A Cabling Formula for the 2-Loop Polynomial of Knots†

By

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Abstract

The 2-loop polynomial is a polynomial presenting the 2-loop part of the logarithm of the Kontsevich invariant of knots. We show a cabling formula for the 2-loop polynomial of knots. In particular, we calculate the 2-loop polynomial for torus knots.

§1. Introduction

The Kontsevich invariant is a very strong invariant of knots (which dominates all quantum invariants and all Vassiliev invariants) and it is expected that the Kontsevich invariant will classify knots. A problem when we study the Kontsevich invariant is that it is difficult to calculate the Kontsevich invariant of an arbitrarily given knot concretely. It has recently been shown [20, 9, 6]1 that the infinite sum of the terms of the logarithm of the Kontsevich invariant with a fixed loop number is presented by using polynomials (after appropriate normalization by the Alexander polynomial). In particular, it is known2 that

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1It was conjectured by Rozansky [20]. The existence of such rational presentations has been proved by Kricker [9] (though such a rational presentation itself is not necessarily a knot invariant in a general loop degree). Further, Garoufalidis and Kricker [6] defined a knot invariant in any loop degree, from which such a rational presentation can be deduced.

2This follows from the theory of [2] on the MMR conjecture. See also [9, 6] and references therein.

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the 1-loop part is presented by the Alexander polynomial. The polynomial giving the 2-loop part is called the 2-loop polynomial. The values of the 2-loop polynomial has been calculated so far only for particular\(^3\) classes of knots.

In this paper, we give a cabling formula for the 2-loop polynomial (Theorem 4.1), which presents the 2-loop polynomial of a cable knot (see Figure 1) of a knot \(K\) in terms of the 2-loop polynomial of \(K\). In particular, we calculate a formula of the 2-loop polynomial for torus knots (Theorem 3.1). This formula and the cabling formula are also obtained independently by Marché [14, 15].

\[\text{Figure 1. A cable knot of a knot}\]

This paper is organized as follows. In Section 1 we review the definition of the 2-loop polynomial. In Section 2 we calculate the 2-loop polynomial of torus knots as the 2-loop part of the primitive part of the cabling formula of the Kontsevich invariant of the trivial knot. In Section 3 we give a cabling formula for the 2-loop polynomial. In Section 4 we show relations to some Vassiliev invariants. In Section 5 we present the \(sl_2\) reduction of the 2-loop polynomial by a 1-variable reduction of it.

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§2. The Kontsevich Invariant and the 2-Loop Polynomial

The 2-loop polynomial is a polynomial presenting the 2-loop part of the logarithm of the Kontsevich invariant. In this section, we review its definition and a cabling formula of the Kontsevich invariant.

An open Jacobi diagram is a uni-trivalent graph such that a cyclic order of the three edges around each trivalent vertex of the graph is fixed. Let \(A(*)\) be

\(^{3}\)A table of the 2-loop polynomial for knots with up to 7 crossings is given by Rozansky [21]. The 2-loop polynomial of knots with the trivial Alexander polynomial can often been calculated by surgery formulas [6, 10].
the vector space over $\mathbb{Q}$ spanned by open Jacobi diagrams subject to the AS and IHX relations; see Figure 2 for the relations.

\[
\text{The AS relation: } \quad \begin{array}{c}
\includegraphics[scale=0.5]{AS_relation.eps}
\end{array} = - \begin{array}{c}
\includegraphics[scale=0.5]{AS_relation.eps}
\end{array}
\]

\[
\text{The IHX relation: } \quad \begin{array}{c}
\includegraphics[scale=0.5]{IHX_relation.eps}
\end{array} = \begin{array}{c}
\includegraphics[scale=0.5]{IHX_relation.eps}
\end{array} - \begin{array}{c}
\includegraphics[scale=0.5]{IHX_relation.eps}
\end{array}
\]

Figure 2. The AS and IHX relations

The Kontsevich invariant $Z^\sigma(K)$ of a framed knot $K$ is defined in $\mathcal{A}(\ast)$; for a definition\footnote{In literatures, the Kontsevich invariant is often defined by $Z(K)$ in the space $\mathcal{A}(S^1)$. The version $Z^\sigma(K)$ is defined to be the image of $Z(K)$ by the inverse map $\sigma$ of the Poincare-Birkhoff-Witt isomorphism $\mathcal{A}(\ast) \to \mathcal{A}(S^1)$.} see e.g. [17]. It is known [12] that the value of the Kontsevich invariant for each knot is group-like, which implies that it is presented by the exponential of some primitive element. That is, $Z^\sigma(K)$ is presented by the exponential of a primitive element, where a primitive element of $\mathcal{A}(\ast)$ is a linear sum of connected open Jacobi diagrams.

For example, it is shown [4] that the Kontsevich invariant of the trivial knot, denoted by $\Omega$, is presented by

$$Z^\sigma(\text{the trivial knot}) = \Omega = \exp_\sqcup(\omega),$$

where $\exp_\sqcup$ denotes the exponential with respect to the disjoint-union product, and $\omega$ is defined by

$$\omega = \frac{1}{2} \log \frac{\sinh(x/2)}{x/2}.$$ 

Here, a label of a power series $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$ implies

$$f(x) = c_0 + c_1 + c_2 + c_3 + \cdots,$$

where a label is put on either of the sides of an edge, and the corresponding
legs are written in the same side of the edge.\(^5\) Note that \(f(x) = \left| f(-x) \right| \) by the AS relation, in the notation of this paper.

