Simultaneous Linearization of Holomorphic Maps with Hyperbolic and Parabolic Fixed Points

By

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Abstract

We study local holomorphic mappings of one complex variable with parabolic fixed points as a limit of a families of mappings with attracting fixed points. We show that the Fatou coordinate for a parabolic fixed point can be obtained as a limit of some linear function of the solutions to Schröder equation for perturbed mappings with attracting fixed points.

§1. Introduction

Let \( g(w) \) be a holomorphic function of one variable of the form

\[
g(w) = \lambda w + \sum_{\nu=2}^{\infty} b_{\nu} w^{\nu}
\]

defined in a neighborhood of the origin 0. If \( 0 < |\lambda| < 1 \), then there are a neighborhood \( V \) of 0 such that \( g(V) \subset V \) and an injective holomorphic function \( \rho(w) \) on \( V \) satisfying the Schröder equation

\[
\rho(g(w)) = \lambda \rho(w).
\]

If \( \lambda = 1 \) and \( b_2 \neq 0 \), then there are a domain \( V \) whose boundary contains 0 and an injective function \( \varphi(w) \) (Fatou coordinate) satisfying the Abel equation

\[
\varphi(g(w)) = \varphi(w) + 1,
\]
which is unique up to an additive constant. (See Schröder [9], Koenigs [5], Leau [6], Fatou [3] and Milnor [8], for these classical results.)

In this paper, we consider families of functions $g_\lambda(w)$ which have $\lambda$ as a parameter and show that, when $\lambda$ tends non-tangentially to 1 from inside of the unit disk, some linear function of $\rho_\lambda(w)$ converges to $\varphi(w)$.

To do this it is convenient to consider the case where the fixed point is $\infty$ on the Riemann sphere. By scaling the coordinate, we consider a family of holomorphic maps of the form

$$f_\tau(z) = \tau z + 1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

defined in a neighborhood of $\infty$. Here the parameter $\tau = 1/\lambda$ varies in a neighborhood of $\tau = 1$. For $|\tau| > 1$, let $\chi_\tau(z)$ denote the unique solution of the equation

$$\chi_\tau(f(z)) = \tau \chi_\tau(z)$$

with $\chi_\tau(\infty) = \infty$ and normalized so that $\chi(z) = z + O(1)$ in a neighborhood of $\infty$. We will show that, when $\tau$ tends to 1 non-tangentially within the domain $|\tau| > 1$, the sequence

$$\chi_\tau(z) - \frac{1}{\tau - 1} - a_1 \log(\tau - 1)$$

converges to a solution to the Abel equation

$$\varphi(f_1(z)) = \varphi(z) + 1.$$

Precise statement and different formulations of the results are given in Theorems 3.3–3.6.

An alternative proof and a generalization is given recently by T. Kawahira [4]. As a related result, we note that T. C. McMullen showed the existence of quasiconformal maps giving conjugacies between $f_\tau$ and linear maps (see [7], Theorem 8.2).

§2. Preliminaries

§2.1. A family of linear maps

We begin with studying the family $\{\ell_\tau\}$ of linear maps

$$\ell_\tau(z) = \tau z + 1$$

on the Riemann sphere $\mathbb{C}$ depending on the complex parameter $\tau$. When $|\tau| > 1$, the map $\ell_\tau$ has $\infty$ as an attracting fixed point and all points except for
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1/(1 - \tau) converge locally uniformly to \infty by the iterates of \ell_\tau. When \tau = 1, then \infty is a parabolic fixed point and all points in \hat{C} converges to \infty, though the convergence is not uniform in the neighborhood of \infty.

We will investigate the uniformity, with respect to the parameter \tau, of the convergence of the iterates \ell^n_\tau, when \tau tends to 1 non-tangentially from outside of the unit disk. So we will restrict the parameter \tau in the closed sector

\[ T_\alpha = \{ \tau \in \mathbb{C} \mid \text{Re} \tau - 1 \geq |\tau - 1| \cos \alpha \}, \]

where \alpha is a real number with 0 < \alpha < \pi/2, fixed throughout this paper.

