Note on the Spectrum of Some Schrödinger Operators*

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§ 0. Introduction

Recently S. T. Kuroda has developed a new stationary method of perturbation of continuous spectra using the technique of factorization of the perturbation term ([2], [3]). The object of this note is to show that his theory can be applied to $n$-dimensional Schrödinger operators which have first order differentiations with variable coefficients. Namely we consider the differential operator:

$$L = \sum_{j=1}^{n} \left( \frac{\partial}{i\partial x_j} + b_j(x) \right)^2 + q(x)$$

where $b_j$ and $q$ are real valued functions. The case of $n \geq 3$ will be treated in this note. This problem has been already treated by Kuroda ([4]) in the case of $b_j(x) = 0$. In this case our work agrees with his result.

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§ 1. Statement of the Results

Let $H_0$ be the self-adjoint realization of $-\Delta$ in $L^2(\mathbb{R}^n)$, where domain $\mathcal{D}(H_0) = \mathcal{D}^2_{L^2}(\mathbb{R}^n)$. We consider the following conditions:

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For some constants $a$, $p_1$, and $p_2$ satisfying

$$a > \frac{n}{2}, \ 2n > p_1 > n \quad \text{and} \quad 2n > p_2 > \max \left( 2, \frac{n}{2} \right),$$

it holds that

$$\begin{align*}
(1 + |x|^a) b_j(x) & \in L^{p_j}(\mathbb{R}^n) \quad (1 \leq j \leq n), \\
(1 + |x|^a) q'(x) & \in L^{p_2}(\mathbb{R}^n),
\end{align*}$$

where $b_j(x)$ are continuously differentiable functions and

$$q'(x) = \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} b_j(x) + b_j'(x) \right) + q(x).$$

For some $p_3$ with $n > p_3 > 2$, it holds that

$$\begin{align*}
(1 + |x|^a) b_j(x) & \in L^{p_j}(\mathbb{R}^n) \quad (1 \leq j \leq n), \\
(1 + |x|^a) q'(x) & \in L^{p_3}(\mathbb{R}^n).
\end{align*}$$

**Theorem.** (a) Under assumption (I), we have:

The restriction $L$ on $S$, the totality of rapidly decreasing functions, has a unique self-adjoint extension $H_1$, with $\mathcal{D}(H_1) = \mathcal{D}^1_2(\mathbb{R}^n)$. The absolutely continuous part of $H_1$ is unitarily equivalent to $H_0$.

(b) Assuming further (II), we have:

The singular spectrum of $H_1$ consists of non zero eigenvalues of finite multiplicity and possibly zero. Negative eigenvalues have not a finite limiting point. Zero is the only possible finite limiting point for positive eigenvalues if they exist.

Now we resume results of Kuroda. Let $H_j$ ($j = 0, 1$) be self-adjoint in a separable Hilbert space $\mathcal{H}$, $R_j(z) = (H_j - z)^{-1}$ be its resolvent for nonreal $z$. Let $\sigma(H_j)$ be its spectrum. Let $\mathcal{H}_{j,ac}(\mathcal{H}_{j,s})$ be the subspace of absolute continuity (of singularity) with respect to $H_j$. These concepts have been defined in Chapt. X of [1]. Then $\mathcal{H}_{j,ac}$ and $\mathcal{H}_{j,s}$ are closed linear subspaces of $\mathcal{H}$, are orthogonal complements to each other and reduce $H_j$. If $\mathcal{H}_{j,ac} = \mathcal{H}$, $H_j$ is said to be absolutely continuous. $H_{j,ac}(H_{j,s})$ be the restriction of $H_j$ to $\mathcal{H}_{j,ac}(\mathcal{H}_{j,s})$. The set $\sigma(H_{j,ac})(\sigma(H_{j,s}))$ is the absolutely continuous (the singular) spectrum of $H_j$ and is denoted by $\sigma_{ac}(H_j)(\sigma_s(H_j))$. We
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denote by $\mathcal{B}$ the space of all bounded linear operators from $\mathcal{D}$ into $\mathcal{D}$ having the uniform operator topology. For any linear operator $T$ its domain (or range) is represented as $\mathcal{D}(T)$ (or $\mathcal{R}(T)$).