Let \(K\) be a framed knot with 0 framing. (Throughout this paper, we often mean a framed knot with 0 framing also by a knot, abusing terminology.) A connected open Jacobi diagram is called an \(n\)-loop diagram when the first Betti number of the uni-trivalent graph of the diagram is equal to \(n\). The loop expansion of the Kontsevich invariant is given by

\[
\log_{\sqcup} Z^n(K) = \frac{1}{2} \log \frac{\sinh(x/2)}{x/2} - \frac{1}{2} \log \Delta_K(e^x)
\]

\[
+ \sum_{i} \left( \frac{p_{i,1}(e^x)/\Delta_K(e^x)}{p_{i,2}(e^x)/\Delta_K(e^x)} \right) + \text{(terms of (≥ 3)-loop)},
\]

where \(\log_{\sqcup}\) denotes the logarithm with respect to the disjoint-union product, and \(\Delta_K(t)\) is the normalized\(^6\) Alexander polynomial of \(K\), and \(p_{i,j}(e^x)\) is a polynomial in \(e^x\). The 2-loop part is characterized by the polynomial,

\[
\Theta_K'(t_1, t_2, t_3) = \sum_{i} p_{i,1}(t_1)p_{i,2}(t_2)p_{i,3}(t_3).
\]

We call its symmetrization,\(^7\)

\[
\Theta_K(t_1, t_2, t_3) = \sum_{\{i,j,k\} = \{1,2,3\}} \Theta_K'(t_1^i, t_2^j, t_3^k) \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]/(t_1t_2t_3 = 1),
\]

the 2-loop polynomial of \(K\), which is an invariant\(^8\) of \(K\). (Note that this normalization of \(\Theta_K(t_1, t_2, t_3)\) is 12 times the usual normalization.) \(\Theta_K(t, t^{-1}, 1)\)

---

\(^5\)Our notation is different from the notation in [6, 10] where a label of an edge is defined by setting a local orientation of the edge that determines the side in which we write the corresponding legs.

\(^6\)We suppose that \(\Delta_K(t)\) is normalized, satisfying that \(\Delta_K(t) = \Delta_K(t^{-1})\) and \(\Delta_K(1) = 1\).

\(^7\)With respect to the symmetry of the theta graph, of order 12.

\(^8\)This is not trivial, since there is another 2-loop trivalent graph, what is called, a “dumbbell diagram”.
is a symmetric polynomial in \( t^{\pm 1} \) divisible by \( t - 1 \) (since \( \Theta_K (1, 1, 1) = 0 \)) and, hence, divisible by \( (t - 1)^2 \). We define the reduced 2-loop polynomial by

\[
\hat{\Theta}_K(t) = \frac{\Theta_K(t, t^{-1}, 1)}{(t^{1/2} + t^{-1/2})^2} \in \mathbb{Q}[t^{\pm 1}],
\]

which is a symmetric polynomial in \( t^{\pm 1} \). This gives the \( sl_2 \) reduction of the 2-loop polynomial; see Proposition 6.1.

Let us review the cabling formula of the Kontsevich invariant of [4]. Another version of the Kontsevich invariant, called the wheeled Kontsevich invariant [3], is defined by

\[
Z^w(K) = \partial^{-1} \Omega Z^\sigma(K),
\]

where \( \partial : A(\ast) \to A(\ast) \) is the wheeling isomorphism; see [4]. Here, for open Jacobi diagrams \( C \) and \( D \), \( \partial_C(D) \) is defined to be 0 if \( C \) has more univalent vertices than \( D \), and the sum of all ways of gluing all univalent vertices of \( C \) to some univalent vertices of \( D \) otherwise. We graphically present it by

\[
\partial_C(D) = \begin{array}{c}
\circlearrowleft \\
C \\
\circlearrowleft \\
\text{[Diagram]}
\end{array} 
\begin{array}{c}
\circlearrowright \\
D \\
\circlearrowright \\
\text{[Diagram]}
\end{array}.
\]

Let \( \Psi(p) : A(\ast) \to A(\ast) \) be the map which takes a diagram with \( k \) univalent vertices to its \( p^k \) multiple. The \( (p, q) \) cable knot of a knot \( K \) is the knot given by a simple closed curve on the boundary torus of a tubular neighborhood of \( K \) which winds \( q \) times in the meridian direction and \( p \) times in the longitude direction (see e.g. [13]); for example see Figure 1. The cabling formula of the Kontsevich invariant is given by

\[
\text{Proposition 2.1} \quad \text{Let ([4], see also [22]). Let } K \text{ be a framed knot with 0 framing, and let } K^{(p,q)} \text{ be the } (p,q) \text{ cable knot of } K \text{ (with 0 framing). Then,}
\]

\[
Z^w(K^{(p,q)}) = \partial^{-1} \Omega Z^\sigma(K) \sqcup \exp\left( \frac{q}{2p} \right) - \frac{q}{48p} \theta \right) 
\sqcup \exp\left( - \frac{pq}{2} \right) + \frac{pq}{48} \theta.
\]

§3. The 2-Loop Polynomial of a Torus Knot

In this section, we calculate the 2-loop polynomial of a torus knot, picking up the 2-loop part of the primitive part of the cabling formula of the Kontsevich
invariant of the trivial knot. The 2-loop part of the logarithm of the Kontsevich invariant for torus knots is also calculated\textsuperscript{10} independently by Marché [14, 15].

Figure 3. The (5, 3) torus knot

The torus knot $T(p, q)$ of type $(p, q)$ is the $(p, q)$ cable knot of the trivial knot (which is isotopic to $T(q, p)$); for example see Figure 3. It is known, see e.g. [13], that the Alexander polynomial of a torus knot is given by

$$\Delta_{T(p, q)}(t) = \frac{(tpq/2 - t^{-pq/2})(t^{1/2} - t^{-1/2})}{(tp/2 - t^{-p/2})(t^{q/2} - t^{-q/2})}.$$  

Theorem 3.1. The 2-loop polynomial of the torus knot $T(p, q)$ of type $(p, q)$ is given by\textsuperscript{11}

$$\Theta_{T(p, q)}(t_1, t_2, t_3) = -\frac{1}{4} \sum_{i,j,k=1,2,3} \psi_{p,q}(t_i)\psi_{q,p}(t_j)\Delta_{T(p, q)}(t_k) \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]/(t_1t_2t_3 = 1),$$

where $\psi_{p,q}$ is defined by

$$\psi_{p,q}(t) = \Delta_{T(p, q)}(t) \cdot \frac{(tp/2 + t^{-p/2})(t^{1/2} - t^{-1/2})}{tp/2 - t^{-p/2} - q(t^{pq/2} + t^{-pq/2})},$$

$$= (tp/2 - t^{-p/2})(t^{q/2} - t^{-q/2}) \times (tp/2 + t^{-p/2} - t^{-pq/2}).$$

In particular, $\Theta_{T(p, q)}(t_1, t_2, t_3)$ is a polynomial in $t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}$ with integer coefficients of degree $t_1(t_1 - 1)^{(p-1)(q-1)}.$

\textsuperscript{10}Bar-Natan has also obtained some presentation of the wheeled Kontsevich invariant for torus knots (private communication).