To measure the rate of convergence to \infty, we introduce the function \( N: \hat{C} \times T_\alpha - \{(\infty, 1)\} \to \mathbb{R} \cup \{\infty\} \) as follows.

\[ N_\tau(z) = \left| z - \frac{1}{1 - \tau} \right| - \left| \frac{1}{1 - \tau} \right| \quad \text{for } (z, \tau) \in \hat{C} \times (T_\alpha - \{1\}); \]

\[ N_1(z) = \sup_{|\theta| \leq \alpha} |\text{Re}(e^{i\theta}z)| \quad \text{for } z \in \mathbb{C}. \]

We will not define \( N_1(\infty) \). It is easy to see that the inequality

\[ |N_\tau(z_1) - N_\tau(z_2)| \leq |z_1 - z_2| \quad z_1, z_2 \in \mathbb{C}, \tau \in T_\alpha \]

holds. In particular we have

\[ N_\tau(z) \leq |z|, \quad z \in \mathbb{C}, \tau \in T_\alpha. \]

**Lemma 2.1.** \( N_\tau(z) \) is upper semi-continuous as a function of two variables \((z, \tau) \in \hat{C} \times T_\alpha - \{z = \infty\} \) and

\[ N_1(z) = \limsup_{T_\alpha \ni \tau \to 1} N_\tau(z). \]

**Proof.** For \( r > 0 \), we let \( \hat{N}_{(r, \theta)}(z) = N_1 + r e^{i\theta}(z) \). Then

\[ \hat{N}_{(r, \theta)}(z) = \left| z + \frac{1}{r e^{i\theta}} \right| - \frac{1}{r} = \frac{1}{r} \left( (1 + 2r \text{Re}(e^{i\theta}z) + r^2 |z|^2)^{1/2} - 1 \right). \]

This can be extended to a continuous function on \( \hat{C} \times \{r \geq 0\} \times \mathbb{R} \), by defining \( \hat{N}_{(0, \theta)}(z) = \text{Re}(e^{i\theta}z) \). Hence

\[ \limsup_{T_\alpha \ni \tau \to 1} N_\tau(z) = \sup_{\{\theta\leq \alpha\}} \hat{N}_{(0, \theta)}(z) = \sup_{\{\theta\leq \alpha\}} \text{Re}(e^{i\theta}z) = N_1(z). \]

This shows the assertion.

To have a uniform estimate of the rate of convergence of the iterats of \( \ell_\tau \), let us first show the following:
Lemma 2.2. For \((z, \tau) \in \mathbb{C} \times T_\alpha = \{ (\infty, 1) \}\), we have

\[ N_\tau(\ell_\tau(z)) \geq |\tau|N_\tau(z) + \cos \alpha. \]

Proof. First, if \(\tau \in T - \{1\}\), then (1) is rewritten as

\[ \ell_\tau(z) = \frac{1}{1 - \tau} = \tau \left( z - \frac{1}{1 - \tau} \right). \]

Hence

\[
N_\tau(\ell_\tau(z)) = \left| \ell_\tau(z) - \frac{1}{1 - \tau} \right| - \left| \frac{1}{1 - \tau} \right|
\]
\[
= |\tau| \left| z - \frac{1}{1 - \tau} \right| - \left| \frac{1}{1 - \tau} \right|
\]
\[
= |\tau|N_\tau(z) + \frac{|\tau| - 1}{|1 - \tau|}
\]
\[
\geq |\tau|N_\tau(z) + \cos \alpha.
\]

Here we have used the fact that

\[
\frac{|\tau| - 1}{|\tau - 1|} \geq \frac{\Re(\tau) - 1}{|\tau - 1|} \geq \cos \alpha.
\]

If \(\tau = 1\), then \(\ell_1(z) = z + 1\), and hence

\[ \Re(e^{i\theta} \ell_1(z)) = \Re(e^{i\theta} z) + \cos \theta \geq \Re(e^{i\theta} z) + \cos \alpha. \]

Therefore

\[ N_1(\ell_1(z)) \geq N_1(z) + \cos \alpha \]

and the lemma is proved. \(\square\)

Let \(R\) be a real number and define

\[
\mathcal{V}_\alpha(R) = \{ (z, \tau) \in \mathbb{C} \times T_\alpha - \{ (\infty, 1) \} \mid N_\tau(z) > R \}.
\]

