The Irregularity of the Direct Image of Some $\mathcal{D}$-modules

By

Céline Roucairol*

Abstract

Let $f$ and $g$ be two regular functions on $U$ smooth affine variety. Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_U$-module. We are interested in the irregularity of the complex $f_+ (\mathcal{M}e^g)$. More precisely, we relate the irregularity number at $c$ of the systems $\mathcal{H}^k f_+ (\mathcal{M}e^g)$ with the characteristic cycles of the systems $\mathcal{H}^k (f, g)_+ (\mathcal{M})$.

§1. Introduction

Let $U$ be a smooth affine variety over $\mathbb{C}$ and $g : U \to \mathbb{C}$ be a regular function on $U$. We denote by $\mathcal{O}_U$ the sheaf of regular functions on $U$ and by $\mathcal{D}_U$ the sheaf of algebraic differential operators on $U$.

Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_U$-module. We denote by $\mathcal{M}e^g$ the $\mathcal{D}_U$-module obtained from $\mathcal{M}$ by twisting by $e^g$. If $\nabla$ is the connection defined by the $\mathcal{D}_U$-module structure of $\mathcal{M}$, $\nabla + dg$ is the one associated with $\mathcal{M}e^g$. Although $\mathcal{M}$ is regular, $\mathcal{M}e^g$ is not regular in general. Here, regular means that there exists a smooth compactification $X$ of $U$ and an extension of $\mathcal{M}e^g$ as $\mathcal{D}_X$-module which is regular holonomic on $X$. In [10], C. Sabbah describes a comparison theorem for these $\mathcal{D}$-modules twisted by an exponential. This theorem gives a relation between the irregularity complex of $\mathcal{M}e^g$ (see [6]) and some topological data given by $g$ and $\mathcal{M}$.

In this paper, we consider two regular functions $f, g : U \to \mathbb{C}$. We are interested in the irregularity of the cohomology modules of the direct image by $f$ of a $\mathcal{D}_U$-module, $\mathcal{M}e^g$, where $\mathcal{M}$ is regular and holonomic.

Communicated by M. Kashiwara. Received June 28, 2005. Revised September 27, 2005. 2000 Mathematics Subject Classification(s): 32C38, 35B40.

*Universität Mannheim, Institut für Mathematik, A5, 6, 68131 Mannheim, Germany.
e-mail: celine.roucairol@uni-mannheim.de

c⃝2006 Research Institute for Mathematical Sciences, Kyoto University. All rights reserved.
• In Section 2, we recall the definitions of a \( D \)-module twisted by an exponential of a meromorphic function. We will need the definition in the case of meromorphic function during the proof of the main theorem.

Then, we will consider the case where \( \mathcal{M} \) is the sheaf of regular function \( \mathcal{O}_U \). According to [4], the system \( \mathcal{H}^k(f_+(\mathcal{O}_U e^g)) \) extends vector bundle with flat holomorphic connection such that the generic fiber of the sheaf of their horizontal sections is canonically isomorphic to the cohomology group \( H^{k+n-1}_0(f^{-1}(e^n), \mathbb{C}) \), where \( \mathcal{O}_t \) is the family of closed subsets of \( f^{-1}(t) \) on which \( e^{-g} \) is rapidly decreasing. Using this result, we will motivate the study of the irregularity of the systems \( \mathcal{H}^k f_+(\mathcal{O}_U e^g) \) by observations on some integrals.

• The main theorem of this paper gives us a formula for the irregularity number of the systems \( \mathcal{H}^k(f_+(\mathcal{M} e^g)) \) at finite distance and at infinity.

In the case where \( f \) and \( g \) are two polynomials in two variables which are algebraically independents and \( \mathcal{M} = \mathcal{O}_{\mathbb{C}^2} \), the complex \( f_+(\mathcal{O}_{\mathbb{C}^2} e^g) \) is concentrated in degree 0 except at a finite number of points (see [9]). Then, the irregularity number at a point \( c \in \mathbb{C} \cup \{\infty\} \) of the system \( \mathcal{H}^k f_+(\mathcal{O}_{\mathbb{C}^2} e^g) \) can be expressed in terms of some geometric data associated with \( f \) and \( g \) (see [9]).