\textsuperscript{11}This value coincides with the value in [14, 15]. However, the values of the 2-loop polynomial for some torus knots in Table 2 of [21] have opposite signs to our values. The signs of some values in Table 2 of [21] might not be correct.
Remark. $\psi_{p,q}(t)$ is not a polynomial, but a rational function, while $\Theta_{T(p,q)}(t_1, t_2, t_3)$ is a polynomial. Rozansky [21] suggests that the 2-loop polynomial is a polynomial with integer coefficients; this holds for torus knots by the theorem. He also suggests a conjectural inequality

$$\text{degree}_{t_1}(\Theta_K(t_1, t_2, t_1^{-1}t_2^{-1})) \leq 2g(K),$$

where $g(K)$ denotes the genus of $K$. Since the genus of $T(p,q)$ equals $(p - 1)(q - 1)/2$ (see e.g. [13]), torus knots give the equality of the above formula.

Remark. The $sl_2$ reduction of the $n$-loop part of the primitive part of the Kontsevich invariant is equal to the $n$th line in the expansion of the colored Jones polynomial; see Section 6. Rozansky [19] has calculated it for torus knots.

For group-like elements $\alpha, \beta \in A(*)$ we write $\alpha \equiv \beta$ if $\log_{\Omega_{\alpha}} \alpha - \log_{\Omega_{\beta}} \beta$ is equal to a linear sum of Jacobi diagrams, either, of $(\geq 3)$-loop, or, having a component of a trivalent graph (i.e., a component with no univalent vertices).

**Proof of Theorem 3.1.** Since the torus knot $T(p,q)$ is obtained from the trivial knot by cabling, we have that

$$Z^w(T(p,q)) \equiv \partial^{-1}_\Omega \psi^{(p)} \partial_\Omega (\Omega \cup \exp_{\Omega_{\Omega}}(\frac{q}{2p}\bigcup)) \cup \exp_{\Omega_{\Omega}}(-\frac{pq}{2}\bigcup),$$

by Proposition 2.1. The first term of the right hand side is calculated as follows. From the definition of $\partial_{\Omega}$,

$$\partial_{\Omega} \left( \exp_{\Omega_{\Omega}} \left( \frac{q}{2p} \bigcup \right) \cup \Omega \right) = \Omega \cup \exp_{\Omega_{\Omega}} \left( \frac{q}{2p} \bigcup \right).$$

Since any component of $\Omega$ has a loop, the $(\leq 1)$-loop part of the primitive part of the right hand side has no edges between the two $\Omega$’s, and, hence, the exponential of this part is presented by

$$\partial_{\Omega} \exp_{\Omega_{\Omega}} \left( \frac{q}{2p} \bigcup \right) \cup \Omega.$$

Further, its first term is given by

$$\partial_{\Omega} \exp_{\Omega_{\Omega}} \left( \frac{q}{2p} \bigcup \right) \equiv \exp_{\Omega_{\Omega}} \left( \frac{q}{2p} \bigcup \right) \cup \Omega_{\frac{1}{2}x},$$
where the equivalence is obtained in the same way as Lemma 6.3 of [4], and, as in [4], \( \Omega_{\frac{q}{p}} \) denotes the element obtained from \( \Omega \) by replacing open Jacobi diagrams with \( l \) legs by their \((q/p)^l\) multiples. The 2-loop part of the primitive part of the right hand side of (3.1) is equal to a linear sum of diagrams, each of which has precisely one edge between the two \( \Omega \)'s. Hence, it is presented by

\[
\begin{array}{c}
\begin{array}{c}
\bigcirc & \bigcirc & \exp\left(\frac{q}{2p}\right)
\end{array}
\end{array}
\]

Since

\[
\begin{array}{c}
\begin{array}{c}
\bigcirc & D & = & \bigcirc
\end{array}
\end{array}
\]

the previous diagram is equivalent to

\[
\begin{array}{c}
\begin{array}{c}
\bigcirc & \bigcirc
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
f(x) & f\left(\frac{q}{p}x\right)
\end{array}
\end{array}
\]

where \( f(x) \) is given by

\[
f(x) = \frac{d}{dx} \left( \frac{1}{2} \log \frac{\sinh x/2}{x/2} \right) = \frac{1}{4} \frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}} - \frac{1}{2x}.
\]

Hence, the \((\leq 2)\)-loop part of the primitive part of (3.1) is presented by

\[
\begin{array}{c}
\begin{array}{c}
\partial_{\Omega} \left( \exp\left(\frac{q}{2p}\right) \right) \cup \Omega
\end{array}
\end{array}
\]

\[
\equiv \exp\left(\frac{q}{2p}\right) \cup \Omega \cup \Omega_{\frac{q}{p}} \cup \exp\left(\frac{q}{p}\right).
\]

The map \( \Psi^{(p)} \) sends this to

\[
\begin{array}{c}
\begin{array}{c}
\exp\left(\frac{pq}{2}\right) \cup \Omega_{px} \cup \Omega_{qx} \cup \exp\left(\frac{pq}{2}\right)
\end{array}
\end{array}
\]