Consider the following conditions on $H_j$.

(K. 1) $\mathcal{D}(H_j)=\mathcal{D}(H_0)=\mathcal{D}$.

(K. 2) There exist linear operators $A$ and $B$ such that: (a) $A$ lies in $\mathcal{B}$, and is invertible with the range being dense in $\mathcal{D}$, (b) $\mathcal{D}(B) \subseteq \mathcal{D}(R_0(z)A)$ for any nonreal $z$, (c) $(H_i-H_0)u=ABu$ for $u \in \mathcal{D}(B) \cap \mathcal{D}$.

(K. 3) $S(z)=A^*\{R_0(z)-R_0(z)\}A$ lies in $\mathcal{B}$, $\lim_{\epsilon \to 0} S(\lambda+i\epsilon)=S(\lambda)$ exists in $\mathcal{B}$, and this convergence is locally uniform in $\lambda$ of the real axis.

(K. 4) $Q(z)=BR_0(z)A$ is in $\mathcal{B}$, and completely continuous for nonreal $z$.

(K. 5) $\lim_{\epsilon \to 0} Q(\lambda \pm i\epsilon)=Q(\lambda \pm i0)$ exists in $\mathcal{B}$ for any real $\lambda$, and $Q(z)$ is a $\mathcal{B}$-valued continuous function on either the upper or the lower half-plane, including the corresponding edge of the real axis.

(K. 6) The operator $\left\{ \frac{1}{2\pi i} S(\lambda) \right\}^{1/2}$ is Hölder continuous with Hölder exponent $\theta>1/2$ on a closed interval $I$ of real axis, and $Q(z)$ is also Hölder continuous with exponent $\theta$ on $I^+$ or $I^-$, where $I^+=\{z: \text{Re} z \in I, \text{Im} z \geq 0\}$ and $I^-=\{z: \text{Re} z \in I, \text{Im} z \leq 0\}$.

We can deduce the following theorem as a corollary of Kuroda’s theory ([2], [3]). The outline of the proof will be sketched in §3.

**Theorem K.** Under conditions (K. 1) to (K. 5) we have:

1° $H_0$ is absolutely continuous.

2° $\sigma_{ac}(H_0)=\sigma(H_0)$, and $\sigma_s(H_0)$ is a closed null set.

3° There exist wave operators $W_\pm \in \mathcal{B}$ such that: $W_\pm^* W_\pm = 1$, $W_\pm W_\pm^* = P_1$, $H_i W_\pm = W_\pm H_0$ and $W_\pm = s\text{-}\lim_{t \to \pm \infty} e^{itH_i} e^{-itH_0}$ where $1$ is the identity operator, and where $P_1$ is the projection to $\mathcal{D}_{1, ac}$.

Assuming further (K. 6), we obtain:

4° $\sigma_s(H_i) \cap I$ consists of at most a countable number of eigenvalues of finite multiplicity which have no accumulation point interior to $I$. 
§ 2. **Proof of the Theorem**

First we write

\[ L = -\Delta + \alpha(x) \left( \sum_{j=1}^{n} \beta_j(x) \frac{\partial}{\partial x_j} + \beta_0(x) \right) \]

where \( \alpha(x) \beta_0(x) = q'(x) = \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} b_j(x) + b_j(x)^2 \right) + q(x) \) and \( \alpha(x) \beta_j(x) = -2ib_j(x) \) \((1 \leq j \leq n)\). Define \((Af)(x) = \alpha(x) \cdot f(x), (Bsf)(x) = \beta_0(x)f(x) (B_jf)(x) = \beta_j(x) \frac{\partial}{\partial x_j} f(x) \) \((1 \leq j \leq n)\) then we have formally \( H_1 = H_0 + AB \) for \( B = \sum_{j=1}^{n} B_j \).