In this paper, we calculate the irregularity number at \( c \in \mathbb{C} \cup \{\infty\} \) of the systems \( \mathcal{H}^k(f_+(\mathcal{M} e^g)) \) with the help of the characteristic cycle of the systems \( \mathcal{H}^k(f, g)_+(\mathcal{M}) \), in the general case where \( f \) and \( g \) are any regular functions.

In the following, we identify \( \mathbb{C} \cup \{\infty\} \) with \( \mathbb{P}^1 \). Let \( i \) be the inclusion of \( \mathbb{C}^2 \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \). Let \( c \in \mathbb{P}^1 \) and \( V = V_1 \times V_2 \subset \mathbb{P}^1 \times \mathbb{P}^1 \) a neighbourhood of \( (c, \infty) \).

Let \( \text{Ch}(c, k) \) be the characteristic cycle of \( \mathcal{H}^k f_+(f, g)_+(\mathcal{M}) \) in the neighborhood \( V \):

\[
\text{Ch}(c, k) = mT_2V + m'T_{(c, \infty)}^*V + m''T_{(c, \infty)}^*V + m'''T_{(c, \infty)}^*V + \sum m_i T_i \mathfrak{Z}V,
\]

where \( Z_i \) are some germs of irreducible curves in a neighbourhood of \( (c, \infty) \) distinct from \( V_1 \times \{\infty\} \) and \( \{c\} \times V_2 \).

**Theorem 1.1.** The irregularity number of \( \mathcal{H}^k f_+(\mathcal{M} e^g) \) at \( c \) is equal to

\[
\sum_i m_i I_{(c, \infty)}(Z_i, \mathbb{P}^1 \times \{\infty\}),
\]

where \( I_{(c, \infty)}(Z_i, \mathbb{P}^1 \times \{\infty\}) \) is the intersection multiplicity of \( Z_i \) and \( \mathbb{P}^1 \times \{\infty\} \) at \( (c, \infty) \).

• The theorem of commutation between the irregularity functor and the direct image functor ([6]) allows us to rephrasing Theorem 1.1 in terms of an irregularity complex of a regular holonomic \( D \)-module twisted by an exponential (cf. Lemma 3.2).

Then, using the comparison theorem of [10], we are led to calculate the Euler characteristic of a germ of a complex of nearby cycles.
§2. The Complex $f_+ (M e^g)$

§2.1. Regular holonomic $\mathcal{D}$-modules twisted by an exponential

Let $X$ be a smooth algebraic variety over $\mathbb{C}$. We identify $\mathbb{P}^1$ with $\mathbb{C} \cup \{ \infty \}$. Let $h : X \to \mathbb{P}^1$ be a meromorphic function.

**Definition 2.1.** We define the $\mathcal{D}_X$-module $\mathcal{O}_X[*h^{-1}(\infty)] e^h$ as a $\mathcal{D}_X$-module which is isomorphic to $\mathcal{O}_X[*h^{-1}(\infty)]$ as $\mathcal{O}_X$-module. The original connection $\nabla$ on $\mathcal{O}_X[*h^{-1}(\infty)]$ is replaced with the connection $\nabla + dh$ on $\mathcal{O}_X[*h^{-1}(\infty)] e^h$.

Let $M$ be a holonomic $\mathcal{D}_X$-module.

**Definition 2.2.** We define the $\mathcal{D}_X$-module $M[*h^{-1}(\infty)] e^h$ as the $\mathcal{D}_X$-module $M \otimes_{\mathcal{O}_X} \mathcal{O}_X[*h^{-1}(\infty)] e^h$.

**Remark.** $\mathcal{O}_X[*h^{-1}(\infty)] e^h$ is the direct image by an open immersion of a vector bundle with integrable connection. Then, it is a holonomic $\mathcal{D}_X$-module as algebraic direct image of a holonomic $\mathcal{D}$-module.

$M[*h^{-1}(\infty)] e^h$ is a holonomic left $\mathcal{D}_X$-module as tensor product of two holonomic left $\mathcal{D}_X$-modules (cf. Theorem 4.6 of [2]).

We have analogous definitions in the analytic case. We just have to transpose in the analytic setting.