Further, \( \partial_{\Omega}^{-1} \) sends this (modulo the equivalence) to

\[
\begin{array}{c}
\partial_{\Omega}^{-1} \left( \exp\left(\frac{pq}{2}\right) \cup \Omega_{px} \cup \Omega_{qx} \right) \cup \exp\left(\frac{pq}{2}\right).
\]

\[
\begin{array}{c}
\begin{array}{c}
f(px) & f(qx)
\end{array}
\end{array}
\]
Its first term is graphically shown as

\[ \exp \left( \frac{pq}{2} \right) \]

\[ \Omega \quad \Omega \]

The 2-loop part of the primitive part of this diagram is calculated similarly as before; for example, when there is precisely one edge between \( \Omega^{-1} \) and \( \Omega_{pq} \), we have the following component,

\[ \exp \left( \frac{pq}{2} \right) \]

\[ f(px) \quad f(pqx) \]

\[ f(qx) \quad f(pqx) \]

Thus, the 2-loop part of the primitive part of (3.3) is equal to

\[ \left( \text{the 2-loop part of the primitive part of } \partial_{\Omega^{-1}} \exp \left( \frac{pq}{2} \right) \right) \]

\[ f(px) \quad f(pqx) \quad f(qx) \quad f(pqx) \]

\[ -p \quad f(pqx) \quad f(pqx) \quad -q \]

\[ = pq \quad f(pqx) \quad f(pqx) \quad -p \quad f(pqx) \quad f(pqx) \]

\[ -q \quad f(pqx) \quad f(pqx) \]

where the equality is obtained from Lemma 3.1 below. Hence, the 2-loop part of the primitive part of \( Z^w(T(p,q)) \) is given by

\[ f(px) \quad f(qx) \]

\[ + pq \quad f(pqx) \quad f(pqx) \]

\[ -p \quad f(pqx) \quad f(pqx) \]

\[ -q \quad f(pqx) \quad f(pqx) \]
\[
\begin{align*}
\phi_{p,q}(t) &= -\frac{1}{8} \phi_{q,p}(t) \\
\phi_{q,p}(t) &= -\frac{1}{8} \phi_{p,q}(t),
\end{align*}
\]

where we put \( t = e^x \) and \( \phi_{p,q} \) is defined by \( \phi_{p,q}(e^x) = 4(f(px) - qf(pqx)) \), that is,

\[
\phi_{p,q}(t) = \frac{p^{p/2} + t^{-p/2}}{p^{p/2} - t^{-p/2}} - q \cdot \frac{p^{q/2} + t^{-q/2}}{p^{q/2} - t^{-q/2}}.
\]

Therefore, from the definition of the 2-loop polynomial, we obtain the required formula.

By Corollary 3.1 below, the degree of \( \hat{\Theta}_{T(p,q)}(t) \) equals \((p - 1)(q - 1) - 1\). Since \((t^{1/2} - t^{-1/2})^2 \hat{\Theta}_{T(p,q)}(t) = \Theta_{T(p,q)}(t, 1, t^{-1}) \) by definition, \( t_1 \)-degree of \( \Theta_{T(p,q)}(t_1, t_2, t_1^{-1} t_2^{-1}) \) is at least \((p - 1)(q - 1)\). We can show that it is exactly \((p - 1)(q - 1)\) in the same way as the proof of Example 1. \(\square\)

**Corollary 3.1.** The reduced 2-loop polynomial of the torus knot \( T(p, q) \) is given by

\[
\hat{\Theta}_{T(p,q)}(t) = \frac{1}{2(t^{1/2} - t^{-1/2})^2} \psi_{p,q}(t) \psi_{q,p}(t)
\]

\[
= \frac{1}{2} \cdot \frac{1}{(t^{p/2} - t^{-p/2})^2} \left( (t^{p/2} + t^{-p/2}) \cdot \frac{t^{p} - t^{-p/2}}{t^{p/2} - t^{-p/2}} - q(t^{p/2} + t^{-p/2}) \right)
\]

\[
\times \frac{1}{(t^{q/2} - t^{-q/2})^2} \left( (t^{q/2} + t^{-q/2}) \cdot \frac{t^{q} - t^{-q/2}}{t^{q/2} - t^{-q/2}} - p(t^{q/2} + t^{-q/2}) \right).
\]

**Lemma 3.1.** For a scalar \( c \),

\[
\partial_{\Omega}^{-1} \exp_{x} \left( \frac{c}{2} \right) \equiv \exp_{x} \left( \frac{c}{2} \right) \sqcup \Omega_{x}^{-1} \sqcup \exp_{x} \left( \frac{f(cx)}{2} \right) \sqcup \exp_{x} \left( \frac{f(cx)}{2} \right).
\]

**Proof.** From the definition of \( \partial_{\Omega} \),

\[
\partial_{\Omega} \left( \exp_{x} \left( \frac{c}{2} \right) \sqcup \Omega_{x}^{-1} \right) = \Omega.
\]
A Cabling Formula for the 2-Loop Polynomial

959

Similarly as in the proof of Theorem 3.1, the $(\leq 1)$-loop part of the primitive part of the right hand side is presented by

$$\partial_{\Omega} \exp_{\alpha}(\frac{c}{2} \bigcup) \sqcup \Omega_{=1}^{-1} \equiv \exp_{\alpha}(\frac{c}{2} \bigcup).$$

Further, the 2-loop part of the primitive part of the right hand side of (3.5) is presented by

$$\omega \exp() \omega \exp_{\alpha}(\frac{c}{2} \bigcup) \equiv -c \exp_{\alpha}(\frac{c}{2} \bigcup).$$

This implies that $\partial_{\Omega}$ takes the right hand side of the formula of the lemma to $\exp_{\alpha}(\frac{c}{2} \bigcup)$.