We note that \(\mathcal{V}_\alpha(R)\) is not open. Slices of \(\mathcal{V}_\alpha(R)\) by \(\tau = \text{const.}\) are open sets given by

\[
\mathcal{V}_\tau(R) = \{ z \in \mathbb{C} \mid N_\tau(z) > R \} \quad (\tau \neq 1);
\]
\[
\mathcal{V}_1(R) = \{ z \in \mathbb{C} \mid N_1(z) > R \} = \bigcup_{|\theta| \leq \alpha} \{ \Re(e^{i\theta} z) > R \}.
\]
Proposition 2.3. We have

\[ |\ell_n^\alpha(z)| \geq n \cos \alpha \quad \text{for } V_\alpha(0) \]

and hence the sequence \( \{\ell_n^\alpha(z)\}_n \) converges to \( \infty \) uniformly \( V_\alpha(0) \).

Proof. \( N_\tau(z) > 0 \) implies \( N_\tau(\ell(z)) \geq N_\tau(z) + \cos \alpha > 0 \) by Lemma 2.2. Hence, if \( \tau \in T_\alpha \) and \( z \in V_\tau(0) \), then \( \ell_n^\alpha(z) \in V_\tau(0) \) and

\[ |\ell_n^\alpha(z)| \geq N_\tau(\ell_n^\alpha(z)) \geq N_\tau(z) + n \cos \alpha \geq n \cos \alpha, \]

for all \( n \). This proves the assertion. \( \square \)

§2.2. Solution to a difference equation

We consider the difference equation

\[ h_\tau(\ell_\tau(z)) - \tau h_\tau(z) = \frac{1}{z} + C_\tau, \]

where \( \ell_\tau(z) = \tau z + 1 \) with \( |\tau| > 1 \) or \( \tau = 1 \); and \( C_\tau \) is a constant depending on \( \tau \), which will be given later.

A solution to this equation is given by

\[ h_\tau(z) = -\frac{1}{\tau z} + \sum_{n=1}^{\infty} \frac{1}{\tau^{n+1}} \left\{ \frac{1}{\ell_n^\alpha(0)} - \frac{1}{\ell_n^\alpha(z)} \right\}. \]

We note that \( \ell_\tau(z) = \tau^n z + \tau^{n-1} + \cdots + \tau + 1 \) and \( \ell_\tau(0) = \tau^n - 1 + \cdots + \tau + 1 \). In the following, we will investigate some properties of this function.

First, for a \( \tau \) fixed, the following properties of \( h_\tau(z) \) can be easily verified:

In the case \( |\tau| > 1 \), the function \( h_\tau(z) \) is meromorphic on \( \hat{C} \) except the essential singularity at \( 1/(1-\tau) \), and has poles at \( (1-\tau^{-n})/(1-\tau) \), \( (n = 0, 1, 2, \ldots) \). This function \( h_\tau(z) \) is holomorphic at \( \infty \). We write

\[ H_\tau = h_\tau(\infty) = \sum_{n=1}^{\infty} \frac{1}{\tau^{n+1} \ell_n^\alpha(0)}. \]

We can easily verify that \( h_\tau(z) \), with \( \tau \neq 1 \), satisfies the equation (3) with the constant

\[ C_\tau = (1-\tau)H_\tau. \]

In the case \( \tau = 1 \), we have \( \ell^n(z) = z + n \) and

\[ h_1(z) = -\frac{1}{z} + \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{z + n} \right\}. \]
This function is meromorphic on $\mathbb{C}$ and has poles at $0, -1, -2, \ldots$. As is easily verified, $h_1(z)$ satisfies the equation (3) with $C_1 = 0$.

We note that

$$h_1(z) = \frac{\Gamma'(z)}{\Gamma(z)} + \gamma$$

where $\Gamma(z)$ denotes the gamma function and $\gamma$ denotes the Euler constant

$$\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right).$$

Now we study the dependence of $h_\tau(z)$ on the parameter $\tau$.

