \(P\).
Let $M$ be a $K[e,f]$-module. Assume that $M$ has a weight decomposition $M = \oplus_{\lambda \in P} M_\lambda$, and $eM_\lambda \subset M_{\lambda+\alpha}$ and $fM_\lambda \subset M_{\lambda-\alpha}$.

Assume the following finiteness conditions:

\begin{equation}
\text{(4.1) \quad for any } \lambda \in P, \dim M_\lambda < \infty \text{ and } M_{\lambda+n\alpha} = 0 \text{ for } n \gg 0.
\end{equation}

Hence for any $u \in M$, we can write $u = \sum_{n \geq 0} f^{(n)} u_n$ with $eu_n = 0$. We define endomorphisms $\tilde{e}$ and $\tilde{f}$ of $M$ by

\begin{align*}
\tilde{e} u &= \sum_{n \geq 1} f^{(n-1)} u_n, \\
\tilde{f} u &= \sum_{n \geq 0} f^{(n+1)} u_n.
\end{align*}

Let $B$ be a crystal with weight decomposition by $P$. In this paper, we consider only the following type of crystals. We have $\text{wt}: B \to P, \tilde{f}: B \to B, \tilde{e}: B \to B \cup \{0\}, \varepsilon: B \to \mathbb{Z} \geq 0$ satisfying the following properties, where $B_\lambda := \text{wt}^{-1}(\lambda)$:

(i) $\tilde{f} B_\lambda \subset B_{\lambda-\alpha}$ and $\tilde{e} B_\lambda \subset B_{\lambda+\alpha} \cup \{0\}$ for any $\lambda \in P$,

(ii) $\tilde{e} \tilde{f} (b) = b$ if $\tilde{e} b \neq 0$, and $\tilde{e} \circ \tilde{f} = \text{id}_B$,

(iii) for any $\lambda \in P$, $B_\lambda$ is a finite set and $B_{\lambda+n\alpha} = \emptyset$ for $n \gg 0$,

(iv) $\varepsilon(b) = \max \{ n \geq 0 \mid \tilde{e}^n b \neq 0 \}$ for any $b \in B$.

Set $\text{ord}(a) = \sup \{ n \in \mathbb{Z} \mid a \in q^n A_0 \}$ for $a \in K$. We understand $\text{ord}(0) = \infty$.

Let $\{ C(b) \}_{b \in B}$ be a system of generators of $M$ with $C(b) \in M_{\text{wt}(b)}$: $M = \sum_{b \in B} K C(b)$.

Let $\xi$ be a map from $B$ to an ordered set. Let $c: \mathbb{Z} \to \mathbb{R}, f: \mathbb{Z} \to \mathbb{R}$ and $e: \mathbb{Z} \to \mathbb{R}$. Assume that a decomposition $B = B' \cup B''$ is given.

Assume that we have expressions:

\begin{align*}
(4.2) & \quad e C(b) = \sum_{b' \in B} E_{b,b'} C(b'), \\
(4.3) & \quad f C(b) = \sum_{b' \in B} F_{b,b'} C(b').
\end{align*}

Now consider the following conditions for these data, where $\ell = \varepsilon(b)$ and $\ell' = \varepsilon(b')$: \begin{equation}
(4.4) \quad c(0) = 0, \text{ and } c(n) > 0 \text{ for } n \neq 0,
\end{equation}
(4.5) \( c(n) \leq n + c(m + n) + c(m) \) for \( n \geq 0 \),
(4.6) \( c(n) \leq c(m + n) + f(m) \) for \( n \leq 0 \),
(4.7) \( c(n) + f(n) > 0 \) for \( n > 0 \),
(4.8) \( c(n) + e(n) > 0 \) for \( n > 0 \),
(4.9) \( \text{ord}(F_{b,b'}) \geq -\ell + f(\ell + 1 - \ell') \),
(4.10) \( \text{ord}(E_{b,b'}) \geq 1 - \ell + e(\ell - 1 - \ell') \),
(4.11) \( F_{b,\bar{b}} \in q^{-\ell}(1 + qA_0) \),
(4.12) \( E_{b,\bar{b}} \in q^{1-\ell}(1 + qA_0) \) if \( \ell > 0 \),
(4.13) \( \text{ord}(F_{b,b'}) > -\ell + f(\ell + 1 - \ell') \) if \( b' \neq \bar{b} \), \( \xi(\bar{f}b) \neq \xi(b') \),
(4.14) \( \text{ord}(F_{b,b'}) > -\ell + f(\ell + 1 - \ell') \) if \( \bar{f}b \in B', b' \neq \bar{f}b \) and \( \ell \leq \ell' - 1 \),
(4.15) \( \text{ord}(E_{b,b'}) > 1 - \ell + e(\ell - 1 - \ell') \) if \( b \in B'' \), \( b' \neq \bar{b} \) and \( \ell \leq \ell' + 1 \).

**Theorem 4.1.** Assume the conditions (4.4)–(4.15). Let \( L \) be the \( A_0 \)-submodule \( \sum_{b \in B} A_0 C(b) \) of \( M \). Then we have \( \bar{e}L \subset L \) and \( \bar{f}L \subset L \). Moreover we have
\[
\bar{e}C(b) \equiv C(\bar{e}b) \mod qL \quad \text{and} \quad \bar{f}C(b) \equiv C(\bar{f}b) \mod qL \quad \text{for any } b \in B.
\]

Here we understand \( C(0) = 0 \).

We shall divide the proof into several steps.

Write
\[
C(b) = \sum_{n \geq 0} f^{(n)}C_n(b) \quad \text{with } \bar{e}C_n(b) = 0.
\]

Set
\[
L_0 = \sum_{b \in B, n \geq 0} A_0 f^{(n)}C_0(b).
\]

Set for \( u \in M, \text{ord}(u) = \sup \{ n \in \mathbb{Z} \mid u \in q^nL_0 \} \). If \( u = 0 \) we set \( \text{ord}(u) = \infty \), and if \( u \notin \bigcup_{n \in \mathbb{Z}} q^nL_0 \), then \( \text{ord}(u) = -\infty \).

We shall use the following two recursion formulas (4.16) and (4.17).

We have
\[
\bar{e}C(b) = \sum_{n \geq 1} q^{1-n} f^{(n-1)}C_n(b)
= \sum_{n \geq 0} E_{b,b'} f^{(n)}C_n(b').
\]
Hence we have

\[ C_n(b) = \sum_{b' \in B_{\lambda+a}} q^{n-1} E_{b,b'} C_{n-1}(b') \quad \text{for } n > 0 \text{ and } b \in B_{\lambda}. \tag{4.16} \]

If \( \ell := \varepsilon(b) > 0 \), then we have

\[
 fC(\tilde{\varepsilon}b) = \sum_{b' \in B, n \geq 0} F_{\tilde{\varepsilon}b,b'} f^{(n)} C_n(b') = \sum_{n \geq 0} [n + 1] f^{(n+1)} C_n(\tilde{\varepsilon}b).
\]

Hence, we have by (4.11)

\[
 \delta_{n \neq 0} [n] C_{n-1}(\tilde{\varepsilon}b) = \sum_{b'} F_{\tilde{\varepsilon}b,b'} C_n(b') \in q^{1-\ell} (1 + qA_0) C_n(b) + \sum_{b' \neq b} F_{\tilde{\varepsilon}b,b'} C_n(b').
\]

Therefore we obtain

\[ C_n(b) \in \delta_{n \neq 0}(1 + qA_0) q^{\ell-n} C_{n-1}(\tilde{\varepsilon}b) + \sum_{b' \neq b} q^{\ell-1} A_0 F_{\tilde{\varepsilon}b,b'} C_n(b') \quad \text{if } \ell > 0. \tag{4.17} \]

**Lemma 4.2.** \( \text{ord}(C_n(b)) \geq c(n-\ell) \) for any \( n \in \mathbb{Z}_{\geq 0} \) and \( b \in B_{\ell} \), where \( \ell := \varepsilon(b) \).

**Proof.** For \( \lambda \in P \), we shall show the assertion for \( b \in B_{\lambda} \) by the induction on \( \sup \{ n \in \mathbb{Z} \mid M_{\lambda+n\alpha} \neq 0 \} \). Hence we may assume

\[ \text{ord}(C_n(b)) \geq c(n-\ell) \] for any \( n \in \mathbb{Z}_{\geq 0} \) and \( b \in B_{\lambda+a} \).

(i) Let us first show \( C_n(b) \in KL_0 \).

Since it is trivial for \( n = 0 \), assume that \( n > 0 \). Since \( C_{n-1}(b') \in KL_0 \) for \( b' \in B_{\lambda+a} \) by the induction assumption (4.18), we have \( C_n(b) \in KL_0 \) by (4.16).

(ii) Let us show that \( \text{ord}(C_n(b)) \geq c(n-\ell) \) for \( n \geq \ell \).

If \( n = 0 \), then \( \ell = 0 \) and the assertion is trivial by (4.4). Hence we may assume that \( n > 0 \).

We shall use (4.16). For \( b' \in B_{\lambda+a} \), we have

\[ \text{ord}(C_{n-1}(b')) \geq c(n-1-\ell') \quad \text{where } \ell' = \varepsilon(b') \]

by the induction hypothesis (4.18). On the other hand, \( \text{ord}(E_{b,b'}) \geq 1 - \ell + c(\ell - 1 - \ell') \) by (4.10). Hence,

\[
 \text{ord}(q^{n-1} E_{b,b'} C_{n-1}(b')) \geq (n-1) + (1 - \ell + c(\ell - 1 - \ell')) + c(n-1-\ell') \\
= (n-\ell) + c(\ell - 1 - \ell') + c((n-\ell) + (\ell - 1 - \ell')) \geq c(n-\ell)
\]
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by (4.5).