**Example 1.** For the $(p, 2)$ torus knot, Theorem 3.1 implies that

$$\Theta_{T(p, 2)}(t_1, t_2, t_3) = \frac{1}{(t_1 + 1)(t_2 + 1)(t_3 + 1)} \times \left( \frac{p-1}{2}(t_1^p + t_1^{-p} + t_2^p + t_2^{-p} + t_3^p + t_3^{-p}) - \frac{t_1^{p-1} - t_1^{-(p-1)}}{t_1 - t_1^{-1}} - \frac{t_2^{p-1} - t_2^{-(p-1)}}{t_2 - t_2^{-1}} - \frac{t_3^{p-1} - t_3^{-(p-1)}}{t_3 - t_3^{-1}} \right).$$

For example, the coefficients of $\Theta_{T(7, 2)}(t_1, t_2, t_3)$ are as shown in Table 1. Further,

$$\Delta_{T(p, 2)}(t) = \frac{t^{p/2} + t^{-p/2}}{t^{1/2} + t^{-1/2}}, \quad \psi_{p, 2}(t) = \frac{t^{p/2} - t^{-p/2}}{t^{1/2} + t^{-1/2}},$$

$$\psi_{2, p}(t) = \frac{1}{(t^{1/2} + t^{-1/2})(t^{p/2} + t^{-p/2})} \cdot \left( (t + t^{-1}) \cdot \frac{t^p - t^{-p}}{t - t^{-1}} - p(t^p + t^{-p}) \right).$$
By Theorem 3.1, we obtain \( \Theta \overset{\text{over (}}{\longrightarrow} \) 

Therefore, when \( \{i, j, k\} = \{1, 2, 3\} \), we have that 

\[
\frac{1}{2} \left( \psi_{p, 2}(t_i) \Delta T_{(p, 2)}(t_k) + \psi_{p, 2}(t_k) \Delta T_{(p, 2)}(t_i) \right) = \frac{t_j^{p/2} - t_j^{-p/2}}{(t_i^{1/2} + t_i^{-1/2})(t_k^{1/2} + t_k^{-1/2})}.
\]

Therefore, 

\[
- \frac{1}{4} \psi_{2,p}(t_j) \cdot \left( \psi_{p, 2}(t_i) \Delta T_{(p, 2)}(t_k) + \psi_{p, 2}(t_k) \Delta T_{(p, 2)}(t_i) \right)
\]

\[
= \frac{1}{(t_i^{1/2} + t_i^{-1/2})(t_j^{1/2} + t_j^{-1/2})(t_k^{1/2} + t_k^{-1/2})} \times \frac{1}{2} \left( (p(t_j^p + t_j^{-p}) - (t_j + t_j^{-1}) \cdot \frac{t_j^{p/2} - t_j^{-p/2}}{t_j - t_j^{-1}} \right)
\]

\[
= \frac{1}{(t_i^{1/2} + t_i^{-1/2})(t_j^{1/2} + t_j^{-1/2})(t_k^{1/2} + t_k^{-1/2})} \times \left( \frac{p - 1}{2} (t_j^p + t_j^{-p}) - \frac{t_j^{p-1} - t_j^{-(p-1)}}{t_j - t_j^{-1}} \right).
\]

By Theorem 3.1, we obtain \( \Theta_{T_{(p, 2)}}(t_1, t_2, t_3) \) as the sum of the above formula over \( \{i, j, k\} = \{1, 2, 3\}, \{2, 3, 1\}, \{3, 1, 2\} \), which gives the required formula. 

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Table 1. The non-zero coefficients of \( t_1^p t_2^p \) in \( \Theta_{T(7, 2)}(t_1, t_2, t_1^{-1} t_2^{-1}) \)
Example 2. In a similar way as the previous example, we have that

\[
\Theta_{T(p,3)}(t_1, t_2, t_3) = \frac{(t_1 - 1)(t_2 - 1)(t_3 - 1)}{(t_1^3 - 1)(t_2^3 - 1)(t_3^3 - 1)} \\
\times \left( (p-1)(t_1^p + t_1^{p-2} + t_2 t_1^{p-2} + t_3 t_1^{p-2}) + t_1^{2p} + t_2^{2p} + t_3^{2p} + t_3 t_2^{2p} + t_3^{(p-1)/2} t_2^{(p-1)/2} + t_1^{(p-1)/2} t_2^{(p-1)/2} + t_1^{(p-1)/2} t_3^{(p-1)/2} + t_2^{(p-1)/2} t_3^{(p-1)/2} \right)
\]

and

\[
\hat{\Theta}_{T(p,3)}(t) = \frac{t^3(t^{p/2} + t^{p/2})}{(t^3 - 1)^2} \\
\times \left( (p-1)(p^{2p/2} + t^{-3p/2}) - 2 \cdot \frac{t^{3(p-1)/2} - t^{-3(p-1)/2}}{t^{3/2} - t^{-3/2}} \right)
\]

\[
= \frac{p^{p/2} + t^{-p/2}}{(t^{3/2} - t^{-3/2})^3} \\
\times \left( (p-1)(t^{3(p+1)/2} - t^{-3(p+1)/2}) - (p+1)(t^{3(p-1)/2} - t^{-3(p-1)/2}) \right).
\]

See also Tables 2 and 3 for the values of \(\Theta_{T(p,q)}\) and \(\hat{\Theta}_{T(p,q)}\) for some \((p, q)\).

§4. A Cabling Formula for the 2-Loop Polynomial

In this section, we give a cabling formula for the 2-loop polynomial. We show the formula by picking up the 2-loop part of the primitive part of the cabling formula of the Kontsevich invariant, modifying the proof of Theorem 3.1. This cabling formula is also obtained independently by Marché [15].