§2.2. On the solutions of the systems $\mathcal{H}^k (f_+ (\mathcal{O}_U e^g))$

The generic fiber of the sheaf of horizontal sections of $\mathcal{H}^k (f_+ (\mathcal{O}_U e^g))$ can be describe as follows:

**Theorem 2.1 ([4]).** There exists a finite subset $\Sigma \subset \mathbb{C}$ such that

- $\mathcal{H}^k (f_+ (\mathcal{O}_U e^g))|_{\mathbb{C} \setminus \Sigma}$ is a vector bundle with flat holomorphic connection.
- For all $t \in \mathbb{C} \setminus \Sigma$, $i_t^* \mathcal{H}^{k-n+1}(f_+ (\mathcal{O}_U e^g)) \simeq H^k_{\phi_t}((f^{-1}(t))^{an}, \mathbb{C})$, where $i_t$ is the inclusion of $\{ t \}$ in $\mathbb{C}$ and $\phi_t$ is the family of closed subsets of $f^{-1}(t)$ on which $e^{-g}$ is rapidly decreasing.

More precisely, the family $\phi_t$ is defined as follow. Let $\pi : \overline{\mathbb{P}^1} \to \mathbb{P}^1$ be the oriented real blow-up of $\mathbb{P}^1$ at infinity. $\overline{\mathbb{P}^1}$ is diffeomorphic to $\mathbb{C} \cup S^1$, where $S^1$ is the circle of directions at infinity. $A$ is in $\phi_t$ if $A$ is a closed subset of $f^{-1}(t)$ and the closure of $g(A)$ in $\mathbb{C} \cup S^1$ intersects $S^1$ in $]-\frac{\pi}{2}, \frac{\pi}{2}[$.
Let us motivate the study of this complex by observations on some integrals. Concerning Gauss-Manin systems, we can express their solutions as period integrals of the type \( \int_{\gamma(t)} w_{f^{-1}(t)} \), where \( \gamma(t) \) is an horizontal family of cycles in the fibres \( f^{-1}(t) \) and \( w \) is a relative algebraic differential form. As the Gauss-Manin connection is regular, these integrals have moderate growth in the neighbourhood of their singularities. In our case, some solutions can also be expressed as integrals.

Let \( \Psi_t \) be the family of closed subsets \( A \) of \( f^{-1}(t) \) such that for all \( R \) big enough, \( A \setminus g^{-1}(\{t \in \mathbb{C} \mid \text{Re}(-t) > R\}) \) is compact. We consider the complex of semi-algebraic chains with support in \( \Psi_t \) (see [8]). We denote by \( H_{k,\Psi_t}(f^{-1}(t)^{an}, \mathbb{C}) \) the \( k \)-th homology group associated with this complex. We can now integrate forms in \( H_k(f^{-1}(t)^{an}, \mathbb{C}) \) on cycles in \( H_{k,\Psi_t}(f^{-1}(t)^{an}, \mathbb{C}) \).

According to Theorem 1.4 of [1], since \( f \) is a submersion outside \( \Sigma \), we have an isomorphism

\[
H^{k-n+1}(f_+(\mathcal{O}_U e^\varphi))|_{\mathcal{C} \setminus \Sigma} \cong R^k f_*(DR_{\mathcal{C}^* / \mathcal{C}}(\mathcal{O}_U e^\varphi))|_{\mathcal{C} \setminus \Sigma}.
\]

Thus, we can extend the integration defined before to a form \( we^\varphi_{f^{-1}(t)} \), where \( w \) is a relative algebraic differential form. Indeed, by the definition of \( \Psi_t \), \( e^\varphi \) is rapidly decreasing on the cycles and semi-algebraic chains with support in \( \Psi_t \) behave well at infinity.

In this way, to \( \gamma(t) \), horizontal family of cycles in \( H_{k,\Psi_t}(f^{-1}(t)^{an}, \mathbb{C}) \), we can associate a solution of the \( \mathcal{D}_{\mathcal{C} \setminus \Sigma} \)-module \( H^{k-n+1}(f_+(\mathcal{O}_U e^\varphi))|_{\mathcal{C} \setminus \Sigma} \). It is a morphism \( \alpha \) of \( \mathcal{D}_{\mathcal{C} \setminus \Sigma} \)-modules defined by \( \alpha([we^\varphi]) = \int_{\gamma(t)} we^\varphi_{f^{-1}(t)} \).