(iii) In the general case, let us set

$$r = \min \{\text{ord}(C_n(b)) - c(n - \varepsilon(b)) \mid b \in B_\lambda, n \geq 0\} \in \mathbb{R} \cup \{\infty\}.$$ 

Assuming $r < 0$, we shall prove

$$\text{ord}(C_n(b)) > c(n - \ell) + r$$

for any $b \in B_\lambda$, which leads a contradiction.

By the induction on $\xi(b)$, we may assume that

$$\text{ord}(C_n(b')) > c(n - \ell') + r$$

for $b' \neq b$.

We shall divide its proof into two cases.

(a) $\xi(b') < \xi(b)$.

In this case, (4.19) implies $\text{ord}(C_n(b')) > c(n - \ell') + r$. Hence

$$\text{ord}(q^{\ell-1} F_{\tilde{e}b,b} C_n(b')) > (\ell - 1) + (1 - \ell + f(\ell - \ell')) + c(n - \ell') + r$$

$$= f(\ell - \ell') + c((n - \ell) + (\ell - \ell')) + r \geq c(n - \ell) + r$$

by (4.9) and (4.6).

(b) Case $\xi(b') \geq \xi(b)$.

In this case, $\text{ord}(F_{\tilde{e}b,b'}) > 1 - \ell + f(\ell - \ell')$ by (4.13), and $\text{ord}(C_n(b')) \geq c(n - \ell') + r$. Hence,

$$\text{ord}(q^{\ell-1} F_{\tilde{e}b,b} C_n(b')) > (\ell - 1) + (1 - \ell + f(\ell - \ell')) + c(n - \ell') + r$$

$$= f(\ell - \ell') + c((n - \ell) + (\ell - \ell')) + r \geq c(n - \ell) + r.$$

\[\square\]

**Lemma 4.3.** $\text{ord}(C_\ell(b) - C_{\ell-1}(\tilde{e}b)) > 0$ for $\ell := \varepsilon(b) > 0$.

**Proof.**

We divide the proof into two cases: $b \in B'$ and $b \in B''$. 

(i) \( b \in B' \).

By (4.17), it is enough to show
\[
\text{ord}(q^{\ell-1}F_{\ell b, b'}C_{\ell}(b')) > 0 \quad \text{for } b' \neq b.
\]

(a) Case \( \ell > \ell' := \varepsilon(b') \).

We have
\[
\text{ord}(q^{\ell-1}F_{\ell b, b'}C_{\ell}(b')) \geq (\ell - 1) + (1 - \ell + f(\ell - \ell')) + c(\ell - \ell') > 0
\]
by (4.7).

(b) Case \( \ell \leq \ell' \).

We have \( \text{ord}(F_{\ell b, b'}) > 1 - \ell + f(\ell - \ell') \) by (4.14). Hence
\[
\text{ord}(q^{\ell-1}F_{\ell b, b'}C_{\ell}(b')) > (\ell - 1) + (1 - \ell + f(\ell - \ell')) + c(\ell - \ell') > 0
\]
by (4.6) with \( n = 0 \).

(ii) Case \( b \in B'' \).

We use (4.16). By (4.12), it is enough to show that
\[
\text{ord}(q^{\ell-1}E_{b, b'}C_{\ell-1}(b')) > 0 \quad \text{for } b' \neq \tilde{e}b.
\]

(a) Case \( \ell - 1 > \ell' \).

\[
\text{ord}(q^{\ell-1}E_{b, b'}C_{\ell-1}(b')) \geq c(\ell - 1 - \ell') + c(\ell - 1 - \ell') > 0
\]
by (4.10) and (4.8).

(b) Case \( \ell - 1 \leq \ell' \).

\[
\text{ord}(E_{b, b'}) > 1 - \ell + e(\ell - 1 - \ell') \quad \text{by (4.15), and } \text{ord}(q^{\ell-1}E_{b, b'}C_{\ell-1}(b')) > c(\ell - 1 - \ell') + c(\ell - 1 - \ell') \geq 0
\]
by (4.5) with \( n = 0 \).

\[\square\]

Hence we have
\[
C_n(b) \equiv 0 \mod qL_0 \quad \text{for } n \neq \ell := \varepsilon(b),
C_\ell(b) \equiv C_0(\tilde{e}b) \mod qL_0,
C(b) \equiv f(\ell)C_\ell(b) \mod qL_0,
fC(b) \equiv C(\tilde{f}b) \mod qL_0,
\tilde{e}C(b) \equiv C(\tilde{e}b) \mod qL_0,
L_0 := \sum_{b \in B, n \geq 0} A_0 f^{(n)}C_0(b) = \sum_{b \in B} A_0 C(b).\]
Indeed, the last equality follows from the fact that \( \{C(b)\}_{b \in B} \) generates \( L_0/qL_0 \).

Thus we have completed the proof of Theorem 4.1.

The following is the special case where \( B' = B'' = B \) and \( \xi(b) = \varepsilon(b) \).

**Corollary 4.4.** Assume (4.4)–(4.12) and

\[
\text{ord}(F_{b, b'}) > -\ell + f(1 + \ell - \ell') \quad \text{if } \ell < \ell' \text{ and } b' \neq \tilde{f}b,
\]

\[
\text{ord}(E_{b, b'}) > 1 - \ell + \varepsilon(\ell - 1 - \ell') \quad \text{if } \ell \leq \ell' + 1 \text{ and } b' \neq \tilde{e}b.
\]

Then the assertions of Theorem 4.1 hold.

§4.2. Crystal structure on \( M_{\theta} \)

We shall define the crystal structure on \( M_{\theta} \).

**Definition 4.5.** Suppose \( k > 0 \). For a \( \theta \)-restricted multisegment \( m = \sum_{-j \leq i \leq j} m_{i, j} \langle i, j \rangle \), we set

\[ \varepsilon_{-k}(m) = \max \left\{ A_j^{(-k)}(m) \mid j \geq -k + 2 \right\}, \]

where

\[
A_j^{(-k)}(m) = \sum_{\ell \geq j} (m_{-k, \ell} - m_{-k+2, \ell+2}) \quad \text{for } j > k,
\]

\[
A_k^{(-k)}(m) = \sum_{\ell > k} (m_{-k, \ell} - m_{-k+2, \ell}) + 2m_{-k, k} + \delta(m_{-k+2, k} \text{ is odd}),
\]

\[
A_j^{(-k)}(m) = \sum_{\ell > k} (m_{-k, \ell} - m_{-k+2, \ell}) + 2m_{-k, k} - 2m_{-k+2, k-2} + \sum_{-k+2 < i \leq j} m_{i, k} - \sum_{-k+2 < i \leq j} m_{i, k-2} \quad \text{for } -k + 2 \leq j \leq k - 2.
\]

(i) Let \( n_f \) be the smallest \( \ell \geq -k + 2 \), with respect to the ordering \( \cdots > k + 2 > k > -k + 2 > \cdots > k - 2 \), such that \( \varepsilon_{-k}(m) = A_{n_f}^{(-k)}(m) \). We define

\[
\tilde{F}_{-k}(m) = \begin{cases} 
- m - \langle -k + 2, n_f \rangle + \langle -k, n_f \rangle & \text{if } n_f > k, \\
- m - \langle -k + 2, k \rangle + \langle -k, k \rangle & \text{if } n_f = k \text{ and } m_{-k+2, k} \text{ is odd,} \\
- m - \delta_{k \neq 1} \langle -k + 2, k - 2 \rangle & \langle -k + 2, k \rangle & \text{if } n_f = k \text{ and } m_{-k+2, k} \text{ is even,} \\
- m - \delta_{n_f \neq k} + 2 \langle n_f + 2, k - 2 \rangle & \langle n_f + 2, k \rangle & \text{if } -k + 2 \leq n_f \leq k - 2.
\end{cases}
\]
(ii) If \( \varepsilon_{-k}(m) = 0 \), then \( \tilde{E}_{-k}(m) = 0 \). If \( \varepsilon_{-k}(m) > 0 \), then let \( n_e \) be the largest \( \ell \geq -k + 2 \), with respect to the above ordering, such that \( \varepsilon_{-k}(m) = A_{\ell}^{(-k)}(m) \). We define

\[
\tilde{E}_{-k}(m) = \begin{cases} 
    m - \langle -k, n_e \rangle + \langle -k + 2, n_e \rangle & \text{if } n_e > k, \\
    m - \langle -k, k \rangle + \langle -k + 2, k \rangle & \text{if } n_e = k \text{ and } m_{-k+2,k} \text{ is even,} \\
    m - \langle -k + 2, k \rangle + \delta_{k\not\equiv1} \langle -k + 2, k - 2 \rangle & \text{if } n_e = k \text{ and } m_{-k+2,k} \text{ is odd,} \\
    m - \langle n_e + 2, k \rangle + \delta_{n_e\not\equiv k-2} \langle n_e + 2, k - 2 \rangle & \text{if } -k + 2 \leq n_e \leq k - 2. 
\end{cases}
\]

Remark 4.6. For \( 0 < k \in I \), the actions of \( \tilde{E}_{-k} \) and \( \tilde{F}_{-k} \) on \( m \in \mathcal{M}_\theta \) are described by the following algorithm.