It is known, see e.g. [13], that a cabling formula for the Alexander polynomial is given by

\[
\Delta_{K(p,q)}(t) = \Delta_{T(p,q)}(t) \Delta_K(t^p).
\]

A cabling formula for the 2-loop polynomial is given by
(p, q) : The non-zero coefficients of \( t_1^n t_2^m \) in \( \Theta_T(p,q) (t_1, t_2, t_1^{-1} t_2^{-1}) \) in the fundamental domain

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Table 2. The non-zero coefficients of \( t_1^n t_2^m \) in \( \Theta_T(p,q) (t_1, t_2, t_1^{-1} t_2^{-1}) \) in a fundamental domain \( \{0 \leq 2m \leq n\} \) (see [21]) for \( (p, q) \) with \( p \leq 7, q \leq 4 \). The array for each \( (p, q) \) is a subset of the full array such as shown in Table 1 and the most left dot is at \( (n, m) = (0, 0) \). We can recover the other coefficients for each \( (p, q) \) from the presented coefficients by the symmetry of \( \Theta_K(t_1, t_2, t_1^{-1} t_2^{-1}) \).
Table 3. The parts of non-negative powers in $\hat{\Theta}_{T(p,q)}(t)$ for $(p, q)$ with $p \leq 10, q \leq 5$. The remaining part for each $(p, q)$ can recover from the presented part by replacing $t$ with $t^{-1}$.

**Theorem 4.1.** Let $K$ be a knot, and let $K^{(p,q)}$ be the $(p,q)$ cable knot of $K$. Then,

$$
\Theta_{K^{(p,q)}}(t_1, t_2, t_3) = \Theta_{T(p,q)}(t_1, t_2, t_3) + \Theta_K(t^p_1, t^p_2, t^p_3) + \frac{1}{2} \Delta_{T(p,q)}(t_1) \Delta_{T(p,q)}(t_2) \Delta_{T(p,q)}(t_3) 
\times \sum_{\{i,j,k\} = \{1,2,3\}} \Delta_K(t^p_i) \cdot t^p_i \cdot \phi_{q,p}(t_j) \Delta_K(t^p_j) \Delta_K(t^p_k).
$$

**Proof.** We show the theorem, modifying the proof of Theorem 3.1. By Proposition 2.1, we have that

$$Z^w(K^{(p,q)}) \equiv \partial^1_{\Omega^p} \Psi^p(\partial_3 \left( Z^w(K) \sqcup \exp_{\Omega^p} \left( \frac{q}{2p} \int \R \right) \right) \sqcup \exp_{\Omega^p} \left( - \frac{pq}{2} \int \R \right),}$$
where $Z^w(K)$ is presented by
$$Z^w(K) = \Omega \sqcup \exp_\cup \left( -\frac{1}{2} \log \Delta_K(e^x) \right) + \text{(terms of (\geq 2)-loop)}.$$  

The 2-loop part of $\log_\cup Z^w(K)$ contributes to the required formula by $\Theta_K(t^p_i, t^p_j, t^p_k)$. We calculate the contribution from the 1-loop part in the following of this proof.

In a similar way as (3.2), we have that
$$\partial_\Omega \left( Z^w(K) \sqcup \exp_\cup \left( \frac{q}{2p} \right) \right) \equiv \exp_\cup \left( \frac{q}{2p} \right) \sqcup \Omega \sqcup \Omega_\frac{2}{x} \sqcup \exp_\cup \left( -\frac{1}{2} \log \Delta_K(e^x) \right) \left( f(x) + g(x) + f\left( \frac{2x}{p} \right) \right),$$
where $g(x)$ is given by
$$g(x) = \frac{d}{dx} \left( -\frac{1}{2} \log \Delta_K(e^x) \right) = -\frac{\Delta_K'(e^x) \cdot e^x}{2\Delta_K(e^x)}.$$  
The map $\Psi^{(p)}$ sends this to
$$\exp_\cup \left( \frac{pq}{2} \right) \sqcup \Omega_{px} \sqcup \Omega_{qx} \sqcup \exp_\cup \left( -\frac{1}{2} \log \Delta_K(e^{px}) \right) \left( f(px) + g(px) + f(qx) \right),$$
Calculating its image by $\partial_\Omega^{-1}$ in a similar way as in the proof of Theorem 3.1, the error term corresponding to the formula (3.4) is as follows,
$$\begin{align*}
g(px) &\quad f(qx) \\
\phi_{q,p}(t) &\quad -p \\
\frac{1}{4} &\quad \phi_{q,p}(t)
\end{align*}$$

This contributes to the required formula by
$$\sum_{\{i,j,k\} = \{1,2,3\}} \frac{\Delta_K'(t^p_i) \cdot t^p_i}{2\Delta_K(t^p_i)} \cdot \Delta_K(p \cdot \omega)(t_i) \phi_{q,p}(t_j) \Delta_K(p \cdot \omega)(t_j) \Delta_K(p \cdot \omega)(t_k).$$
Noting that \( \Delta_{K(p,q)}(t) = \Delta_{T(p,q)}(t) \Delta_K(t^p) \), we obtain the required formula.

A cabling formula for the reduced 2-loop polynomial is given by

**Corollary 4.1.** For the notation in Theorem 4.1,

\[
\hat{\Theta}_{K(p,q)}(t) = \hat{\Theta}_{T(p,q)}(t) + \frac{(t)p/2 - t-\nu/2)^2}{(t^2 - t^{-1/2})^2} \cdot \hat{\Theta}_K(t^p) \]

\[ - \frac{t^p}{(t^2 - t^{-1/2})^2} \cdot \Delta_{T(p,q)}(t) \Delta_K(t^p) \Delta'_K(t) \psi_{q,p}(t). \]

**Proof.** The required formula is obtained from the formula of Theorem 4.1 by putting \( t_1 = t, \ t_2 = 1/t \), and \( t_3 = 1 \). \( \square \)

§5. Relations to Vassiliev Invariants

In this section we show some relations to Vassiliev invariants of degree 2, 3. A leading part of the Kontsevich invariant is presented by

\[
\log Z^K = \frac{v_2(K)}{2} + \frac{v_3(K)}{4} + \text{(terms of degree } \geq 4) \]

where the degree of a Jacobi diagram is half the number of univalent and trivalent vertices of the diagram, and \( v_2, v_3 \) are \( \mathbb{Z} \)-valued primitive Vassiliev invariants of degree 2, 3 respectively (see [17]). Since \( \hat{\Theta}_K(1) \) can be presented by the Alexander polynomial; in fact, from the formula of the loop expansion,

\[
v_2(K) = -(\text{the coefficient of } x^2 \text{ in the expansion of } \Delta_K(x^2))
\]

\[ = -\frac{1}{2} \Delta''_K(1). \]