From now on we will take \( \alpha(x) = (1 + |x|)^{-\alpha} \), then the operator \( A \) belongs to \( \mathcal{B} \) and satisfies conditions (a) of (K. 2). Operators \( B_j \) with domains \( \{f(x) | f(x) \in L^2, (B_j f)(x) \in L^2 \} \) have closed extensions, which are also denoted by \( B_j \). For the proof of the first part of (a), we have only to show that \( \mathcal{D}(H_0) \) is contained in \( \mathcal{D}(AB) \) and that for any \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that for any \( u \in \mathcal{D}(H_0) \)

\[ ||ABu|| \leq \varepsilon ||H_0 u|| + c_\varepsilon ||u|| \]

(Chapt. V of [1]).

Assume that \( D(B_j) \supset D(H_0^{\mu_j}) \) for \( \mu_j \) with \( 0 < \mu_j < 1 \), then we have for \( u \in D(H_0) \) and \( k > 1 \)

\[ ||B_j u|| = ||B_j R_0 (H_0 + (-1)^{\mu_j} R_0 (H_0 + k)) u|| \leq ||B_j R_0 (H_0 + (-1)^{\mu_j}) || \cdot ||(H_0 + 1)^{\mu_j} R_0 (-k) || \cdot \{ ||H_0 u|| + k ||u|| \} . \]

By the closed graph theorem, \( B_j R_0 (-1)^{\mu_j} \) is bounded. Noticing that for \( k > 1 \)

\[ ||(H_0 + 1)^{\mu_j} R_0 (-k)|| \leq k^{\mu_j - 1} , \]

we have the estimate that there exists \( C_\varepsilon > 0 \)

\[ ||B_j u|| \leq \varepsilon ||H_0 u|| + C_\varepsilon ||u|| \]

for \( \varepsilon > 0 \) and \( u \in \mathcal{D}(H_0) \). Since \( A \) is bounded, we will obtain the desired statements if \( \mathcal{D}(B_j) \supset \mathcal{D}(H_0^{\mu_j}) \).

Now we will show the validity of this inclusion for \( 1 \leq j \leq n \). Since \( p_i > n \) (condition I), we can choose \( \mu_j \) such that \( p_i > \frac{n}{2 \mu_j - 1} \) and
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<\mu_j<1$. If \( u \in D(H_0^{\mu_j}) \), the Fourier transform of \( u(x) \), \( \hat{u}(\xi) \) satisfies 
\[(1 + |\xi|^2)^{-\mu_j} \hat{u}(\xi) \in L^q(\mathbb{R}^n) \]. Since \( p_i > n \), we have \( \xi_j (1 + |\xi|^2)^{-\mu_j} \hat{u}, \xi_j \hat{u} \in L^q(\mathbb{R}^n) \) for \( \frac{1}{q} = \frac{1}{p_i} + \frac{1}{2} \). Therefore
\[
\frac{\partial}{\partial x_j} u \in L^q \quad \text{with} \quad \frac{1}{q} = 1 - \frac{1}{q} = 1 - \frac{1}{p_1}. \]
As \( \beta_j \in L^{p_1} \) and \( \frac{1}{p_1} + \frac{1}{2} = \frac{1}{2} \), we have \( B_j u = \beta_j \frac{\partial}{\partial x_j} u \in L^2(\mathbb{R}^n) \).

Next we treat the case \( j = 0 \). Since \( \mu_0 > \frac{n}{2} \), we can choose \( \mu_0 \) satisfying \( \mu_0 > \frac{n + 1}{2} \) and \( 0 < \mu_0 < 1 \). Then \( (1 + |\xi|^2)^{-\mu_0} \in L^{p_2}(\mathbb{R}^n) \). As \( \hat{u} = (1 + |\xi|^2)^{-\mu_0} (1 + |\xi|^2)^{\mu_0} \hat{u} \), \( \hat{u} \in L^q(\mathbb{R}^n) \) for \( \frac{1}{q} = \frac{1}{p_2} + \frac{1}{2} \). Therefore \( u \in L^q(\mathbb{R}^n) \) with \( \frac{1}{q} = 1 - \frac{1}{q} = 1 - \frac{1}{p_2} \). And finally we have \( B_0 u = \beta_0 u \in L^2(\mathbb{R}^n) \). Thus the first part of (a) has been proved.