Step 1. Arrange segments in \( m \) of the form \( \langle -k, j \rangle (j > k) \), \( \langle -k + 2, j \rangle (j > k) \), \( \langle i, k \rangle (-k \leq i \leq k) \), \( \langle i, k - 2 \rangle (-k + 2 \leq i \leq k - 2) \) in the order

\[
    \cdots, \langle -k, k + 2 \rangle, \langle -k + 2, k + 2 \rangle, \langle -k, k \rangle, \langle -k + 2, k \rangle, \langle -k + 2, k - 2 \rangle, \\
    \langle -k + 4, k \rangle, \langle -k + 4, k - 2 \rangle, \cdots, \langle k - 2, k \rangle, \langle k - 2, k - 2 \rangle, \langle k \rangle.
\]

Step 2. Write signatures for each segment contained in \( m \) by the following rules.

(i) If a segment is not \( \langle -k + 2, k \rangle \), then

- For \( \langle -k, k \rangle \), write \(--\),
- For \( \langle -k, j \rangle \) with \( j > k \), write \( -\),
- For \( \langle -k + 2, k - 2 \rangle \) with \( k > 1 \), write \( ++\),
- For \( \langle -k + 2, j \rangle \) with \( j > k \), write \( +\),
- For \( \langle j, k \rangle \) with \( -k + 2 < j \leq k \), write \( -\),
- For \( \langle j, k - 2 \rangle \) with \( -k + 2 < j \leq k - 2 \), write \( +\),
- Otherwise, write no signature.

(ii) For segments \( m_{-k+2,k}\langle -k + 2, k \rangle \), if \( m_{-k+2,k} \) is even, then write no signature, and if \( m_{-k+2,k} \) is odd, then write \( --\).

Step 3. In the resulting sequence of \( + \) and \( - \), delete a subsequence of the form \( ++-- \) and keep on deleting until no such subsequence remains.
Then we obtain a sequence of the form \(- - \cdots - + + \cdots +\).

(1) \(\varepsilon_{-k}(m)\) is the total number of \(-\) in the resulting sequence.

(2) \(\tilde{F}_{-k}(m)\) is given as follows:

(i) if the leftmost \(+\) corresponds to a segment \((-k + 2, j)\) for \(j > k\), then replace it with \((-k, j)\),

(ii) if the leftmost \(+\) corresponds to a segment \((j, k - 2)\) for \(-k + 2 \leq j \leq k - 2\), then replace it with \((j, k)\),

(iii) if the leftmost \(+\) corresponds to segment \(m_{-k+2,k}(-k + 2, k)\), then replace one of the segments with \((-k, k)\),

(iv) if no \(+\) exists, add a segment \((k, k)\) to \(m\).

(3) \(\tilde{E}_{-k}(m)\) is given as follows:

(i) if the rightmost \(-\) corresponds to a segment \((-k, j)\) for \(j \geq k\), then replace it with \((-k + 2, j)\),

(ii) if the rightmost \(-\) corresponds to a segment \((j, k)\) for \(-k + 2 < j < k\), then replace it with \((j, k - 2)\),

(iii) if the rightmost \(-\) corresponds to segments \(m_{-k+2,k}(-k + 2, k)\), then replace one of the segments with \((-k + 2, k - 2)\),

(iv) if the rightmost \(-\) corresponds to a segment \((k, k)\) for \(k > 1\), then delete it,

(v) if no \(-\) exists, then \(\tilde{E}_{-k}(m) = 0\).

**Example 4.7.**

(1) We shall write \(\{a, b\}\) for \(a(-1, 1) + b(1)\). The following diagram is the part of the crystal graph of \(B_\theta(0)\) that concerns only the 1-arrows and the \((-1)\)-arrows.

\[
\begin{array}{cccccccc}
\phi & 1 & \{0, 1\} & \leftarrow & 1 & \{0, 2\} & \leftarrow & 1 & \{0, 4\} & \leftarrow & 1 & \{0, 5\} & \cdots \\
-1 & \leftarrow & \{1, 0\} & \leftarrow & 1 & \{1, 2\} & \leftarrow & 1 & \{1, 3\} & \cdots \\
& & \{1, 1\} & \leftarrow & & \{2, 0\} & \leftarrow & & & \{2, 1\} & \cdots \\
\end{array}
\]
Especially the part of \((-1)\)-arrows is the following diagram.

\[
\begin{array}{c}
0,2n \xrightarrow{-1} 0,2n+1 \xrightarrow{-1} 1,2n \xrightarrow{-1} 1,2n+1 \xrightarrow{-1} 2,2n \ldots
\end{array}
\]

(2) The following diagram is the part of the crystal graph of \(B_\theta(0)\) that concerns only the \((-1)\)-arrows and the \((-3)\)-arrows. This diagram is, as a graph, isomorphic to the crystal graph of \(A_2\).

(3) Here is the part of the crystal graph of \(B_\theta(0)\) that concerns only the \(n\)-arrows and the \((-n)\)-arrows for an odd integer \(n \geq 3\):

\[
\phi \xrightarrow{n} \langle n \rangle \xrightarrow{n} 2\langle n \rangle \xrightarrow{n} 3\langle n \rangle \xrightarrow{n} \ldots
\]

**Lemma 4.8.** For \(k \in I_{>0}\), the data \(\tilde{E}_{-k}, \tilde{F}_{-k}, \varepsilon_{-k}\) define a crystal structure on \(\mathcal{M}_\theta\), namely we have
(i) \( \tilde{F}_{-k} \mathcal{M}_\theta \subset \mathcal{M}_\theta \) and \( \tilde{E}_{-k} \mathcal{M}_\theta \subset \mathcal{M}_\theta \sqcup \{0\} \),
(ii) \( \tilde{F}_{-k} \tilde{E}_{-k} (m) = m \) if \( \tilde{E}_{-k} (m) \neq 0 \), and \( \tilde{E}_{-k} \circ \tilde{F}_{-k} = \text{id} \),
(iii) \( \varepsilon_{-k}(m) = \max \left\{ n \geq 0 \mid \tilde{E}^{n}_{-k}(m) \neq 0 \right\} \) for any \( m \in \mathcal{M}_\theta \).

**Proof.** We shall first show that, for 

\[ (i) \quad \tilde{F}_{-k}(m) \] 

\[ (ii) \quad \tilde{F}_{-k} \tilde{E}_{-k} (m) = m \] 

\[ (iii) \quad \varepsilon_{-k}(m) = \max \left\{ n \geq 0 \mid \tilde{E}^{n}_{-k}(m) \neq 0 \right\} \] 

for any \( m \in \mathcal{M}_\theta \).

Let \( \tilde{F}_{-k}(m) \) be \( \theta \)-restricted, \( \tilde{E}_{-k} \tilde{F}_{-k}(m) = m \) and \( \varepsilon_{-k}(\tilde{F}_{-k}m) = \varepsilon_{-k}(m) + 1 \). Let 

\[ A_j := \tilde{A}_{j}(-k)(m) \] 

\[ (j \geq -k + 2) \] 

and let \( n_f \) be as in Definition 4.5. Set \( m' = \tilde{F}_{-k}m \).

(i) Assume \( n_f > k \). Since \( A_{n_f} > A_{n_f-2} = A_{n_f} + m_{-k,n_f-2} - m_{-k+2,n_f} \), we have \( m_{-k,n_f-2} < m_{-k+2,n_f} \). Hence \( m' = m - \langle -k + 2, n_f \rangle + \langle -k, n_f \rangle \) is \( \theta \)-restricted. Then we have

\[
A'_j = \begin{cases} 
A_j & \text{if } j > n_f, \\
A_j + 1 & \text{if } j = n_f, \\
A_j + 2 & \text{if } j < n_f.
\end{cases}
\]

Hence \( \varepsilon_{-k}(m') = A_{n_f} + 1 = \varepsilon_{-k}(m) + 1 \) and \( n'_e = n_f \), which implies \( m = \tilde{E}_{-k}(m') \).

(ii) Assume \( n_f = k \).

(a) If \( m_{-k+2,k} \) is odd, then \( m' = m - \langle -k + 2, k \rangle + \langle -k, k \rangle \) is \( \theta \)-restricted. We have

\[
A'_j = \begin{cases} 
A_j & \text{if } j > k, \\
A_j + 1 & \text{if } j = k, \\
A_j + 2 & \text{if } j < k.
\end{cases}
\]

Hence \( \varepsilon_{-k}(m') = \varepsilon_{-k}(m) + 1 \) and \( n'_e = k \), which implies \( m = \tilde{E}_{-k}(m') \).

(b) Assume that \( m_{-k+2,k} \) is even. If \( k \neq 1 \), then \( A_k > A_{-k+2} = A_k - 2m_{-k+2,k-2} \), and hence \( m_{-k+2,k-2} > 0 \). Therefore \( m' = m - \delta_{k \neq 1}(-k + 2, k - 2) + \langle -k + 2, k \rangle \) is \( \theta \)-restricted. We have

\[
A'_j = \begin{cases} 
A_j & \text{if } j > k, \\
A_j + 1 & \text{if } j = k, \\
A_j + 2 & \text{if } j < k.
\end{cases}
\]

Hence \( \varepsilon_{-k}(m') = \varepsilon_{-k}(m) + 1 \) and \( n'_e = k \), which implies \( m = \tilde{E}_{-k}(m') \).
(iii) Assume $-k + 2 \leq n_f < k - 2$. Since $A_{n_f} > A_{n_f+2} = A_{n_f} + m_{n_f+4,k} - m_{n_f+2,k-2}$, we have $m_{n_f+2,k-2} > m_{n_f+4,k}$. Hence $m' = m - \langle n_f + 2, k - 2 \rangle + \langle n_f + 2, k \rangle$ is $\theta$-restricted. Then we have

$$A'_{j} = \begin{cases} 
A_j & \text{if } j > n_f, \\
A_j + 1 & \text{if } j = n_f, \\
A_j + 2 & \text{if } j < n_f.
\end{cases}$$

(Here the ordering is as in Definition 4.5 (i).) Hence $\varepsilon_{-k}(m') = \varepsilon_{-k}(m) + 1$ and $n'_f = n_f$, which implies $m = \tilde{E}_{-k}m'$.