The study of the irregularity of the systems \( H^{k-n+1}(f_+(\mathcal{O}_U e^\varphi)) \) gives us informations about the growth of these integrals in the neighbourhood of their singularities.

§3. On the Irregularity of the Complex \( f_+(\mathcal{M}e^\varphi) \)

In the following, we will identify \( \mathbb{C} \cup \{\infty\} \) with \( \mathbb{P}^1 \) and we consider the canonical immersion \( j : \mathbb{C} \to \mathbb{P}^1 \). Let us fix \( k \in \mathbb{Z} \) and \( c \in \mathbb{P}^1 \).

We are interested in the number \( IR_{c,k} \), it being the irregularity number at \( c \in \mathbb{P}^1 \) of the system \( H^k j_* f_*(\mathcal{M}e^\varphi) \).

The first step of the proof of Theorem 1.1 consists in rephrasing it using an irregularity complex of a \( \mathcal{D}_{\mathbb{C}} \)-module twisted by an exponential.

For the definition of irregularity complex along an hypersurface, we refer the reader to [6] and [7]. We adopt the following notations. If \( \mathfrak{M} \) is a complex of \( \mathcal{D}_X \)-modules and \( Z \) is an hypersurface of \( X \), we denote by \( IR_Z(\mathfrak{M}) \) the irregu-
Irregularity complex of $\mathfrak{M}$ along $Z$. For simplicity of notations, we write $IR^k_Z(\mathfrak{M})$ instead of $\mathcal{H}^k(IR_Z(\mathfrak{M}))$.

According to [5], the irregularity number $IR_{c,k}$ is equal to the dimension of the $\mathbb{C}$-vector space $IR^0(\mathcal{H}^k j_+ f_+(\mathcal{M}e^\theta))_c$.

Denote by $\mathcal{N}^\bullet$ the complex of $D_{\mathbb{P}^1 \times \mathbb{P}^1}$-modules $i_+(f,g)_+(\mathcal{M})$. In the way of rephrasing Theorem 1.1, we need the following lemma:

**Lemma 3.1.** Let $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the second projection and $D = (\mathbb{P}^1 \times \mathbb{P}^1) \setminus i(\mathbb{C}^2)$.

$$IR_c(\mathcal{H}^k j_+ f_+(\mathcal{M}e^\theta))_c = IR_{\{c\} \times \mathbb{P}^1}(\mathcal{H}^k(\mathcal{N}^\bullet)(\star D)[e^{\pi_2}])_{(c,\infty)}[+1].$$

**Proof.**
- Reduction to the case of the two projections.
- Let $p_1 : \mathbb{C}^2 \to \mathbb{C}$ and $p_2 : \mathbb{C}^2 \to \mathbb{C}$ be the two canonical projections.
- As $f = p_1 \circ (f,g)$, we have $f_+(\mathcal{M}e^\theta) = p_1(f,g)_+(\mathcal{M}e^\theta)$. Moreover, $(f,g)_+(\mathcal{M}e^\theta) = (f,g)_+(\mathcal{M})e^{p_2}$.
- Finally, we obtain that $f_+(\mathcal{M}e^\theta) \simeq p_1((f,g)_+(\mathcal{M})e^{p_2})$.
- In this paragraph, we denote by $\mathcal{P}^h$ the perverse cohomology. According to Corollary 2-1-8 of [6], we have:

$$IR_c(\mathcal{H}^k j_+ f_+(\mathcal{M}e^\theta)) = \mathcal{P}^h IR_c(j_+ f_+(\mathcal{M}e^\theta)).$$

Consider the following diagrams:

$$\begin{array}{cc}
\mathbb{C}^2 \xrightarrow{i} \mathbb{P}^1 \times \mathbb{P}^1 & \mathbb{C}^2 \xrightarrow{i} \mathbb{P}^1 \times \mathbb{P}^1 \\
\downarrow p_1 & \downarrow p_2 \\
\mathbb{C} \xrightarrow{j} \mathbb{P}^1 & \mathbb{C} \xrightarrow{j} \mathbb{P}^1 \\
\end{array}$$