**Proposition 2.4.** The function $h_\tau(z)$ is continuous on $V_\alpha(0)$.

**Proof.** The continuity at the points $(z, \tau)$ with $\tau \neq 1$ is clear. Using $\ell_\tau^n(z) - \ell_\tau^n(0) = \tau^n z$ and the estimate (2), we have

$$\left| \frac{1}{\tau^{n+1}} \left\{ \frac{1}{\ell_\tau^n(0)} - \frac{1}{\ell_\tau^n(z)} \right\} \right| = \left| \frac{z}{\tau^{n+1} \ell_\tau^n(0) \ell_\tau^n(z)} \right| \leq \frac{|z|}{\tau^{n+1} n^2 \cos^2 \alpha}.$$

This shows that the series (4) is locally uniformly convergent on $V_\alpha(0) - \{z = \infty\}$ and hence $h_\tau(z)$ is continuous there. \[ \Box \]

**Corollary 2.5.** The constant $C_\tau$ is a continuous function of $\tau \in T_\alpha$.

**Proof.** By the difference equation (3), we have $C_\tau = h_\tau(\ell_\tau(z)) - \tau h_\tau(z) - 1/z$, which is continuous on $V_\alpha(0)$ by Proposition 2.4. Hence $C_\tau$ is continuous on $T_\alpha$. \[ \Box \]

**Proposition 2.6.** For any $\varepsilon > 0$, there is a constant $M$ such that

$$|h'_\tau(z)| \leq \frac{M}{N_\tau(z)}$$

on $V_\alpha(\varepsilon)$.

**Proof.** Differentiation of (3) with respect to $z$ yields

$$h'_\tau(z) = \frac{1}{\tau} \sum_{n=0}^{\infty} \frac{1}{(\ell_\tau^n(z))^2}.$$

Hence

$$|h'_\tau(z)| \leq \sum_{n=0}^{\infty} \frac{1}{|\ell_\tau^n(z)|^2} \leq \sum_{n=0}^{\infty} \frac{1}{(N_\tau(z) + n \cos \theta)^2} \leq \int_{0}^{\infty} \frac{dx}{(N_\tau(z) + x \cos \theta)^2}.$$

Therefore $|h'_\tau(z)|$ is bounded by $M/N_\tau(z)$ with some constant $M$. \[ \Box \]
§2.3. Behavior of \( H_\tau \)

Now we look at the behavior of the function \( H_\tau \) defined by (5), when \( \tau \to 1 \) within the sector \( T \). It is clear from the expression (5) that \( H_\tau \) is unbounded, while \( C_\tau = (1 - \tau)H_\tau \) tends to 0 by Corollary 2.5. Here we give a more precise description of its behavior.

**Proposition 2.7.** \( \) We have

\[
H_\tau = -\log(\tau - 1) + \gamma - 1 + o(1)
\]

as \( \tau \to 1 \) within the sector \( T_\alpha \). Here \( \gamma \) denotes the Euler constant.

**Proof.** To begin with, letting \( \lambda = 1/\tau \), we have

\[
H_{1/\lambda} = \sum_{n=1}^\infty \frac{\lambda^{2n}}{1 + \lambda + \cdots + \lambda^{n-1}}
\]

\[
= (1 - \lambda) \sum_{n=1}^\infty \left( \frac{\lambda^n}{1 - \lambda^n} - \lambda^n \right)
\]

\[
= (1 - \lambda)L(\lambda) - \lambda.
\]

Here \( L(\lambda) \) denotes the Lambert series defined by

\[
L(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{1 - \lambda^n}.
\]

This series \( L(\lambda) \) defines a holomorphic function on \( |\lambda| < 1 \). We want to know the behavior of this function when \( \lambda \) tends to 1 non-tangentially within the unit disk.

\( L(\lambda) \) is developed into the power series

\[
L(\lambda) = \sum_{n=1}^\infty d(n)\lambda^n = \lambda + 2\lambda^2 + 2\lambda^3 + 3\lambda^4 + \cdots,
\]

where \( d(n) \) denotes the number of divisors of \( n \). We write

\[ D(n) = d(1) + \cdots + d(n). \]

Then

\[
\frac{L(\lambda)}{1 - \lambda} = \sum_{n=1}^\infty D(n)\lambda^n.
\]
The asymptotic behavior of $D(n)$ is given by a theorem of Dirichlet (see Apostol [1], Chandrasekharan [2]):

$$D(n) = n \log n + (2\gamma - 1)n + O(\sqrt{n}) \quad (n \to \infty).$$

From this and the fact that

$$\sum_{k=1}^{n} \frac{1}{k} - \log n = \gamma + O\left(\frac{1}{n}\right),$$

it follows that

$$D(n) = n \sum_{k=1}^{n} \frac{1}{k} + (\gamma - 1)n + p_n$$

where $p_n = O(\sqrt{n})$ as $n \to \infty$. Therefore, noting that

$$\frac{\lambda}{(1-\lambda)^2} = \sum_{n=1}^{\infty} n\lambda^n, \quad \log(1-\lambda) = -\sum_{n=1}^{\infty} \frac{\lambda^n}{n}$$