Now we start to check conditions (K.1) to (K.5). For \( r > 0 \) we define
\[
R_n(r, z) = c_n \sqrt{z}^{(n/2)-1} r^{1-(n/2)} H_{(n/2)-1}^{(1)} (\sqrt{z} r) \]
with
\[
c_n = i 2^{-(n/2)-1} r^{1-(n/2)}, \quad \Im \sqrt{z} \geq 0
\]
and the \( \nu \)-th Hankel function of first kind \( H_{(n/2)-1}^{(1)}(\zeta) \). Then it holds that for a nonreal or negative number \( z \),
\[
(H_0 - z)^{-1} f(x) = \int_{\mathbb{R}^n} R_n(|x - y|, z) f(y) dy.
\]

The following asymptotic representations are well known:

(H.1) \[ H_{(n/2)-1}^{(1)}(\zeta) = -\pi^{-1/2} \Gamma(\nu) \zeta^{-\nu} + O(\zeta^{-\nu}) \quad \text{as} \quad |\zeta| \to 0 \quad \text{with} \quad \Im \zeta \geq 0, \]
(H.2) \[ H_{(n/2)-1}^{(1)}(\zeta) = \sqrt{2} (\pi \zeta)^{-(n/2)} e^{i(\zeta - (3\nu - 1)/2) - (\zeta^2/4)} + O(\zeta^{-(n/2)}) \quad \text{as} \quad |\zeta| \to \infty \quad \text{with} \quad \Im \zeta \geq 0. \]

Taking \( \phi(\zeta) \in C^\infty \) such that \( \phi(\zeta) = 1 \) for \( |\zeta| \leq 1 \) and \( \phi(\zeta) = 0 \) for \( |\zeta| \geq 2 \), we have
\[
R_n(r, z) = \phi(\sqrt{z} r) R_n(r, z) + (1 - \phi(\sqrt{z} r)) R_n(r, z) = S_n(r, z) + S_n(r, z).
\]

Let \( \Pi \) be the complex plane which has a cut along the positive real axis from 0 to \( \infty \), including both edges of the cut. Then \( S_n^{(k)}(r, z) \)
can be regarded to be continuous in \((r, z) \in (0, \infty) \times (\Pi \setminus \{0\})\). From (H.1) and (H.2) it holds that:

\[ |S_n^{(1)}(r, z)| \leq \text{const}, \quad S_n^{(3)}(r, z) = 0 \quad \text{for} \quad |\sqrt{z} r| \geq 2, \]

and that:

\[ |S_n^{(2)}(r, z)| \leq \text{const} \cdot |z|^{(n-2)/4}, \quad S_n^{(2)}(r, z) = 0 \quad \text{for} \quad |\sqrt{z} r| \leq 1. \]

By the identity \( \frac{d}{d\xi}(\zeta^{-\nu} H^{(1)}_\nu(\xi)) = -\zeta^{-\nu} H^{(1)}_{\nu+1}(\xi) \), we deduce that for \( r = \left( \sum_{j=1}^n x_j^2 \right)^{1/2} \),

\[
\frac{\partial}{\partial x_j} R_n(r, z) = -\frac{c_n}{c_{n+2}} \frac{x_j}{r} R_{n+1}(r, z)
\]

where \( S_{n,j}(x; z) = -\frac{c_n}{c_{n+2}} \frac{x_j}{r} S_{n+1}(r, z) \) (\( k = 1, 2, 1 \leq j \leq n \)).

These functions satisfy the estimates of the same type as \( S_{n+1}(r, z) \).