(iv) Assume $n_f = k - 2$. It is obvious that $m' = m + \langle k \rangle$ is $\theta$-restricted. We have

$$A'_{j} = \begin{cases} 
A_j & \text{if } j \neq n_f, \\
A_j + 1 & \text{if } j = n_f.
\end{cases}$$

Hence $\varepsilon_{-k}(m') = \varepsilon_{-k}(m) + 1$ and $n'_f = n_f$, which implies $m = \tilde{E}_{-k}(m')$. Similarly, we can prove that if $\varepsilon_{-k}(m) > 0$, then $\tilde{E}_{-k}(m)$ is $\theta$-restricted and $\tilde{F}_{-k}\tilde{E}_{-k}(m) = m$. Hence we obtain the desired results.

**Definition 4.9.** For $k \in I_{>0}$, we define $\tilde{F}_{k}, \tilde{E}_{k}$ and $\varepsilon_{k}$ by the same rule as in Definition 3.7 for $\tilde{f}_{k}, \tilde{e}_{k}$ and $\varepsilon_{k}$.

Since it is well-known that it gives a crystal structure on $M$, we obtain the following result.

**Theorem 4.10.** By $\tilde{F}_{k}, \tilde{E}_{k}, \varepsilon_{k} (k \in I)$, $M_{\theta}$ is a crystal, namely, we have

(i) $\tilde{F}_{k}M_{\theta} \subset M_{\theta}$ and $\tilde{E}_{k}M_{\theta} \subset M_{\theta} \sqcup \{0\},$

(ii) $\tilde{F}_{k}\tilde{E}_{k}(m) = m$ if $\tilde{E}_{k}(m) \neq 0$, and $\tilde{E}_{k} \circ \tilde{F}_{k} = \text{id},$

(iii) $\varepsilon_{k}(m) = \max \left\{ n \geq 0 \mid \tilde{E}_{n}^{\alpha}(m) \neq 0 \right\}$ for any $m \in M_{\theta}$.

The crystal $M_{\theta}$ has a unique highest weight vector.

**Lemma 4.11.** If $m \in M_{\theta}$ satisfies that $\varepsilon_{k}(m) = 0$ for any $k \in I$, then $m = 0$. Here $0$ is the empty multisegment. In particular, for any $m \in M_{\theta}$, there exist $\ell \geq 0$ and $i_1, \ldots, i_{\ell} \in I$ such that $m = \tilde{F}_{i_1} \cdots \tilde{F}_{i_{\ell}} 0$. 
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Proof. Assume $m \neq \emptyset$. Let $k$ be the largest $k$ such that $m_{k,j} \neq 0$ for some $j$. Then take the largest $j$ such that $m_{k,j} \neq 0$. Then $j \geq |k|$. Moreover, we have $m_{k+2,\ell} = 0$ for any $\ell$, and $m_{k,\ell} = 0$ for any $\ell > j$. Hence we have

$$A_j^{(k)}(m) = \begin{cases} 2m_{k,j} & \text{if } k = -j, \\ m_{k,j} & \text{otherwise.} \end{cases}$$

Hence $\varepsilon_k(m) \geq A_j^{(k)}(m) > 0$.

§4.3. Estimates of the order of coefficients

By applying Theorem 4.1, we shall show that $\{P_\theta(m)\phi\}_{m \in M_\theta}$ is a crystal basis of $V_\theta(0)$ and its crystal structure coincides with the one given in §4.2.

Let $k$ be a positive odd integer. We define $c, f, e: \mathbb{Z} \to \mathbb{Q}$ by $c(n) = |n/2|$ and $f(n) = e(n) = n/2$. Then the conditions (4.4)–(4.8) are obvious. Set $\xi(m) = (-1)^{m_{-k+2,k}m_{-k,k}}$ and

$$B'' = \{ m \in M_\theta \mid -k + 2 \leq n_e(m) < k \} \cup \{ m \in M_\theta \mid m_{-k+2,k}(m) \text{ is odd} \},$$

$$B' = M_\theta \setminus B''.$$  

Here $n_e(m)$ is $n_e$ given in Definition 4.5 (ii). If $\varepsilon_{-k}(m) = 0$, then we understand $n_e(m) = \infty$.

We define $F_{m,m'}^{-k}$ and $E_{m,m'}^{-k}$ by the coefficients of the following expansion:

$$F_{-k}P_\theta(m)\tilde{\phi} = \sum_{m'} F_{m,m'}^{-k}P_\theta(m')\tilde{\phi},$$

$$E_{-k}P_\theta(m)\tilde{\phi} = \sum_{m'} E_{m,m'}^{-k}P_\theta(m')\tilde{\phi},$$

as given in Theorems 3.21 and 3.22. Put $\ell = \varepsilon_{-k}(m)$ and $\ell' = \varepsilon_{-k}(m')$.

**Proposition 4.12.** The conditions (4.9), (4.11), (4.13) and (4.14) are satisfied for $\tilde{E}_{-k}, \tilde{F}_{-k}, \varepsilon_{-k}$, namely, we have

(a) if $m' = \tilde{F}_{-k}(m)$, then $F_{m,m'}^{-k} \in q^{-\ell}(1 + qA_0),$

(b) if $m' \neq \tilde{F}_{-k}(m)$, then $\text{ord}(F_{m,m'}^{-k}) \geq -\ell + f(\ell + 1 - \ell') = -(\ell + \ell' - 1)/2,$

(c) if $m' \neq \tilde{F}_{-k}(m)$ and $\text{ord}(F_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2$, then the following two conditions hold:

1. $\xi(\tilde{F}_{-k}(m)) > \xi(m'),$
(2) $\ell \geq \ell'$ or $\tilde{F}_{-k}(m) \in B''$.

Proof. We shall write $A_j$ for $A_j^-(m)$. Let $n_f$ be as in Definition 4.5 (i). Note that $F_{m,F_{-k}(m)}^m \neq 0$.

If $F_{m,m'}^m \neq 0$, we have the following four cases. We shall use $[n] \in q^{1-n}(1 + qA_0)$ for $n > 0$.

Case 1. $m' = m - \langle -k + 2, n \rangle + \langle -k, n \rangle$ for $n > k$.

In this case, we have

$$F_{m,m'}^{-k} = [m_{-k,n} + 1]q^{\sum_{j} \langle m_{-k+2,j} - m_{-k,j} \rangle} \in q^{-A_k}(1 + qA_0)$$

and

$$\ell = \max\{A_j (j \geq -k + 2)\},$$

$$\ell' = \max\{A_j (j > n), A_n + 1, A_j + 2 (j < n)\}.$$

If $m' = \tilde{F}_{-k}(m)$, then $\ell = A_n$ and we obtain (a). Assume $m' \neq \tilde{F}_{-k}(m)$. Since $A_n \leq \ell, \ell' - 1$, we have $\operatorname{ord}(F_{m,m'}^{-k}) = -A_n \geq -(\ell + \ell' - 1)/2$. Hence we obtain (b). If $\operatorname{ord}(F_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2$, then we have $A_n = \ell = \ell' - 1$. Since $A_j + 2 \leq \ell' = A_n + 1$ for $j < n$, we have $n_f = n$ and $m' = \tilde{F}_{-k}(m)$, which is a contradiction.

Case 2. $m' = m - \langle -k + 2, n \rangle + \langle -k, k \rangle$.

In this case we have

$$F_{m,m'}^{-k} = [2m_{-k,k} + 2]q^{\sum_{j} \langle m_{-k+2,j} - m_{-k,j} \rangle} \in q^{-A_k - \delta(m_{-k+2,k} \text{ is even})}(1 + qA_0).$$

(i) Assume that $m_{-k+2,k}$ is odd. We have $F_{m,m'}^{-k} \in q^{-A_k}(1 + qA_0)$ and

$$\ell' = \max\{A_j (j > k), A_k + 1, A_j + 2 (j < k)\}.$$

If $m' = \tilde{F}_{-k}(m)$, then $\ell = A_k$ and (a) holds. Assume that $m' \neq \tilde{F}_{-k}(m)$. We have $A_k \leq \ell, \ell' - 1$ and hence $\operatorname{ord}(F_{m,m'}^{-k}) = -A_k \geq -(\ell + \ell' - 1)/2$.

If $\operatorname{ord}(F_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2$, then $A_k = \ell = \ell' - 1$, and we have $m' = \tilde{F}_{-k}(m)$, which is a contradiction.