Then,

$$IR_c(j_+ f_+(\mathcal{M}e^\theta)) = IR_c(j_+ p_1 \circ ((f,g)_+(\mathcal{M})e^{p_2}))$$
$$= IR_c(\pi_1 + i_+(f,g)_+(\mathcal{M})e^{p_2}))$$
$$= IR_c(\pi_1 + i_+(\mathcal{N}^\bullet[\star D][e^{\pi_2}])).$$

Then, $IR_c(\mathcal{H}^k j_+ f_+(\mathcal{M}e^\theta)) = \mathcal{P}^h IR_c(\pi_1 + i_+(\mathcal{N}^\bullet[\star D][e^{\pi_2}])).$

- According to Proposition 3-6-4 of [7], the irregularity functor commutes with the direct image functor. Thus:

$$IR_c(\pi_1 + i_+(\mathcal{N}^\bullet[\star D][e^{\pi_2}]))_c = R\pi_1 IR_{\{c\} \times \mathbb{P}^1}(\mathcal{N}^\bullet[\star D][e^{\pi_2}])_{[+1]}$$
$$= R\Gamma(\{c\} \times \mathbb{P}^1, IR_{\{c\} \times \mathbb{P}^1}(\mathcal{N}^\bullet[\star D][e^{\pi_2}])[+1].$$
• Then, we remark that $\pi_2$ is holomorphic out of $(c, \infty)$ and $N^*$ is regular holonomic (direct image complex of an algebraic regular holonomic $\mathcal{D}$-module). Then, $IR_{(c)\times \mathbb{P}^1}(N^*[-2])$ has its support in $(c, \infty)$ and we have an isomorphism of complexes of vector spaces

$$IR_c(\pi_1 + i_+ (N^*[-2])) = IR_{(c)\times \mathbb{P}^1}(N^*[-2])_{(c, \infty)}[+1].$$

• We conclude that

$$IR_c(\mathcal{H}^l j_+ f_+ (Me^p)) = \mathcal{H}^l IR_c(\pi_1 + i_+ (N^*[-2]))_{(c, \infty)}[+1] = IR_{(c)\times \mathbb{P}^1}(\mathcal{H}^l (N^*[-2]))_{(c, \infty)}[+1] = IR_{(c)\times \mathbb{P}^1}(\mathcal{H}^l (N^*[-2]))_{(c, \infty)}[+1].$$

Now, we can rephrase Theorem 1.1.

Let us choose some local coordinates $(x, z)$ of $\mathbb{P}^1 \times \mathbb{P}^1$ in a neighbourhood of $(c, \infty)$ such that:

1. $(c, \infty)$ has for coordinates $(0, 0)$,
2. $(c) \times \mathbb{P}^1$ has equation $x = 0$ in a neighbourhood of $(c, \infty)$,
3. $\mathbb{P}^1 \times \{\infty\}$ has equation $z = 0$ in a neighbourhood of $(c, \infty)$.

In these coordinates, $\pi_2$ is equal to $\frac{1}{z}$ in a neighbourhood of $(c, \infty)$. Then, according to Lemma 3.1, we are led to prove the following lemma:

**Lemma 3.2.** Let $\mathcal{M}$ be a holonomic regular $\mathcal{D}_{\mathcal{C}^2}$-module. We denote the characteristic cycle of $\mathcal{M}$ in a neighbourhood of $(0, 0)$ by:

$$Ceh(\mathcal{M}) = mT_{c=0}^x \mathcal{C}^2 + m' T_{(c, \infty)}^x \mathcal{C}^2 + m'' T_{z=0}^z \mathcal{C}^2 + \sum m_i T_{Z_i}^z \mathcal{C}^2,$$

where $Z_i$ are some germs of irreducible curves in a neighbourhood of $(0, 0)$ distinct from $x = 0$ and $z = 0$.

Then,

$$\chi(IR_{z=0}(\mathcal{M}[-1])^e(0, 0)) = -\sum m_i I(c, \infty)(Z_i, \{z = 0\}).$$
\section{Proof of Lemma 3.2}

We break up the proof of Lemma 3.2 in three steps:

**Lemma 4.1.** \( \chi(IR_{x=0}(\mathcal{M}[\frac{1}{x}] e^\cdot_\ast)(0,0)) = \chi(IR_{z=0}(\mathcal{M}[\frac{1}{x}] e^\cdot_\ast)(0,0)) \).