we have

$$L(\lambda) = \frac{\lambda}{1-\lambda} - \sum_{n=1}^{\infty} D(n)\lambda^n = -\frac{\lambda \log(1-\lambda)}{(1-\lambda)^2} + \frac{\gamma\lambda}{(1-\lambda)^2} + P(\lambda)$$

where $P(\lambda) = \sum_{n=1}^{\infty} p_n\lambda^n$. Since $p_n = O(\sqrt{n}) = o(n)$, we have

$$P(\lambda) = o((1-\lambda)^{-2}) \quad \text{as } \lambda \to 1 \text{ non-tangentially.}$$

Thus we obtain

$$H_\tau = (1-\lambda)L(\lambda) - \lambda$$

$$= -\lambda \log(1-\lambda) + (\gamma - 1)\lambda + (1-\lambda)^2 P(\lambda)$$

$$= -\log(\tau - 1) + \gamma - 1 + o(\tau - 1)$$

and the proposition is proved. \(\square\)
§3. Families of Maps with Attracting/Parabolic Fixed Points

§3.1. Domain of convergence

Let \( U(R) = \{ z \in \hat{\mathbb{C}} \mid R < |z| \leq \infty \} \) be a neighborhood of \( \infty \in \hat{\mathbb{C}} \) and we consider a family of holomorphic maps \( f_\tau : U(R) \to \hat{\mathbb{C}} \) of the form

\[
f_\tau(z) = \tau z + 1 + A_\tau(z)
\]

with

\[
A_\tau(z) = \frac{a_1 \tau}{z} + \frac{a_2 \tau}{z^2} + \cdots.
\]

We suppose that \( f_\tau \) depends holomorphically on the parameter \( \tau \in \Delta_\rho = \{ \tau \in \mathbb{C} \mid |\tau - 1| < \rho \} \).

As in the previous section, we choose and fix \( \alpha \) so that \( 0 < \alpha < \pi / 2 \) and let \( \delta = \frac{1}{2} \cos \alpha \). By shrinking the neighborhoods \( U(R) \) and \( \Delta_\rho \), we assume that there is a constant \( K \) such

\[
|A_\tau(z)| < K \frac{|z|}{|z|} < \delta
\]

for \( (z, \tau) \in U(R) \times \Delta_\rho \). Further we assume that \( f_\tau(z) \) is injective in \( z \) for every \( \tau \in \Delta_\rho \).

Now we have results on uniformity of convergence for \( f_\tau^n(z) \), corresponding to Lemma 2.2 and Proposition 2.3 for \( \ell_\tau(z) \). We set

\[
T_{\alpha, \rho} = T_{\alpha} \cap \Delta_\rho = \{ \tau \in \mathbb{C} \mid \text{Re}(\tau - 1) \leq |\tau - 1| \cos \alpha, \ |\tau - 1| < \rho \}.
\]

**Lemma 3.1.** For \( (z, \tau) \in U(R) \times T_{\alpha, \rho} - \{(\infty, 1)\} \) we have

\[
N_\tau(f_\tau(z)) \geq |\tau| N_\tau(z) + \delta.
\]

**Proof.** From \( f_\tau(z) = \ell_\tau(z) + A_\tau(z) \), it follows that

\[
N_\tau(f_\tau(z)) \geq N_\tau(\ell_\tau(z)) - |A_\tau(z)|
= \geq |\tau| N_\tau(z) + \cos \alpha - \delta
= \geq |\tau| N_\tau(z) + \delta,
\]

which proves the lemma. \( \square \)

Now let

\[
V_{\alpha, \rho}(R) = \{ (z, \tau) \in V_\alpha(R) \mid \tau \in T_{\alpha, \rho} \}.
\]

We note that \( V_{\alpha, \rho}(R) \subset U(R) \times T_{\alpha, \rho} \) since \( N_\tau(z) \leq |z| \).
Proposition 3.2. If \( \tau \in T_{\alpha,\rho} \) and \( z \in V_{\tau}(R) \), then \( f_{\tau}(z) \in V_{\tau}(R) \).