(ii) Assume that $m_{-k+2,k}$ is even. Then $m' \neq \tilde{F}_{-k}(m)$, $F_{m,m'}^{-k} \in q^{-A_k - 1}(1 + qA_0)$ and

$$\ell' = \max\{A_j (j > k), A_k + 3, A_j + 2 (j < k)\}.$$

We have $A_k \leq \ell, \ell' - 3$ and hence $\operatorname{ord}(F_{m,m'}^{-k}) = -A_k - 1 \geq -(\ell + \ell' - 1)/2$. Hence (b) holds. Let us show (c). Assume $m' \neq \tilde{F}_{-k}(m)$, and
ord($F_{m}^{-k}$) = $- (t^2  + t' - 1)/2$. Then we have $A_k = t = t' - 3$. Hence $n_f \leq k$ and we have either $\tilde{F}_{-k}(m) = m - \delta_{i \neq k} \langle i, k - 2 \rangle + \langle i, k \rangle$ or $\tilde{F}_{-k}(m) = m - \delta_{k \neq 1} \langle -k + 2, k - 2 \rangle + \langle -k + 2, k \rangle$. Hence we have $\xi(\tilde{F}_{-k}(m)) = \pm m_{-k,k} > -m_{-k,k} - 1 = \xi(m')$. Hence we obtain (c) (1).

(1) Assume $\tilde{F}_{-k}(m) = m - \delta_{\neq k} \langle i, k - 2 \rangle + \langle i, k \rangle$ with $-k + 2 < i \leq k$. Then $k \neq 1$ and $\tilde{E}_{-k}(\tilde{F}_{-k}(m)) = \tilde{F}_{-k}(m) - \langle i, k \rangle + \delta_{i \neq k} \langle i, k - 2 \rangle$. Hence $n_e(\tilde{F}_{-k}(m)) = i - 2 < k$. Hence $\tilde{F}_{-k}(m) \in B''$. Therefore we obtain (c) (2).

(2) Assume $\tilde{F}_{-k}(m) = m - \delta_{k \neq 1} \langle -k + 2, k - 2 \rangle + \langle -k + 2, k \rangle$. Then $m_{-k+2,k} (\tilde{F}_{-k}(m)) = m_{-k+2,k} + 1$ is odd. Hence $\tilde{F}_{-k}(m) \in B''$.

Case 3. $m' = m - \delta_{k \neq 1} \langle -k + 2, k - 2 \rangle + \langle -k + 2, k \rangle$. In this case, we have

$$F_{m,m'}^{-k} = [m_{-k+2,k} + 1] q^{\sum_{j > k} (m_{-k+2,j} - m_{-k,j}) + m_{-k+2,k} - 2 m_{-k,k}}$$

$\in q^{-A_k + \delta (m_{-k+2,k} \text{ is odd})} (1 + q^A_0)$. 

(i) If $m_{-k+2,k}$ is odd, then $m' \neq \tilde{F}_{-k}(m)$, $F_{m,m'}^{-k} \in q^{-A_k + 1} (1 + q^A_0)$, and

$$\ell' = \max \{ A_j \ (j > k), A_k - 1, A_j + 2 \ (j < k) \}.$$ 

We have $A_k \leq \ell, \ell' + 1$ and hence $\text{ord}(F_{m,m'}^{-k}) = -A_k + 1 \geq -(\ell + \ell' - 1)/2$. If $\text{ord}(F_{m,m'}^{-k}) = - (\ell + \ell' - 1)/2$, then $A_k = \ell - \ell' + 1$, and $n_f = k$.

Hence we obtain (c) (2), and $\tilde{F}_{-k}(m) = m - \langle -k + 2, k \rangle + \langle -k, k \rangle$. Hence $\xi(\tilde{F}_{-k}(m)) = m_{-k,k} + 1 > m_{-k,k} = \xi(m')$. Hence we obtain (c) (1).

(ii) If $m_{-k+2,k}$ is even, then $F_{m,m'}^{-k} \in q^{-A_k} (1 + q^A_0)$ and

$$\ell' = \max \{ A_j \ (j > k), A_k + 1, A_j + 2 \ (j < k) \}.$$ 

If $m' = \tilde{F}_{-k}(m)$, then $\ell = A_k$ and (a) is satisfied. Assume $m' \neq \tilde{F}_{-k}(m)$. We have $A_k \leq \ell, \ell' - 1$ and hence $\text{ord}(F_{m,m'}^{-k}) = -A_k \geq -(\ell + \ell' - 1)/2$. If $\text{ord}(F_{m,m'}^{-k}) = - (\ell + \ell' - 1)/2$, then $A_k = \ell - \ell' - 1$, and hence $m' = \tilde{F}_{-k}(m)$, which is a contradiction.

Case 4. $m' = m - \delta_{i \neq k} \langle i, k - 2 \rangle + \langle i, k \rangle$ for $-k + 2 < i \leq k$. We have

$$F_{m,m'}^{-k} = [m_{i,k} + 1] \times q^{\sum_{j > k} (m_{-k+2,j} - m_{-k,j}) + 2 m_{-k+2,k} - 2 m_{-k,k} + \sum_{-k+2 < j} (m_{j,k} - m_{k,j})}$$

$\in q^{-A_k} (1 + q^A_0)$,
and
\[ \ell' = \max\{A_j (j \geq k), A_j (j < i - 2), A_{i-2} + 1, A_j + 2 (i - 2 < j \leq k - 2)\}. \]

If \( m' = \tilde{F}_{-k}(m) \), then \( \ell = A_{i-2} \) and (a) holds. Assume \( m' \neq \tilde{F}_{-k}(m) \). Since \( A_{i-2} \leq \ell, \ell' - 1 \), we have \( \text{ord}(E_{m, m'}^{-k}) = -A_{i-2} \geq -(\ell + \ell' - 1)/2 \). Hence we obtain (b). If \( \text{ord}(E_{m, m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then we have \( A_{i-2} = \ell = \ell' - 1 \). Hence \( m' = \tilde{F}_{-k}(m) \), which is a contradiction.

\[ \square \]

**Proposition 4.13.** Suppose \( k > 0 \). The conditions (4.10), (4.12), and (4.15) hold, namely, we have

(a) if \( m' = \tilde{E}_{-k}(m) \), then \( E_{m, m'}^{-k} \in q^{1-\ell}(1 + qA_0) \),

(b) if \( m' \neq \tilde{E}_{-k}(m) \), then \( \text{ord}(E_{m, m'}^{-k}) \geq 1 - \ell + e(\ell - 1 - \ell') = -(\ell + \ell' - 1)/2 \),

(c) if \( m' \neq \tilde{E}_{-k}(m) \), \( \ell \leq \ell' + 1 \) and \( \text{ord}(E_{m, m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then \( b \not\in B'' \).

**Proof.** The proof is similar to the one of the above proposition.

We shall write \( A_j \) for \( A_j^{-k}(m) \). Let \( n_c \) be as in Definition 4.5 (ii).

Note that \( E_{m, \tilde{E}_{-k}(m)}^{-k} \neq 0 \) if \( \tilde{E}_{-k}(m) \neq 0 \). If \( E_{m, m'}^{-k} \neq 0 \), we have the following five cases.

**Case 1.** \( m' = m - \langle -k, n \rangle + \langle -k + 2, n \rangle \) for \( n > k \).

In this case, we have

\[ E_{m, m'}^{-k} = (1 - q^2)[m_{-k+2, n} + 1]q^{1+\sum\gamma_{m \geq 2, j = m_{-k, j}} - m_{-k, j}} \in q^{1-A_n}(1 + qA_0) \]

and

\[ \ell = \max\{A_j (j \geq -k + 2)\}, \]

\[ \ell' = \max\{A_j (j > n), A_n - 1, A_{j < n} \}. \]

If \( m' = \tilde{E}_{-k}(m) \), then \( \ell = A_n \) and we obtain (a). Assume \( m' \neq \tilde{E}_{-k}(m) \). Since \( A_n \leq \ell, \ell' + 1 \), we have \( \text{ord}(E_{m, m'}^{-k}) = 1 - A_n \geq -(\ell + \ell' - 1)/2 \). Hence we obtain (b). If \( \text{ord}(E_{m, m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then we have \( A_n = \ell = \ell' + 1 \).

Since \( A_j \leq \ell' = A_n - 1 \) for \( j > n \), we have \( n_c = n \) and \( m' = \tilde{E}_{-k}(m) \), which is a contradiction.

**Case 2.** \( m' = m - \langle -k, k \rangle + \langle -k + 2, k \rangle \).

In this case we have

\[
E_{m, m'}^{-k} = (1 - q^2)[m_{-k+2, k} + 1]q^{1+\sum\gamma_{m \geq 2, j = m_{-k, j}} - m_{-k, j} + m_{-k, 2} + 2m_{-k, 2} - 2m_{-k, k}} \in q^{1-A_k + \delta_{m_{-k+2, k} \text{ is odd}}}(1 + qA_0).
\]
(i) Assume that \( m_{-k+2,k} \) is odd. Then \( m' \neq \tilde{E}_{-k}(m) \), \( E_{m,m'}^{-k} \in q^{-A_k}(1+qA_0) \) and
\[
\ell' = \max\{A_j \mid (j > k), A_k - 3, A_j - 2 \mid (j < k)\}.
\]
We have \( A_k \leq \ell, \ell' + 3 \) and hence \( \text{ord}(E_{m,m'}^{-k}) = 2 - A_k \geq -(\ell + \ell' - 1)/2 \).
Hence (b) holds. If \( \text{ord}(E_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then \( A_k = \ell = \ell' + 3 \).
Hence \( \ell > \ell' + 1 \) and (c) holds.

(ii) Assume that \( m_{-k+2,k} \) is even. Then \( E_{m,m'}^{-k} \in q^{1-A_k}(1+qA_0) \) and
\[
\ell' = \max\{A_j \mid (j > k), A_k - 1, A_j - 2 \mid (j < k)\}.
\]
If \( m' = \tilde{E}_{-k}(m) \), then \( \ell = A_k \), and we obtain (a). Assume \( m' \neq \tilde{E}_{-k}(m) \).
We have \( A_k \leq \ell, \ell' + 1 \) and hence \( \text{ord}(E_{m,m'}^{-k}) = 1 - A_k \geq -(\ell + \ell' - 1)/2 \).
If \( \text{ord}(E_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then \( A_k = \ell = \ell' + 1 \) and \( n_e = k \). Hence \( m' = \tilde{E}_{-k}(m) \), which is a contradiction.