We denote by \( \Psi_z(\mathcal{M}[\frac{1}{x}]) \) the complex of nearby cycles of \( \mathcal{M}[\frac{1}{x}] \) relative to \( z \). It is a complex of constructible sheaves on \( \mathbb{C} \times \{0\} \) defined as follows.

Let \( \eta \) small enough. We denote by \( D^*(0,\eta) \) the universal covering of \( D^*(0,\eta) \). Let \( (E,\pi,\tilde{z}) \) be the fiber product over \( D^*(0,\eta) \) of \( \mathbb{C} \times D^*(0,\eta) \) and \( D^*(0,\eta) \). Then, we have the following diagram:

\[
\begin{array}{c}
\mathbb{C} \times \{0\} \xrightarrow{\alpha} \mathbb{C}^2 \xrightarrow{i} \mathbb{C} \times D^*(0,\eta) \xrightarrow{\pi} E \\
\mathbb{C} \times D^*(0,\eta) \xrightarrow{\pi} D^*(0,\eta) \xrightarrow{\tilde{\pi}} D^*(0,\eta).
\end{array}
\]

\[
\Psi_z(\mathcal{M}[\frac{1}{x}]) = \alpha^{-1} R(\tilde{i} \circ \pi)_{\ast}(\tilde{i} \circ \pi)^{-1}(DR(\mathcal{M}[\frac{1}{x}])).
\]

**Lemma 4.2.** \( \chi(IR_{x=0}(\mathcal{M}[\frac{1}{x}] e^\cdot_\ast)(0,0)) = \chi(\Psi_z(\mathcal{M}[\frac{1}{x}])(0,0)) \).

**Lemma 4.3.** \( \chi(\Psi_z(\mathcal{M}[\frac{1}{x}])(0,0)) = - \sum_i m_i I_{(c,\infty)}(Z_i, \mathbb{P}^1 \times \{\infty\}) \).

**Proof of Lemma 4.1.** Let us first show that

\[
IR_{x=0}(\mathcal{M}[\frac{1}{x}] e^\cdot_\ast) = R\Gamma_{x=0}(IR_{z=0}(\mathcal{M}[\frac{1}{x}] e^\cdot_\ast)).
\]

Let \( \eta \) be the inclusion of \( \mathbb{C} \times \mathbb{C}^* \) in \( \mathbb{C}^2 \). By definition,

\[
IR_{z=0}(\mathcal{M}[\frac{1}{x}] e^\cdot_\ast) = \text{cone} \left( DR(\mathcal{M}[\frac{1}{x}] e^\cdot_\ast) \to R\eta_* \eta^{-1}(DR(\mathcal{M}[\frac{1}{x}] e^\cdot_\ast)) \right)
\]

\[
= \text{cone} \left( DR(\mathcal{M}[\frac{1}{x}] e^\cdot_\ast) \to R\eta_* (DR(\mathcal{M}[\frac{1}{x}] e^\cdot_\ast))_{\mathbb{C} \times \mathbb{C}^*} \right).
\]

Now, consider the following diagram:

\[
\begin{array}{c}
\mathbb{C} \times \mathbb{C}^* \xrightarrow{\eta} \mathbb{C}^2 \\
\mathbb{C}^* \times \mathbb{C}^* \xrightarrow{\eta'} \mathbb{C}^* \times \mathbb{C}.
\end{array}
\]
As \( \mathfrak{M} \) is regular, we have:

\[
R\eta_* (\text{DR}(\mathfrak{M}[\frac{1}{x}])(\mathbb{C} \times \mathbb{C})) = R\eta_* R\eta'_* (\text{DR}(\mathfrak{M})(\mathbb{C} \times \mathbb{C}))
\]

\[
= RJ_* R\eta'_* (\text{DR}(\mathfrak{M})(\mathbb{C} \times \mathbb{C})).
\]

But \( R\Gamma_{x=0} R\eta_* = 0 \). Then:

\[
R\Gamma_{x=0} (IR_{z=0}(\mathfrak{M}[\frac{1}{xz}])(\mathbb{C} \times \mathbb{C}))) = R\Gamma_{x=0} (\text{DR}(\mathfrak{M}[\frac{1}{xz}](\mathbb{C} \times \mathbb{C}))
\]

\[
= IR_{z=0}(\mathfrak{M}[\frac{1}{x}](\mathbb{C} \times \mathbb{C})),
\]

by definition of irregularity complex.