Define \( \Pi_N = \{ z : N^{-1} \leq |z| \leq N, \text{Im} z \geq 0 \} \) for \( N > 1 \), and similarly \( \Pi_N \) for \( \text{Im} z \leq 0 \). Noticing that \( S_n^{(1)}(r, z) = r^{n-2} \phi(\sqrt{z} r) R_n(r, z) \) and that \( S_n^{(2)}(r, z) = r^{n-1/2}(1 - \phi(\sqrt{z} r)) R_n(r, z) \), using the identity \( \frac{d}{d\xi}(\zeta^{-\nu} H^{(1)}_\nu(\xi)) = -\zeta^{-\nu} H^{(1)}_{\nu+1}(\xi) \), we have the following estimates:

\[
\left| \frac{\partial}{\partial z} S_n^{(1)}(r, z) \right| \leq c_N, \quad \text{and} \quad \left| \frac{\partial}{\partial z} S_n^{(2)}(r, z) \right| \leq c_N r
\]

for \( z \in \Pi_N \) where \( c_N \) is some constant depending only on \( N \). Therefore we have

\[
\left| \frac{\partial}{\partial z} S_{n,j}^{(1)}(x, z) \right| \leq c'_N, \quad \text{and} \quad \left| \frac{\partial}{\partial z} S_{n,j}^{(2)}(x, z) \right| \leq c'_N r
\]

for \( z \in \Pi_N \) and \( 1 \leq j \leq n \).

Letting \( S_{n,k}(x, z) = S_n^{(k)}(|x|, z) \), we define integral kernels:

\[
Q_{j,k}(x, y; z) = \beta_j(x) S_{n,j}^{(k)}(x - y, z) \alpha(y) |x - y|^{\lambda_{j,k}}
\]

where \( \lambda_{0,j} = n - 2, \lambda_{j,1} = n - 1 (1 \leq j \leq n) \) and \( \lambda_{j,2} = \frac{n - 1}{2} (0 \leq j \leq n) \) and integral operators:
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\[(Q_j^{(\lambda)}(z)f)(x) = \int_{\mathbb{R}^n} Q_j^{(\lambda)}(x, y ; z)f(y)\,dy.\]

Formally we have \(Q(z) = BR_\delta(z) A = \sum_{j=1}^n \sum_{k=1}^n Q_j^{(\lambda)}(z)\). We need:

**Lemma.** For a given \(\lambda\) satisfying \(n > \lambda > 0\), assume that there exist \(p, q\) such that \(\frac{1}{p} + \frac{1}{q} = 1 - \frac{\lambda}{n}\). Consider an integral operator:

\[(K_\lambda f)(x) = \int_{\mathbb{R}^n} \frac{\beta(x) \gamma(x, y) \alpha(y)}{|x-y|^\lambda} f(y)\,dy,\]

where \(\gamma(x, y)\) is continuous and bounded on \(\mathbb{R}^n\).

If \(\alpha(x) \in L^p(\mathbb{R}^n)\) and \(\beta(x) \in L^q(\mathbb{R}^n)\), then \(K_\lambda\) is a bounded operator in \(L^2(\mathbb{R}^n)\) satisfying

\[\|K_\lambda\| \leq C(p, q) \|\gamma\|_\infty \cdot \|\beta\|_q \cdot \|\alpha\|_p\]

where \(\|\gamma\|_\infty = \sup_{x, y} |\gamma(x, y)|\) and \(\|u\|_p = \left(\int_{\mathbb{R}^n} |u(x)|^p\,dx\right)^{1/p}\).

Moreover if we assume further that \(\alpha(x) \in L^{p'}(\mathbb{R}^n)\) and \(\beta(x) \in L^{q'}(\mathbb{R}^n)\) where \(p'\) and \(q'\) satisfy

\[\frac{1}{2} - \frac{\lambda}{n} \leq \frac{1}{p'} < \frac{1}{2} \quad \text{and} \quad \frac{1}{p'} + \frac{1}{q'} < 1 - \frac{\lambda}{n},\]

then \(K_\lambda\) is completely continuous (due to Kuroda, see lemma 5.3 of [4]).

**Proof.** Sobolev’s inequality shows that

\[\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(x)f(y)\,dx\,dy \right| \leq C(P, Q) \|g\|_Q \|f\|_P\]

for \(P > 1, Q > 1, \frac{1}{P} + \frac{1}{Q} > 1\) and \(\lambda = n\left(2 - \frac{1}{P} - \frac{1}{Q}\right)\) ([6]). Let \(\frac{1}{P} = \frac{1}{2} + \frac{1}{p'}\) and \(\frac{1}{Q} = \frac{1}{2} + \frac{1}{q'}\). Substituting the above inequality for \(g = \beta v\) and \(f = \alpha u\) where \(u, v \in L^2(\mathbb{R}^n)\), we have readily the first part of the lemma.