Case 3. \( m' = m - (k + 2) + \delta_{k \neq 1}(-k + 2, k - 2) \). If \( k \neq 1 \), we have
\[
E_{m,m'}^{-k} = (1 - q^2)[2(m_{-k+2,k-2} + 1)]q^{1+\sum_{j > k}(m_{-k+2,j} - m_{-k,j})+2m_{-k+2,k-2} - 2m_{-k,k}}
\in q^{-A_k + \delta(m_{-k+2,k} \text{ odd})} (1 + qA_0).
\]
If \( k = 1 \), we have
\[
E_{m,m'}^{-k} = q^{\sum_{j > k}(m_{-k+2,j} - m_{-k,j}) - 2m_{-k,k}} = q^{-A_k + \delta(m_{-k+2,k} \text{ odd})}.
\]
In the both cases, we have
\[
E_{m,m'}^{-k} \in q^{-A_k + \delta(m_{-k+2,k} \text{ odd})} (1 + qA_0).
\]

(i) If \( m_{-k+2,k} \) is odd, then \( E_{m,m'}^{-k} \in q^{1-A_k}(1+qA_0) \) and
\[
\ell' = \max\{A_j \mid (j > k), A_k - 1, A_j - 2 \mid (j < k)\}.
\]
If \( m' = \tilde{E}_{-k}(m) \), then \( \ell = A_k \) and (a) is satisfied. We have \( A_k \leq \ell, \ell' + 1 \) and hence \( \text{ord}(E_{m,m'}^{-k}) = 1 - A_k \geq -(\ell + \ell' - 1)/2 \). Assume \( m' \neq \tilde{E}_{-k}(m) \).
If \( \text{ord}(E_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then \( A_k = \ell = \ell' + 1 \) and \( n_e = k \). Hence \( m' = \tilde{E}_{-k}(m) \), which is a contradiction.

(ii) If \( m_{-k+2,k} \) is even, then \( m' \neq \tilde{E}_{-k}(m) \), \( E_{m,m'}^{-k} \in q^{-A_k}(1+qA_0) \), and
\[
\ell' = \max\{A_j \mid (j > k), A_k + 1, A_j - 2 \mid (j < k)\}.
\]
We have \( A_k \leq \ell, \ell' - 1 \) and hence \( \text{ord}(E_{m,m'}^{-k}) = -A_k \geq -(\ell + \ell' - 1)/2 \).
Hence we obtain (b). If \( \text{ord}(E_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then \( A_k = \ell = \ell' - 1 \).
Hence \( n_e(m) \geq k \) and \( m_{-k+2,k} \) is even. Hence \( m \notin B'' \).
\textbf{Case 4.} \( m' = m - (i, k) + (i, k - 2) \text{ for } -k + 2 < i \leq k - 2 \).

We have
\[
E_{m,m'}^{k} = (1 - q^2)[m_i,k-2 + 1] \\
\times q^{1 + \sum_{j \geq k}(m_{-k+2,j} - m_{-k,j}) + 2m_{-k,k-2} - 2m_{-k,k} + \sum_{-k+2 \leq j < k}(m_{j,k-2} - m_{j,k})} \\
\in q^{1-A_{i-2}}(1 + qA_0),
\]
and
\[\ell' = \max\{A_j \mid j \geq k\}, A_j \mid j < i - 2\}, A_{i-2} - 1, A_j - 2 \mid i \leq j \leq k - 2\}].\]

If \( m' = \tilde{E}_{-k}(m) \), then \( \ell = A_{i-2} \) and (a) holds. Assume \( m' \neq \tilde{E}_{-k}(m) \). Since \( A_{i-2} \leq \ell, \ell' + 1 \), we have \( \text{ord}(E_{m,m'}^{k}) = 1 - A_{i-2} \geq -(\ell + \ell' - 1)/2 \). Hence we obtain (b). If \( \text{ord}(E_{m,m'}^{k}) = -(\ell + \ell' - 1)/2 \), then we have \( A_{i-2} = \ell = \ell' + 1 \).

Hence \( m' = \tilde{E}_{-k}(m) \), which is a contradiction.

\textbf{Case 5.} \( k \neq 1 \) and \( m' = m - (k) \). In this case,
\[
E_{m,m'}^{k} = q^{\sum_{i \geq k}(m_{-k+2,j} - m_{-k,j}) - 2m_{-k,k} + 1 - m_{k,k} + 2m_{-k,k-2} + \sum_{-k+2 \leq j \leq k-2}(m_{j,k-2} - m_{j,k})} \\
\in q^{1-A_{k-2}}(1 + qA_0),
\]
and
\[\ell' = \max\{A_j \mid j \neq k - 2, A_{k-2} - 1\}].\]

If \( m' = \tilde{E}_{-k}(m) \), then \( \ell = A_{k-2} \) and (a) holds. Assume \( m' \neq \tilde{E}_{-k}(m) \). Since \( A_{k-2} \leq \ell, \ell' + 1 \), we have \( \text{ord}(E_{m,m'}^{k}) = 1 - A_{k-2} \geq -(\ell + \ell' - 1)/2 \). Hence we obtain (b). If \( \text{ord}(E_{m,m'}^{k}) = -(\ell + \ell' - 1)/2 \), then we have \( A_{k-2} = \ell = \ell' + 1 \).

Hence \( m' = \tilde{E}_{-k}(m) \), which is a contradiction.

\textbf{Proposition 4.14.} Let \( k \in I_{>0} \). Then the conditions in Corollary 4.4 holds for \( \tilde{E}_k, \tilde{F}_k \) and \( \epsilon_k \), with the same functions \( c, e, f \).

Since the proof is similar to and simpler than the one of the preceding two propositions, we omit the proof.

As a corollary we have the following result. We write \( \phi \) for the generator \( \phi_0 \) of \( V_0(0) \) for short.

\textbf{Theorem 4.15.}

(i) \textbf{The morphism}
\[
\tilde{V}_0(0) := U_q^-(g)/\sum_{k \in \ell} U_q^-(g)(f_k - f_{-k}) \rightarrow V_0(0)
\]
is an isomorphism.
(ii) $\{P_\theta(m)\phi\}_{m \in M_\theta}$ is a basis of the $K$-vector space $V_\theta(0)$.

(iii) Set

$$L_\theta(0) := \sum_{i_0, i_1, \ldots, i_{\ell} \in I} A_{i_0} \tilde{F}_{i_1} \cdots \tilde{F}_{i_{\ell}} \phi \subset V_\theta(0),$$

$$B_\theta(0) = \left\{ \tilde{F}_{i_1} \cdots \tilde{F}_{i_{\ell}} \phi \mod qL_\theta(0) \mid \ell \geq 0, i_1, \ldots, i_{\ell} \in I \right\}.$$

Then, $B_\theta(0)$ is a basis of $L_\theta(0)/qL_\theta(0)$ and $(L_\theta(0), B_\theta(0))$ is a crystal basis of $V_\theta(0)$, and the crystal structure coincides with the one of $M_\theta$.

(iv) More precisely, we have

(a) $L_\theta(0) = \bigoplus_{m \in M_\theta} A_{0} P_\theta(m)\phi$,

(b) $B_\theta(0) = \{ P_\theta(m)\phi \mod qL_\theta(0) \mid m \in M_\theta \}$,

(c) for any $k \in I$ and $m \in M_\theta$, we have

1. $\tilde{F}_k P_\theta(m)\phi \equiv P_\theta(\tilde{F}_k(m))\phi \mod qL_\theta(0),$

2. $\tilde{E}_k P_\theta(m)\phi \equiv P_\theta(\tilde{E}_k(m))\phi \mod qL_\theta(0)$,

where we understand $P_\theta(0) = 0$,

3. $\tilde{E}_k^n P_\theta(m)\phi \in qL_\theta(0)$ if and only if $n > \varepsilon_k(m)$.

**Proof.** Let us recall that $P_\theta(m)\phi \in V_\theta(0)$ is the image of $\tilde{P}_\theta(m) \in \tilde{V}_\theta(0)$. By Theorem 3.2, $\{\tilde{P}_\theta(m)\}_{m \in M_\theta}$ generates $\tilde{V}_\theta(0)$. Let us set $\tilde{L} = \sum_{m \in M_\theta} A_{0} \tilde{P}_\theta(m) \subset \tilde{V}_\theta(0)$. Then Theorem 4.1 implies that

$$\tilde{F}_k \tilde{P}_\theta(m) \equiv \tilde{P}_\theta(\tilde{F}_k(m)) \mod q\tilde{L} \text{ and } \tilde{E}_k \tilde{P}_\theta(m) \equiv \tilde{P}_\theta(\tilde{E}_k(m)) \mod q\tilde{L}.$$

Hence the similar results hold for $L_\theta := \sum_{m \in M_\theta} A_{0} P_\theta(m)\phi \subset V_\theta(0)$ and $P_\theta(m)\phi$.