Then, we are led to prove that the complexes \( R\Gamma_{x=0} (IR_{z=0}(\mathfrak{M}[\frac{1}{xz}](\mathbb{C} \times \mathbb{C}))) \) and \( IR_{z=0}(\mathfrak{M}[\frac{1}{x}](\mathbb{C} \times \mathbb{C})) \) have the same characteristic function at \((0,0)\).

Using the following distinguished triangle,

\[
\xymatrix{ R\Gamma_{x=0}(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}](\mathbb{C} \times \mathbb{C}))) \ar[rr] & & R\Gamma_{x=0}(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}](\mathbb{C} \times \mathbb{C}))) \ar[r]^{[+1]} & IR_{x=0}(\mathfrak{M}[\frac{1}{xz}](\mathbb{C} \times \mathbb{C})),
}
\]

it is sufficient to show that the characteristic function on \( \{x=0\} \) of the complex \( R\Gamma_{x=0}(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}](\mathbb{C} \times \mathbb{C}))) \) is zero.

Now, if \( \mathcal{F} \) is a constructible sheaf on \( X \) and \( P \in \{x=0\} \),

\[
\chi((R\jmath_* J^{-1}\mathcal{F})_P) = \chi((\mathbb{D}(J^{-1}\mathcal{F}))_P) = \chi((J^{-1} \mathbb{D} \mathcal{F})_P) = 0,
\]

where \( \mathbb{D} \) is the Verdier duality (see [11]).

\( \square \)

**Proof of Lemma 4.2.** This is a particular case of a result of C. Sabbah (cf. Corollary 5-2 of [10]).

\( \square \)

**Proof of Lemma 4.3.** Denote by \( C^\bullet \) the complex \( \Psi_x(\mathfrak{M}[\frac{1}{x}]) \). By definition,

\[
C^\bullet_{(0,0)} = R(\hat{i} \circ \pi)_* (\hat{i} \circ \pi)^{-1} (\text{DR}(\mathfrak{M}[\frac{1}{x}])(0,0)).
\]

Then, for all \( k \in \mathbb{Z} \),

\[
\mathcal{H}^k C^\bullet_{(0,0)} = \text{indlim}_{(0,0) \in U_{\text{open}}} \text{R}^k \Gamma(U, R(\hat{i} \circ \pi)_* (\hat{i} \circ \pi)^{-1} (\text{DR}(\mathfrak{M}[\frac{1}{x}]))).
\]

As \( \{D(0, \eta_1) \times D(0, \eta_2)\}_{\eta_1, \eta_2} \) is a fundamental system of neighbourhoods of \((0,0)\), we have

\[
\mathcal{H}^k C^\bullet_{(0,0)} = \text{indlim}_{\eta_1, \eta_2} \text{R}^k \Gamma(D(0, \eta_1) \times D(0, \eta_2), R(\hat{i} \circ \pi)_* (\hat{i} \circ \pi)^{-1} (\text{DR}(\mathfrak{M}[\frac{1}{x}]))).
\]
Let $\Sigma$ be a Whitney stratification associated with the constructible sheaf $DR(\mathcal{M}[1])$. Then, for $\eta_1$ and $\eta_2$ small enough, there exists a homotopy equivalence $p : (\hat{i} \circ \pi)^{-1}(D(0, \eta_1) \times D(0, \eta_2)) \rightarrow D(0, \eta_1) \times \{\tilde{\eta}\}$ compatible with $\Sigma$. Thus, $\mathcal{H}^{0}\mathcal{C}_{(0,0)}^{*} = \text{indlim}_{\eta_1, \eta_2} \Gamma(D(0, \eta_1) \times \{\tilde{\eta}\}, DR(\mathcal{M}[1])).$