Let \(\chi_N(x)\) be the characteristic function of \(D_N = \{x \mid |x| \leq N\}\). Define \((K_\lambda^{(N,M)}f)(x)\) as

\[\int_{\mathbb{R}^n} \frac{\chi_M(x) \beta(x) \gamma(x, y) \alpha(y) \chi_N(y)}{|x-y|^\lambda} f(y)\,dy.\]

By our assumption on \(p'\) and \(q'\), Kondrašev’s Theorem ([7]) asserts that an integral kernel, \(\frac{\gamma(x, y)}{|x-y|^\lambda}\), determines a completely continuous integral operator from \(L^{p'}(D_N)\) to \(L^{q'}(D_N)\) for \(N < \infty\) with \(\frac{1}{P'} = \frac{1}{p'} + \frac{1}{2}\) and \(\frac{1}{Q'} = \frac{1}{2} + \frac{1}{q'}\). Since we can regard \(\alpha \cdot \chi_N\) as a bounded operator from \(L^2(\mathbb{R}^n)\) to \(L^{p'}(D_N)\) and \(\beta \cdot \chi_N\) as that from \(L^{q'}(D_N)\) to
$L^2(R^n)$, we have that $K_\lambda^{(N,N)}$ is completely continuous from $L^2(R^n)$ to $(L^2(R^n))$. But from the first part of the lemma, we have

$$||K_\lambda - K_\lambda^{(N,N)}|| \leq ||K_\lambda - K_\lambda^{(N,N)}|| + ||K_\lambda^{(N,N)} - K_\lambda^{(N,N)}||$$

$$\leq C(p, q) \{||1 - \chi_N\|_p \|\beta\|_q + ||\chi_N\|_p \|1 - \chi_N\|_q\}$$

$$\to 0 \quad \text{(as N tends to } \infty).$$

Therefore $K_\lambda$ is completely continuous. Thus the lemma has been proved.

In the above lemma, if we take $\alpha(x) = (1 + |x|)^{-a}$ where $a > \frac{n}{2}$, and $\beta(x) \in L^q$ where $q$ satisfies $\frac{1}{2} - \frac{\lambda}{n} < \frac{1}{q} < \frac{1}{2}$ for $\frac{1}{2} - \frac{\lambda}{n} > 0$ and $\frac{1}{q} < 1 - \frac{\lambda}{n}$ for $\frac{1}{2} - \frac{\lambda}{n} \leq 0$, then $\alpha(x) \in L^p$ for any $p$ with $\frac{1}{p} \leq \frac{1}{2}$.

Since we can choose $p$ and $p'$ appropriately, the above lemma asserts that $K_\lambda$ is completely continuous in $L^2(R^n)$. More precisely, if $\beta(x) \in L^q$, where $q_j$ satisfy that:

- for $\lambda_1 = n - 1, \quad q_1 > n$,
- for $\lambda_2 = n - 2$, if $n = 3$, then $6 > q_2 > 2$, if $n > 3$, then $q_2 > \frac{n}{2}$,
- for $\lambda_3 = \frac{n - 1}{2}, \quad 2n > q_3 > 2$,

then $K_\lambda$ are completely continuous. Therefore $Q_j^{(i)}(z)$ are completely continuous if $\beta_0(x) \in L^{q_1} \cap L^{q_2}$ and if $\beta_j(x) \in L^{q_1} \cap L^{q_2}$ ($1 \leq j \leq n$). These being assumed in condition (I), we will obtain (K.4) if the operator $Q(z)$, treated above, coincides with $BR_c(z)A$. We will show this fact. From Sobolev’s inequality it holds that if $f \in L^2$, then

$$\int_{R^n} \frac{\partial}{\partial x_j} R_n(|x - y|, z)\alpha(y)f(y)dy \in L^p \quad \text{for } 1 \geq \frac{1}{p} \geq \frac{1}{2}.$$ 

This implies that $\mathcal{D}(B) \supseteq \mathfrak{N}(R_c(z)A) ((b) \text{ of (K.2)})$ and that our integral operator $Q(z)$ is equal to $BR_c(z)A$.