Let us show that

(A) $\{P_\theta(m)\phi \mod qL_\theta(0)\}_{m \in M_\theta}$ is linearly independent in $L_\theta/qL_\theta$,

by the induction of the $\theta$-weight (see Remark 2.12). Assume that we have a linear relation $\sum_{m \in S} a_m P_\theta(m)\phi \equiv 0 \mod qL_\theta$ for a finite subset $S$ and $a_m \in \mathbb{Q} \setminus \{0\}$. We may assume that all $m$ in $S$ have the same $\theta$-weight. Take $m_0 \in S$. If $m_0$ is the empty multisegment $\emptyset$, then $S = \{\emptyset\}$ and $P_\theta(m_0)\phi = \phi$ is non-zero, which is a contradiction. Otherwise, there exists $k$ such that $\varepsilon_k(m_0) > 0$ by Lemma 4.11. Applying $\tilde{E}_k$, we have $\sum_{m \in S} a_m \tilde{E}_k P_\theta(m)\phi \equiv \sum_{m \in S, \tilde{E}_k(m) \neq 0} a_m P_\theta(\tilde{E}_k(m))\phi \equiv 0 \mod qL_\theta$. Since $\tilde{E}_k(m)$ ($\tilde{E}_k(m) \neq 0$) are mutually distinct, we have $a_{m_0} = 0$ by the induction hypothesis. It is a contradiction.
Thus we have proved (A). Hence \( \{P_\theta(m)\phi\}_{m \in M_\theta} \) is a basis of \( V_\theta(0) \), which implies that \( \{\tilde{P}_\theta(m)\} \) is a basis of \( \tilde{V}_\theta(0) \). Thus we obtain (i) and (ii).

Let us show (iv) (a). Since \( \tilde{F}_i \cdots \tilde{F}_{i'} \phi \equiv P_\theta(\tilde{F}_i \cdots \tilde{F}_{i'} \theta) \mod qL_0 \), we have \( L_0(0) \subset L_0 \subset L_0(0) + qL_0 \). Hence Nakayama’s lemma implies \( L_0 = L_0(0) \). The other statements are now obvious.

\[ \square \]

§5. Global Basis of \( V_\theta(0) \)

§5.1. Integral form of \( V_\theta(0) \)

In this section, we shall prove that \( V_\theta(0) \) has a lower global basis. In order to see this, we shall first prove that \( \{P_\theta(m)\phi\}_{m \in M_\theta} \) is a basis of the \( A \)-module \( V_\theta(0) \). Recall that \( A = \mathbb{Q}[q, q^{-1}] \), and \( V_\theta(0) \) is a \( \mathfrak{gl}_\infty \) \( A \)-module.

**Lemma 5.1.** \( V_\theta(0) = \bigoplus_{m \in M_\theta} A P_\theta(m) \).

**Proof.** It is clear that \( \bigoplus_{m \in M_\theta} A P_\theta(m) \) is stable by the actions of \( F^{(n)}_k \) by Proposition 3.20. Hence we obtain \( V_\theta(0) \subset \bigoplus_{m \in M_\theta} A P_\theta(m) \).

We shall prove \( P_\theta(m) \phi \in U_q^- (\mathfrak{gl}_\infty) A \phi \). It is well-known that \( \langle i, j \rangle^{(n)} \) is contained in \( U_q^- (\mathfrak{gl}_\infty) A \), which is also seen by Proposition 3.20 (3). We divide \( m \) as \( m = m_1 + m_2 \), where \( m_1 = \sum_{-j < i < j} m_{ij} \langle i, j \rangle \) and \( m_2 = \sum_{k \geq 0} m_k \langle -k, k \rangle \). Then \( P_\theta(m) = P(m_1)P_\theta(m_2) \) and \( P(m_1) \in U_q^- (\mathfrak{gl}_\infty) A \). Hence we may assume from the beginning that \( m = \sum_{0 < k \leq a} m_k \langle -k, k \rangle \). We shall show that \( P_\theta(m) \phi \in V_\theta(0) \) by the induction on \( a \).

Assume \( a > 1 \). Set \( m' = \sum_{0 < k \leq a - 4} m_k \langle -k, k \rangle \) and \( v = P_\theta(m') \phi \). Then \( \langle -a + 2, a - 2 \rangle^{[m]} v \in V_\theta(0) \) for any \( m \) by the induction hypothesis.

We shall show that \( \langle -a, a \rangle^{[n]} \langle -a + 2, a - 2 \rangle^{[m]} v \) is contained in \( V_\theta(0) \) by the induction on \( n \). Since \( P_\theta(m') \) commutes with \( \langle a \rangle, \langle -a \rangle, \langle -a + 2, a - 2 \rangle, \langle -a + 2, a \rangle \) and \( \langle -a, a \rangle \), Proposition 3.20 (2) implies

\[
\langle -a \rangle^{(2n)} \langle -a + 2, a - 2 \rangle^{[n+m]} v = \sum_{i+j+2t=2n, j+t=a} q^{2(n+m)i+j(1-j)/2-i(t+u)} \langle a \rangle^{(i)} \langle -a + 2, a \rangle^{(j)} \langle -a, a \rangle^{[t]} \langle -a + 2, -2 \rangle^{[n+m-u]} v,
\]

which is contained in \( V_\theta(0) \). Since we have

\[
\langle a \rangle^{(i)} \langle -a + 2, a \rangle^{(j)} \langle -a, a \rangle^{[t]} \langle -a + 2, a - 2 \rangle^{[n+m-u]} v \in V_\theta(0) \]

if \( (i, j, t, u) \neq (0, 0, n, n) \) by the induction hypothesis on \( n \), \( \langle -a, a \rangle^{[n]} \langle -a + 2, a - 2 \rangle^{[m]} v \) is contained in \( V_\theta(0) \).
If \( a = 1 \), we similarly prove \( P_\theta(m) \in V_\theta(0)_A \) using Proposition 3.20 (1) instead of (2).

\[ \square \]

§ 5.2. Conjugate of the PBW basis

We will prove that the bar involution is upper triangular with respect to the PBW basis \( \{ P_\theta(m) \}_{m \in M_\theta} \).

First we shall prove Theorem 3.10 (4).

For \( a, b \in M \) such that \( a \leq b \), we denote by \( M_{[a,b]} \) (resp. \( M_{\leq b} \)) the set of \( m \in M \) of the form \( m = \sum_{i \leq j \leq b} m_{i,j} (i, j) \) (resp. \( m = \sum_{i \leq j \leq b} m_{i,j} (i, j) \)). Similarly we define \( (M_\theta)_{\leq b} \). For a multisegment \( m \in M_{\leq b} \), we divide \( m = m_0 + m_{<b} \), where \( m_0 = \sum_{i \leq b} m_{i,j} (i, b) \) and \( m_{<b} = \sum_{i \leq j < b} m_{i,j} (i, j) \).

**Lemma 5.2.** For \( n \geq 0 \) and \( a, b \in I \) such that \( a \leq b \), we have

\[
\overline{\langle a, b \rangle}^{(n)} \in \langle a, b \rangle^{(n)} + \sum_{m < cry(a, b)} K P(m).
\]

**Proof.** We shall first show

\[
\overline{\langle a, b \rangle} = \langle a \rangle (a + 2, b) - q^{-1} \langle a, b \rangle \langle a \rangle + \sum_{a + 2 < k \leq b} \langle k, b \rangle U_q^-(g)
\]

by the induction on \( b - a \). If \( a = b \), it is trivial. If \( a < b \), we have

\[
\overline{\langle a, b \rangle} = \langle a \rangle (a + 2, b) - q^{-1} \langle a, b \rangle \langle a \rangle + \sum_{a + 2 < k \leq b} \langle k, b \rangle U_q^-(g) \langle a \rangle
\]

\[
\subset \langle a, b \rangle + (q - q^{-1}) \langle a + 2, b \rangle \langle a \rangle + \sum_{a + 2 < k \leq b} \langle k, b \rangle \langle a \rangle U_q^-(g) + \langle k, b \rangle U_q^-(g).
\]

Hence we obtain (5.1). We shall show the lemma by the induction on \( n \). We may assume \( n > 0 \) and

\[
\overline{\langle a, b \rangle}^{n-1} \in \langle a, b \rangle^{n-1} + \sum_{m < cry(n-1)(a,b)} K P(m).
\]

Hence we have

\[
\overline{\langle a, b \rangle} = \overline{\langle a, b \rangle}^{(n-1)} \in \langle a, b \rangle^{(n)} + \sum_{a < k \leq b} \langle k, b \rangle U_q^-(g) + \sum_{m < cry(n-1)(a,b)} K \langle a, b \rangle P(m).
\]
For $a < k \leq b$ and $m \in M$ such that $\text{wt}(m) = \text{wt}(n(a, b)) - \text{wt}(k, b)$, we have $m \in M_{[a, b]}$ and $m = \sum_{a \leq i \leq b} m_{i, b}(i, b)$ with $\sum_{i} m_{i, b} = n - 1$. In particular, $m_{a, b} \leq n - 1$. Hence $(k, b)P(m) \in KP(m + (k, b))$ and $m + (k, b) \leq n(a, b)$.

If $m < (n - 1)(a, b)$, then $(a, b)P(m) \in KP((a, b) + m)$ and $(a, b) + m < n(a, b)$. □

**Proposition 5.3.** For $m \in M$,

$$\overline{P(m)} \in P(m) + \sum_{n \in m} KP(n).$$

**Proof.** Put $m = \sum_{a \leq i \leq b} m_{i, j}(i, j)$ and divide $m = m_{b} + m_{< b}$. We prove the claim by the induction on $b$ and the number of segments in $m_{b}$. Suppose $m_{b} = m(a, b) + m_{1}$ with $m = m_{a, b} + 0$, where $m_{1} = \sum_{a < i \leq b} m_{i, b}(i, b)$.