Now, as $\mathcal{M}$ is regular, $DR(\mathcal{M}[1]) = RJ, J^{-1}(DR(\mathcal{M})).$ Then,
\[
\mathcal{H}^{0}\mathcal{C}_{(0,0)}^{*} = \text{indlim}_{\eta_1, \eta_2} \Gamma(D(0, \eta_1) \times \{\tilde{\eta}\}, RJ, J^{-1}(DR(\mathcal{M})))
= \text{indlim}_{\eta_1, \eta_2} \Gamma(D^{*}(0, \eta_1) \times \{\tilde{\eta}\}, J^{-1}(DR(\mathcal{M}))).
\]

Let fix $\eta_1$ and $\tilde{\eta}$ small enough such that the singular support of $\mathcal{M}$ in $D^{*}(0, \eta_1) \times \{\tilde{\eta}\}$ is a finite number of points. Denote by $P_1, \ldots, P_s$ these points. They are the intersection points of $D^{*}(0, \eta_1) \times \{\tilde{\eta}\}$ and $\cup Z_i$. As $DR(\mathcal{M})|_{P^{*}(0,\eta_1)\times\{\tilde{\eta}\}}$ is a complex of constructible sheaves with respect to the stratification $\{D^{*}(0, \eta_1) \times \{\tilde{\eta}\}\{P_1, \ldots, P_1, P_1, \ldots, P_1\}$, the Euler characteristic of $\Gamma(D^{*}(0, \eta_1) \times \{\tilde{\eta}\}, J^{-1}(DR(\mathcal{M})))$ is equal to:
\[
\chi(\Gamma(D^{*}(0, \eta_1) \times \{\tilde{\eta}\} - \{P_1, \ldots, P_1\}, DR(\mathcal{M}))) + \sum_{i=1}^{l} \chi(D(\mathcal{M})|_{P_i}).
\]

Then, according to the index theorem of Kashiwara (cf. [3]),
\[
\begin{align*}
\chi(\Gamma(D^{*}(0, \eta_1) \times \{\tilde{\eta}\}, J^{-1}(DR(\mathcal{M})))
&= rk(\mathcal{M}) \sum_{i} I_{(0,0)}(Z_i, \{z = 0\}) + \sum_{i} (rk(\mathcal{M}) - m_i)I_{(0,0)}(Z_i, \{z = 0\}) \\
&= - \sum_{i} m_i I_{(0,0)}(Z_i, \{z = 0\}).
\end{align*}
\]

Remark. If $f$ and $g$ are two polynomials in two variables, we can compare Theorem 1.1 and Theorem 1 of [9]. Let us recall this theorem:

Let $X$ be a smooth projective compactification of $\mathbb{C}^2$ such that there exists $F, G : X \rightarrow \mathbb{P}^1$, two meromorphic maps, which extend $f$ and $g$. Let us denote by $D$ the divisor $X \setminus \mathbb{C}^2$. Let $\Gamma$ be the critical locus of $(F, G)$.

Let $c \in \mathbb{P}^1$. We denote by $\Delta_1$ the cycle in $\mathbb{P}^1 \times \mathbb{P}^1$ which is the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of $(F, G)(\Gamma) \cap (\mathbb{C}^2 \setminus \{c\} \times \mathbb{C})$, where the image is counted with multiplicity and by $\Delta_2$ the cycle in $\mathbb{P}^1 \times \mathbb{P}^1$ which is the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of $(F, G)(D) \cap (\mathbb{C}^2 \setminus \{c\} \times \mathbb{C})$, where the image is counted with multiplicity.

**Theorem 4.1.** If $f$ and $g$ are algebraically independent, the irregularity number of $\mathcal{H}^{0}(f_{*}(\mathcal{O}_{\mathbb{C}^2}^c))$ is equal to
\[
I_{(c, \infty)}(\Delta_1, \mathbb{P}^1 \times \{\infty\}) + I_{(c, \infty)}(\Delta_2, \mathbb{P}^1 \times \{\infty\}).
\]
Then, we can prove that the germs $Z'_l$ of irreducible curves in Theorem 1.1 are the germs at $(c, \infty)$ of the irreducible branches of $\Delta_1 \cup \Delta_2$. The multiplicity $m_l$ of $i_+(f, g)_+ (\mathcal{O}_{C^2})$ on $T_{Z'_l} V$ are the multiplicity of $Z_k$ in $\Delta_1 \cup \Delta_2$.

References