The sequence \( \{f_{\tau}^n(z)\}_n \) converges uniformly on \( V_{\alpha,\rho}(R) \) to \( \infty \).

Proof. If \( \tau \in T_{\alpha,\rho} \) and \( z \in V_{\tau}(R) \), then \( N_{\tau}(z) > R \). Hence \( N_{\tau}(f_{\tau}(z)) \geq N_{\tau}(z) + \delta > R + \delta \) and \( f_{\tau}(z) \in V_{\tau}(R) \). Further

\[ |f_{\tau}^n(z)| \geq N_{\tau}(f_{\tau}^n(z)) \geq N_{\tau}(z) + n\delta > R + n\delta. \]

This shows the uniform convergence of \( \{f_{\tau}^n(z)\}_n \) to \( \infty \) on \( V_{\alpha,\rho}(R) \).

§3.2. Schröder-Abel equation

We recall that \( C_{\tau} \) is continuous on \( T_{\alpha} \) and holomorphic in the interior of \( T_{\alpha} \) and that \( C_{\tau} = (1 - \tau)H_{\tau} \) when \( \tau \neq 1 \). Let

\( B_{\tau} = 1 - a_{1,\tau}C_{\tau}. \)

The following theorem constitutes the main ingredient of this paper.

Theorem 3.3. There exists a function \( \varphi_{\tau}(z) \) on \( V_{\alpha,\rho}(R) \) with values in \( \hat{\mathbb{C}} \) satisfying the following conditions:

(i) \( \varphi_{\tau}(z) \) is continuous on \( V_{\alpha,\rho}(R) \) and holomorphic in its interior as a mapping to \( \hat{\mathbb{C}} \).

(ii) For each \( \tau \in T_{\alpha,\rho} - \{1\} \) fixed, the function \( \varphi_{\tau}(z) \) is holomorphic in \( V_{\tau}(R) \) except for a simple pole at \( \infty \); and \( \varphi_{1}(z) \) is holomorphic in \( V_{1}(R) \). Further \( \varphi_{\tau}(z) \) satisfies the functional equation

\[ \varphi_{\tau}(f_{\tau}(z)) = \tau\varphi_{\tau}(z) + B_{\tau}. \]

(iii) For each \( \tau \in T_{\alpha,\rho} - \{1\} \) fixed, the function \( \varphi_{\tau}(z) \) is of the form

\[ \varphi_{\tau}(z) = z - a_{1,\tau}H_{\tau} + o(1) \]

in a neighborhood of \( z = \infty \).

The proof is given in the next subsection.

This theorem implies in particular the following: Suppose that \( \tau \) tends to \( 1 \) from outside of the unit disk with direction \( \theta \), i.e., \( \tau = 1 + re^{i\theta} \) which fixed \( \theta \) and \( r \) tending to \( 0 \). Then the domain \( V_{\tau}(R) \) of \( \varphi_{\tau}(z) \) converges to the half plane \( \{\text{Re}e^{i\theta}z > R\} \subset V_{1}(R) \), and \( \varphi_{\tau}(z) \) converges to \( \varphi_{1}(z) \) on this half plane. This remark applies also to \( \psi_{\tau} \) given below.

To make clear the meaning of this theorem, we will give the relation between \( \varphi_{\tau}(z) \) and the solution to Schröder equation.
Suppose $|\tau| > 1$ and consider the equation

\begin{equation}
\chi_\tau(f_\tau(z)) = \tau \chi_\tau(z),
\end{equation}

which is a variant of the Schröder equation formulated for the case where the fixed point is $\infty$. It is classical that this equation has a unique solution $\chi_\tau(z)$ of the form $\chi_\tau(z) = z + O(1)$ in a neighborhood of $\infty$. By comparing the coefficients of the Laurent expansion we can see that

$$\chi_\tau(z) = z + \frac{1}{\tau - 1} + O(1/z).$$

On the other hand, we can easily verify that $\varphi(z) + B_\tau/(\tau - 1)$ satisfies the equation (11). Since $B_\tau = 1 - a_{1,\tau}C_\tau = 1 - a_{1,\tau}(1 - \tau)H_\tau$ by (9) and (6), we have the following.