Further, since \( \hat{\Theta}_K(1) \) can be presented by the 2-loop polynomial; in fact, we have

**Proposition 5.1.**

\[
v_3(K) = \frac{1}{2} \hat{\Theta}_K(1).
\]

**Proof.** Let us consider the map

\[
\begin{align*}
& \begin{cases}
  f_2(x) \\
  f_1(x)
\end{cases} \\
& \begin{cases}
  f_2(x) \\
  f_1(x)
\end{cases} \\
& \begin{cases}
  f_2(x) \\
  f_1(x)
\end{cases} \\
& \begin{cases}
  f_2(x) \\
  f_1(x)
\end{cases}
\end{align*}
\]

\[
\begin{align*}
& \rightarrow \begin{cases}
  f_3(0) \\
  f_2(0)
\end{cases} \\
& \begin{cases}
  f_2(x) \\
  f_1(x)
\end{cases} \\
& \begin{cases}
  f_2(x) \\
  f_1(x)
\end{cases} \\
& \begin{cases}
  f_2(x) \\
  f_1(x)
\end{cases}
\end{align*}
\]
\[ \mapsto \frac{1}{6} \sum_{\{i,j,k\}=\{1,2,3\}} f_i(x)f_j(-x)f_k(0). \]

This map takes the 2-loop part of \( \log_{\mathfrak{sl}_2} Z^\sigma(K) \) to \( \frac{1}{12}(e^{x/2} - e^{-x/2})^2 \hat{\Theta}_K(e^x)/\Delta_K(e^x)^2 \), whose coefficient of \( x^2 \) equals \( \frac{1}{12} \hat{\Theta}_K(1) \). Since \( \frac{1}{6} v_3(K) = \frac{1}{12} \hat{\Theta}_K(1) \), this implies the required formula.

**Example 3.** A cabling formula for \( v_3 \) is given by
\[ v_3(K^{(p,q)}) = p^2 \cdot v_3(K) + \frac{1}{12} p(p^2 - 1)q \cdot \Delta'_K(1) + \frac{1}{144} p(p^2 - 1)q(q^2 - 1). \]

**Proof.** From Proposition 5.1 and Corollary 4.1 putting \( t = 1 \), we have that
\[ v_3(K^{(p,q)}) = v_3(T(p,q)) + p^2 \cdot v_3(K) - \frac{p}{2} \Delta'_K(1) \phi'_{q,p}(1). \]

The required formula follows from it, by using
\[ v_3(T(p,q)) = \frac{1}{2} \hat{\Theta}_{T(p,q)}(1) = \frac{1}{144} p(p^2 - 1)q(q^2 - 1), \]
\[ \phi'_{q,p}(1) = \frac{1}{6} q(1 - p^2). \]

For the value of the first formula, see also [22].

§6. The \( \mathfrak{sl}_2 \) Reduction of the 2-Loop Polynomial

The aim of this section is to show Proposition 6.1, which implies that the \( \mathfrak{sl}_2 \) reduction of the 2-loop part of the logarithm of the Kontsevich invariant is presented by the reduced 2-loop polynomial.

The loop expansion of the colored Jones polynomial

Let us denote by \( J(L; t) \) the Jones polynomial [8] of a link \( L \) defined by
\[ t^{-1}V\left( \begin{array}{c} \hline \hline \end{array} ; t \right) - tV\left( \begin{array}{c} \hline \hline \end{array} ; t \right) = (t^{1/2} - t^{-1/2})V\left( \begin{array}{c} \hline \hline \end{array} ; t \right) \]

and by the normalization\(^{12}\) \( J(\text{the trivial knot}; t) = t^{1/2} + t^{-1/2} \), where the three pictures in the above formula denote three oriented links, which are identical.

\(^{12}\)This normalization is the normalization of the quantum \( \mathfrak{sl}_2 \) invariant (see e.g. [17]), which differs from the usual normalization where the value of the trivial knot is 1.
A Cabling Formula for the 2-Loop Polynomial

except for a ball, where they differ as shown in the pictures. The colored Jones polynomial [16], which we denote by $J_k(K; t)$, of a knot $K$ is defined by

$$J(K^{(n)}; t) = \sum_{0 \leq k \leq n/2} c_{n,k} J_{n+1-2k}(K; t)$$

where $K^{(n)}$ denotes the disconnected $n$ cable of $K$ with 0 framing, and $c_{n,k}$’s are scalars characterized\(^{13}\) by $V_2^{\otimes n} = \bigoplus_{0 \leq k \leq n/2} c_{n,k} V_{n+1-2k}$; in particular $J_1(K; t) = 1$ and $J_2(K; t) = J(K; t)$. The colored Jones polynomial in another normalization, which we denote by $V_n(K; t)$, is defined by

$$V_n(K; t) = \frac{J_n(K; t)}{J_n(\text{the trivial knot}; t)} = \frac{t^{1/2} - t^{-1/2}}{t^{n/2} - t^{-n/2}} J_n(K; t).$$

As in [19], based on the expansion

$$V_n(K; e^h) = \sum_{l \geq 0} b^l \sum_{k \geq 0} d_{l,k}(nh)^k,$$

the 1-loop and 2-loop parts of the colored Jones polynomial are given by

$$V^{(1\text{-loop})}(K; e^{nh}) = \sum_{k \geq 0} d_{0,k}(nh)^k,$$
$$V^{(2\text{-loop})}(K; e^{nh}) = \sum_{k \geq 0} d_{1,k}(nh)^k,$$

where the right hand sides are rational functions of $e^{nh}$, as discussed in [19]. The aim of this section is to present $V^{(2\text{-loop})}(K; t)$ by the reduced 2-loop polynomial of $K$.