(i) Let us first show that

$$(5.2) \quad \overline{P(m_{b})} \in P(m_{b}) + \sum_{m' \leq m_{b}} KP(m').$$

We have $\overline{P(m_{b})} = \overline{P(m_{1})} \cdot \langle a, b \rangle^{(m_{b})}$. Since $\overline{P(m_{1})} \in P(m_{1}) + \sum_{m_{cry} \leq m_{1}} KP(m_{1})$ by the induction hypothesis, and $\langle a, b \rangle^{(m_{b})}$ is a sum of terms $m'_{cry} \leq m_{a, b}$.

If $m_{cry}' < m_{1}$ and $m_{cry}' \in M_{[a+2, b]}$, then $P((m_{1})_{cry})$ and $(a, b)^{(m_{b})}$ commute. Hence $P(m_{cry}')(a, b)^{(m_{b})} = P(m_{cry}') + m(a, b)$ and $m_{cry}' + m(a, b) \leq m_{b}$.

If $m_{cry}' \leq m_{1}$, then $m_{cry}' \in M_{[a+2, b]}$ and $m_{cry}' < m(a, b)$, then we can write $m_{b}' = j(a, b) + m_{cry}$ with $j < m$ and $m_{cry} \in M_{[a+2, b]}$. Hence we have

$$P(m_{cry}')(a, b)^{(m_{b})} \in KP((m_{1})_{cry})P(j(a, b))P((m_{1})_{cry})P(m_{2})P(m_{b}').$$

Since $(m_{1})_{cry}$, $m_{2} \in M_{[a+2, b]}$, we have $P((m_{1})_{cry})P(m_{2})P(m_{b}'_{cry}) \in \sum_{n \in M_{[a+2, b]}} KP(n)$. Hence we have $P(m_{cry}')(a, b)^{(m_{b})} \in \sum_{n \in M_{[a+2, b]}} KP((m_{1})_{cry})P(j(a, b) + n)$ and $(m_{1})_{cry} + j(a, b) + n < m_{b}$. Hence we obtain (5.2).
By the induction hypothesis, \[ \overline{P(m_{\leq b})} \in P(m_{<b}) + \sum_{m'' < m_{\leq b}} KP(m''). \]

Since \[ \overline{P(m)} = P(m_{\leq b}) P(m_{<b}), \]

(5.2) implies that

\[ \overline{P(m)} \in P(m) + \sum_{m' < m_{\leq b}, m'' \in M_{<b}} KP(m') P(m'') + \sum_{m'' < m_{\leq b}} KP(m_b) P(m''). \]

For \( m' < m_b \) and \( m'' \in M_{<b} \), we have

\[ P(m') P(m'') = P(m_b) P(m'_{<b}) P(m'') \in \sum_{n \in M_{<b}, n_b = m_b} KP(n) \subset \sum_{m' \in M_{<b}, m' < m} KP(n). \]

For \( m'' < m_{<b} \), we have \( P(m_b) P(m'') = P(m_b + m'') \) and \( m_b + m'' < m \). Thus we obtain the desired result.

**Proposition 5.4.** For \( m \in M_{\theta} \), we have

\[ \overline{P_{\theta}(m) \phi} \in P_{\theta}(m) + \sum_{m' \in M_{\theta}, m' < m} KP_{\theta}(m') \phi. \]

**Proof.** First note that

\[ (5.3) \quad P(m) \phi \in \sum_{n \in (M_{\theta})_{\leq b}} KP_{\theta}(n) \phi \quad \text{for any } b \in I_{>0} \text{ and } m \in M_{[-b,b]}, \]

by the weight consideration.

For \( m \in M_{\theta} \), \( P_{\theta}(m) \) and \( P(m) \) are equal up to a multiple of bar-invariant scalar. Thus we have

\[ \overline{P_{\theta}(m)} \in P_{\theta}(m) + \sum_{m' \in M, m' < m} KP(m') \]

by Proposition 5.3. Hence it is enough to show that

\[ (5.4) \quad P(m') \phi \in \sum_{n \in M_{\theta}, n < m_{\text{cry}}} KP_{\theta}(n) \phi \]

for \( m' \in M \) such that \( m' < m \) and \( \text{wt}(m') = \text{wt}(m) \). Put \( m = \sum_{i < j \leq b} m_{i,j}(i,j) \) and write \( m = m_b + m_{<b} \). We prove (5.4) by the induction on \( b \). By the assumption on \( m' \), we have \( m' \in M_{[-b,b]} \) and \( m'_{\text{cry}} \leq m_b \). Thus \( m'_{\text{cry}} \in M_{\theta} \). Hence \( KP(m') \phi = KP_{\theta}(m'_{\text{cry}}) P(m'_{<b}) \phi. \)
If \( m'_b = m_b \), then \( m'_{\leq b} = m_{\leq b} \), and the induction hypothesis implies
\[
P(m'_{\leq b}) \phi = \sum_{n \leq M_b, n < m_{\leq b}} K P_b(n) \phi.
\]
Since \( P_b(m'_b) P_b(n) = P_b(m'_b + n) \) and \( m'_b + n < m \), we obtain (5.4).

If \( m'_b < m_b \), write \( m' = \sum_{b' \leq i \leq j < b} m'_{i,j} \langle i, j \rangle \). Set \( s = m_{-b,b} - m'_{-b,b} \geq 0 \).

Since \( wt(m') = wt(m) \), we have \( \sum_{j < b} m'_{-b,j} = s \). If \( s = 0 \), then \( m'_{\leq b} \in \mathcal{M}_{[-b, b]} \), and \( P(m'_{\leq b}) \phi = \sum_{n \in (\mathcal{M}_b)_{<b}} K P_b(n) \phi \) by (5.3). Then (5.4) follows from \( m'_b + n < m \).

Assume \( s > 0 \). Since \( m'_{\leq b} \in \mathcal{M}_{[-b, b]} \), we have \( P(m'_{\leq b}) \phi \in \sum_{n \in (\mathcal{M}_b)_{<b}} K P_b(n) \phi \) by (5.3). We may assume \((1 + \theta) wt(m'_{\leq b}) = (1 + \theta) wt(n) \) (see Remark 2.12).

Hence, we have \( s = 2m_{-b,b}(n) + \sum_{b' < i \leq b} m_{i,b}(n) \). In particular, \( m_{-b,b}(n) \leq s/2 \).

We have \( m'_b + n \in \mathcal{M}_b \) and \( P_b(m'_b) P_b(n) \phi = P_b(m'_b + n) \phi \). Since \( m_{-b,b}(m'_b + n) \leq (m_{-b, b} - s) + s/2 < m_{-b,b} \), we have \( m'_b + n < m \). Hence we obtain (5.4).

\[\Box\]

§5.3. Existence of a global basis

As a consequence of the preceding subsections, we obtain the following theorem.

**Theorem 5.5.**

(i) \((L_0(0), L_0(0)^-, V_0(0)_A)\) is balanced.

(ii) For any \( m \in \mathcal{M}_b \), there exists a unique \( \bar{G}^{low}_b(m) \in L_0(0) \cap V_0(0)_A \) such that \( \bar{G}^{low}_b(m) = G^{low}_b(m) \) and \( \bar{G}^{low}_b(m) \equiv P_0(m) \phi \mod q L_0(0) \).

(iii) \( G^{low}_b(m) \in P_0(m) \phi + \sum_{m < m} q \mathbb{Q}[q] P_0(n) \phi \) for any \( m \in \mathcal{M}_b \).

(iv) \( \{G^{low}_b(m)\}_{m \in \mathcal{M}_b} \) is a basis of the \( A \)-module \( V_0(0)_A \), the \( A_0 \)-module \( L_0(0) \) and the \( K \)-vector space \( V_0(0) \).

**Proof.** We have already seen that
\[
\bar{P}_0(m) \phi = \sum_{m' \leq m} \bar{P}_0(m') \phi \text{ for } \bar{e}_{m,m'} \in A \text{ with } e_{m,m} = 1.
\]
Let us denote by \( C \) the matrix \( (e_{m,m'})_{m,m' \in \mathcal{M}_b} \). Then \( \overline{CC} = id \) and it is well-known that there is a matrix \( A = (a_{m,m'})_{m,m' \in \mathcal{M}_b} \) such that \( \mathbb{F}C = A \), \( a_{m,m'} = 0 \) unless \( m' \leq m, a_{m,m} = 1 \) and \( a_{m,m'} \in q \mathbb{Q}[q] \) for \( m' < m \), \( m \). Set \( \bar{G}^{low}_b(m) = \sum_{m' \leq m} a_{m,m'} P_0(m') \phi \). Then we have \( \bar{G}^{low}_b(m) = \bar{G}^{low}_b(m) \) and \( \bar{G}^{low}_b(m) \equiv P_0(m) \phi \mod q L_0(0) \). Since \( \bar{G}^{low}_b(m) \) is a basis of \( V_0(0)_A \), we obtain the desired results. \( \Box \)

(i) In Conjecture 3.8, $\lambda = \Lambda_{p_0} + \Lambda_{p_0^{-1}}$ should be read as $\lambda = \sum_{a \in A} \Lambda_a$, where $A = I \cap \{p_0, p_0^{-1}, -p_0, -p_0^{-1}\}$. We thank S. Ariki who informed us that the original conjecture is false.

(ii) In the two diagrams of $B_\theta(\lambda)$ at the end of §2, $\lambda$ should be 0.

(iii) Throughout the paper, $A_{\ell}^{(1)}$ should be read as $A_{\ell-1}^{(1)}$.

References