**Theorem 3.4.** For $\tau \in T_{\alpha,\rho} - \{1\}$ we have

$$\varphi_\tau(z) = \chi_\tau(z) - \frac{B_\tau}{\tau - 1} = \chi_\tau(z) - \frac{1}{\tau - 1} - a_{1,\tau}H_\tau.$$

Our result may be stated, without referring to $\varphi_\tau(z)$, as follows.

**Theorem 3.5.** When $\tau$ tends to $1$ non-tangentially from outside of the unit disk, the function

$$\chi_\tau(z) - \frac{1}{\tau - 1} + a_{1,\tau} \log(\tau - 1)$$

for $\tau \in T - \{1\}$ converges to a solution to the Abel equation for $f_1(z)$.

Here we may replace $a_{1,\tau}$ by $a_{1,1}$, since $a_{1,\tau}H_\tau$ and $-a_{1,1} \log(\tau - 1)$ differ only by a continuous function on $T_{\alpha, \rho}$.

We may normalize $\varphi_\tau(z)$ by letting $\varphi_\tau^*(z) = \varphi_\tau(z)/B_\tau$. Then $\varphi_\tau^*(z)$ satisfies the conditions of Theorem 3.3, replacing $B_\tau$ by $1$. For $\tau \neq 1$, we have

$$\varphi_\tau^*(z) = \frac{\chi_\tau(z)}{B_\tau} - \frac{1}{\tau - 1}.$$

Now we give another reformulations of the result. The function $\varphi_\tau$ has pole on $z = \infty$. By a linear fractional transformation, we obtain a function which is holomorphic on $\mathcal{V}_{\alpha,\rho}(R)$. 

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Theorem 3.6. There exists a function $\psi_\tau(z)$ on $\mathcal{V}_{\alpha,\rho}(R)$ satisfying the following conditions:

(i) $\psi_\tau(z)$ is continuous on $\mathcal{V}_{\alpha,\rho}(R)$ and holomorphic in its interior, as a function of two variables.

(ii) For each $\tau \in T_{\alpha,\rho}$ fixed, the function $\psi_\tau(z)$ is holomorphic in $V_\tau(R)$ and satisfies the functional equation

$$\psi_\tau(f_\tau(z)) = \frac{1}{\tau} \psi_\tau(z) + 1.$$ 

(iii) For each $\tau \in T_{\alpha,\rho}$ fixed, the function $\psi_\tau(z)$ is of the form

$$\psi_\tau(z) = \frac{\tau \phi_\tau(z)}{(\tau - 1)\phi_\tau(z) + B_\tau}.$$ 

In the neighborhood of $z = \infty$,

Proof. We define

$$\psi_\tau(z) = \frac{\tau \phi_\tau(z)}{(\tau - 1)\phi_\tau(z) + B_\tau}.$$ 

Then

$$\psi_\tau(z) = \frac{\tau}{\tau - 1} - \frac{\tau B_\tau}{(\tau - 1)^2} + O\left(\frac{1}{z^2}\right).$$

We can easily verify that $\psi_\tau(z)$ satisfies the required conditions. \qed

§3.3. Proof of Theorem 3.3

To simplify the notation, we omit the subscript $\tau$ for $f_\tau$ etc.

We rewrite the expression (7) in the form

$$f(z) = \tau z + 1 + \frac{a_1 \tau}{z} + A_1(z)$$

with $A_{1,\tau}(z) = a_2(\tau)/z^2 + \ldots$. There exists some constant $K_1$ such that

$$|A_1(z)| \leq \frac{K_1}{|z|^2}.$$

To make clear the idea of the proof, we will first consider the case where $a_{1,\tau} = 0$ identically. Replacing $z$ by $f^{n-1}(z)$ for in (11) and dividing by $\tau^n$, we obtain

$$\frac{1}{\tau^n} f^n(z) = \frac{1}{\tau^{n-1}} f^{n-1}(z) + \frac{1}{\tau^n} + \frac{1}{\tau^n} A_1(f^{n-1}(z)).$$
We define
\[ \varphi_n(z) := \frac{1}{\tau^n} f^n(z) - \sum_{k=1}^n \frac{1}{\tau^k} = z + \sum_{k=1}^n \frac{1}{\tau^k} A_1(f^{k-1}(z)). \]

Since
\[ \left| \frac{1}{\tau^k} A_1(f^{n-1}(p)) \right| \leq \frac{K_1}{|f^{n-1}(z)|^2} \leq \frac{K_1}{N(f^{n-1}(z))^2} \leq \frac{K_1}{(N(z) + (n-1)\delta)^2}, \]
we conclude that \( \varphi_n(z) \) converges uniformly as \( n \to \infty \). Therefore the limit
\[ \varphi(z) := \lim_{n \to \infty} \varphi_n(z) \]
is continuous on \( V_{\alpha,\rho}(R) \). From
\[ \varphi_n(f(z)) = \tau \phi_{n+1}(z) + 1, \]
it follows that \( \varphi(z) \) satisfies the equation (10) with \( B_\tau = 1 \).