The colored Jones polynomial is obtained from the Kontsevich invariant by\(^{14}\)

$$J_n(K; e^{-h}) = W_{sl_2,V_n}(Z(K)),$$

where $W_{sl_2,V_n}$ denotes the weight system derived from the Lie algebra $sl_2$ and its $n$-dimensional irreducible representation $V_n$, which can be calculated recursively (see [5, 17]) by

$$\begin{align*}
(6.1) & \quad = 2h \left( \begin{array}{c} \hline \rule{0cm}{0.5cm} \hline \end{array} \right) \left( \begin{array}{c} \hline \rule{0cm}{0.5cm} \hline \end{array} \right),
\end{align*}$$

\(^{13}\)This characterization is based on the disconnected cabling formula of quantum invariants (see e.g. [17]). There scalars are concretely presented by $c_{n,k} = \left( \begin{array}{c} n+1 \end{array} \right) - \left( \begin{array}{c} n+1-k \end{array} \right)$.

\(^{14}\)In the left hand side, we put, not $t = e^h$, but $t = e^{-h}$. This difference is derived from the difference of normalization between the colored Jones polynomial and the quantum $sl_2$ invariants.
(6.2) \[
\begin{array}{c}
\otimes \\
\text{sl}_2
\end{array}
= 4h,
\]

(6.3) \[
\begin{array}{c}
\bigtriangledown \\
\alpha \\
\text{sl}_2
\end{array}
= hC \cdot \alpha,
\]

where we write \( \alpha = \beta \) if \( W_{\text{sl}_2,V_n}(\alpha) = W_{\text{sl}_2,V_n}(\beta) \), and \( C \) denotes the Casimir element of \( \text{sl}_2 \), whose eigenvalue on \( V_n \) is equal to \( \frac{n^2 - 1}{2} \). We apply these recursive relations to

\[
Z^w(K) = \frac{Z^w(O)}{\exp_u \left( -\frac{1}{2} \log \Delta_K(e^x) \right)}
\]

\[
+ \sum_{i} \left( \frac{p_i(e^x)/\Delta_K(e^x)}{p_{i,1}(e^x)/\Delta_K(e^x)} + (\geq 3)-\text{loop part} \right).
\]

The 1-loop part

**Lemma 6.1.** For a positive integer \( l \),

\[
\begin{array}{c}
\bigotimes \\
\text{sl}_2
\end{array}
= (2C)^{l/2}h^l(1 + (-1)^l).
\]

**Proof.** If \( l \) is odd, the diagram is equal to 0 by the AS relation, and, hence, the lemma holds. If \( l \) is even, the lemma is proved by induction on \( l \) using (6.1) and (6.3).

Putting \( -\frac{1}{2} \log \Delta_K(e^x) = \sum_{k \geq 0} a_k x^{2k} \), we have that

\[
\exp_u \left( \begin{array}{c}
\bigotimes \\
\text{sl}_2
\end{array} \right) = \sum_{k \geq 0} \exp \left( 2a_k(2C)^kh^{2k} \right)
\]
\[ \equiv \exp \left( 2 \sum_{k \geq 0} a_k (nh)^{2k} \right) = \frac{1}{\Delta_K(e^{nh})}, \]

where we write \( \alpha \equiv \beta \) if \( \log \alpha - \log \beta \) is equal to a linear sum of contributions from \((\geq 3)\)-loop diagrams. Hence,

\[ V^{(1\text{-loop})}(K; t) = \frac{1}{\Delta_K(t)}. \]

This is nothing but the Melvin-Morton-Rozansky conjecture proved in [2].

**The 2-loop part**

**Lemma 6.2.** Let \( l_1, l_2, l_3 \) be non-negative integers such that at least one of them is positive. Then,

\[ L_{l_1, l_2, l_3} = \begin{cases} 0 & \text{if } l_1l_2l_3 \neq 0, \\ 2h(2C)^{l_1+1/2}h^{l_1+1/2}((-1)^{l_1}+(-1)^{l_2}) & \text{if } l_1l_2 \neq 0 \text{ and } l_3 = 0, \\ 4h(2C)^{l_1/2}h^{l_1+1}(1+(-1)^{l_1}) & \text{if } l_1 \neq 0 \text{ and } l_2 = l_3 = 0, \end{cases} \]

where \( \{i, j, k\} = \{1, 2, 3\} \).

**Proof.** We assume that \( l_1 \geq l_2 \geq l_3 \) without loss of generality. If \( l_1 > l_2 = l_3 = 0 \), then the lemma is obtained from (6.2) and Lemma 6.1. If \( l_2 > l_3 = 0 \), then the lemma is obtained from (6.1) and Lemma 6.1. If \( l_3 > 0 \), then we obtain the lemma by induction on \( l_3 \); we can decrease \( l_3 \) by moving one of \( l_3 \) legs to upper edges by the IHX relation. \( \square \)

By Lemma 6.2,

\[ \frac{f_i(x)}{f_j(x)} \equiv 2h \left( f_3(0) \frac{f_j(x)}{f_i(x)} + f_2(0) \frac{f_j(x)}{f_i(x)} + f_1(0) \frac{f_j(x)}{f_i(x)} \right) \]

\[ \equiv 2h \sum_{\{i,j,k\} = \{1,2,3\}} f_i(nh)f_j(-nh)f_k(0). \]

Hence, similarly as in the proof of Proposition 5.1, the \( sl_2 \) reduction of the 2-loop part of \( \log_{\mathbb{E}}(Z^w(K)/Z^w(O)) \) is equal to \( h(e^{nh/2} - e^{-nh/2})^2 \hat{\Theta}_K(e^{nh})/(\Delta_K(e^{nh}))^2 \). Therefore, we obtain
Proposition 6.1.

\[ V^{(2\text{-loop})}(K; t) = -\frac{(t^{1/2} - t^{-1/2})^2}{(\Delta_K(t))^3} \hat{\Theta}_K(t). \]

This gives a concrete presentation of the formula of [19, Conjecture 2] in terms of the reduced 2-loop polynomial.

References