Now, in the general case where \( a_1, \tau \) does not vanish identically, we have to modify the above construction to have convergent sequence. Let us recall the function \( h(z) \) satisfying the difference equation (3) in the previous section. We set
\[ A_2(z) = h(f(z)) - h(\ell(z)). \]

Then
\[ h(f(z)) = \tau h(z) + C_\tau + \frac{1}{z} + A_2(z). \]

Combining this with (12), we get
\[ f(z) - a_1 h(f(z)) = \tau \{ z - a_1 h(z) \} + B_\tau + \tilde{A}(z) \]
with \( B_\tau = 1 - a_1 C_\tau \), where we have set
\[ \tilde{A}(z) = A_1(z) - a_1 A_2(z). \]

In the same manner as in (13), we obtain
\[ \frac{1}{\tau^n} (f^n(z) - a_1 h(f^n(z))) = \frac{1}{\tau^n} \{ f^{n-1}(z) - a_1 h(f^{n-1}(z)) \} + \frac{B_\tau}{\tau} + \frac{1}{\tau^n} \tilde{A}(f^{n-1}(z)). \]

We define
\[ \varphi_n(z) = \frac{1}{\tau^n} \{ f^n(z) - a_1 h(f^n(z)) \} - B_\tau \sum_{k=1}^n \frac{1}{\tau^k} \]
\[ = z - a_1 h(z) + \sum_{k=1}^n \frac{1}{\tau^k} \tilde{A}(f^{k-1}(z)). \]

The sum on the right is
\[ \sum_{k=1}^n \frac{1}{\tau^k} A_1(f^{k-1}(z)) - a_1 \tau \sum_{k=1}^n \frac{1}{\tau^k} A_2(f^{k-1}(z)). \]
When $n \to \infty$, the first sum is uniformly convergent by the estimate (14). The convergence of the second sum follows from Lemma 3.7 below. Thus $\varphi_n(z)$ converges uniformly on $V_{\alpha, \rho}(R)$ as $n \to \infty$. Hence the limit $\varphi(z) = \lim_{n \to \infty} \varphi_n(z)$ is continuous on $V_{\alpha, \rho}(R)$. From $\varphi_n(f(z)) = \tau \varphi_{n+1}(z) + B_\tau$ it follows that $\varphi(z)$ satisfies the equation (10).

Since $\tilde{A}(z)$ vanishes at $z = \infty$, we have $\varphi_n(z) = z - a_1 H_\tau + o(1)$ in the neighborhood of $z = \infty$. Letting $n \to \infty$ yields the assertion (iii). □

**Lemma 3.7.** We have

$$|A_2(z)| \leq \frac{KM}{|z|N(z)}$$

on $(z, \tau) \in V_{\alpha, \rho}(R)$.

**Proof.** Let $z$ be a point with $N(z) > R$ and let $[\ell(z), f(z)]$ denote the segment joining $\ell(z)$ and $f(z)$ in $\mathbb{C}$. The length of this segment is

$$|f(z) - \ell(z)| = |A(z)| < \frac{K}{|z|} < \delta,$$

by (8). For any $\zeta$ in this segment, we have $|N(\zeta) - N(\ell(z))| \leq |\zeta - \ell(z)| < \delta$. Hence $N(\zeta) > N(\ell(z)) - \delta > N(z)$. Hence, by Proposition 2.6 we have

$$|h'(\zeta)| \leq \frac{M}{N(\zeta)} \leq \frac{M}{N(z)}$$

on this segment. Thus we have

$$|A_2(z)| = \left| \int_{[\ell(z), f(z)]} h'(\zeta)d\zeta \right| \leq \frac{KM}{|z|N(z)},$$

which proves the assertion. □

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**References**

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