Naturally we can define $Q(\lambda \pm i0)$ for $\lambda > 0$ as the integral operator with the kernel $\sum_j Q_j^{(i)}(x, y; \lambda \pm i0) = \lim_{\epsilon \to 0} \sum_j Q_j^{(i)}(x, y; \lambda \pm i\epsilon)$ for $x \neq y$. Also $Q(0)$ can be defined by putting $R_n(r, 0) = (n - 2)^{-1}(2\pi)^{-\frac{3}{2}}$
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\[ Q^{CN}(z) \]

We denote by \( Q^{CN}(z) \) the operator which is obtained by the replacing \( \beta_j(x) \) by \( \chi_N(x) \beta_j(x) \) and \( \alpha(x) \) by \( \chi_N(x) \alpha(x) \) in \( Q(z) \). As in the proof of the lemma, we can show \( \lim_{N \to \infty} Q^{CN}(z) = Q(z) \) in \( \mathfrak{B} \) uniformly in \( z \) belonging to a bounded set of either the upper or the lower half-plane including the real axis. The asymptotic formula (H.1) asserts that if \( \text{Im} z \), \( \text{Im} z_1 \geq 0 \) (or \( \leq 0 \)) and if \( |z_1|, |z_2| \) and \( |r| \leq N \), we have \( R_n(r, z_1) - R_n(r, z_2) = o(1) \) uniformly as \( |z_1 - z_2| \) tends to zero. This assures that \( Q^{CN}(z) \) is a \( \mathfrak{B} \)-valued continuous function on \( \text{Im} z \geq 0 \) (or \( \text{Im} z \leq 0 \)). Therefore \( Q(z) \) is a \( \mathfrak{B} \)-valued continuous function, which implies (K.5).

As for (K.3), we must consider integral operators \( A^{(k)}(z) \) with kernels: \( A^{(k)}(x, y; z) = \frac{\alpha(x) S_n^{(k)}(x - y, z) \alpha(y)}{|x - y|^{\lambda_{N,k}}} \) \((k = 1, 2)\). Since \( A^{(k)}(z) \) behave similarly to \( Q_0^{(k)}(z) \), we omit to describe the check of this condition.

Finally we will show that (K.6) holds under conditions (I) and (II). Let \( I \) be a closed interval on the real axis which does not contain zero. Since \( \frac{\partial}{\partial z} S_{n,j}^{(1)}(x, z) \) is bounded in \( z \in \Pi_n^\pm \), \( Q_j^{(1)}(z) \) \((0 \leq j \leq n)\) is Lipschitz continuous on \( I^\pm \). Next we notice that for \( z_1, z_2 \in \Pi_n^\pm \)

\[
\left| S_{n,j}^{(2)}(x, z_1) - S_{n,j}^{(2)}(x, z_2) \right| \leq C_n |r|^{1/2}
\]

since \( \left| \frac{\partial}{\partial z} S_{n,j}^{(2)} \right| \leq C_N |r| \). Putting \( \theta = \frac{1}{2} + \varepsilon \) and \( \lambda_\varepsilon = \frac{n}{2} - 1 - \varepsilon \) \((\varepsilon > 0)\), the integral kernel of \( Q_j^{(2)}(z_1) - Q_j^{(2)}(z_2) \) is estimated by const \( |z_1 - z_2|^{(1/2) + \varepsilon} \beta_j(x) \alpha(y) \) for \( z_1, z_2 \in \Pi_n^\pm \). By the lemma if \( \beta_j \in L^q_\varepsilon \) with \( \frac{1 + \varepsilon}{n} < \frac{1}{q_\varepsilon} < \frac{1}{2} \), the above kernel defines a bounded operator \( \mathfrak{B} \). Since we can choose \( \varepsilon \) arbitrarily small, condition (II) asserts that \( Q_j^{(2)}(z) \) is Hölder continuous on \( \Pi_n^\pm \) with the exponent greater than \( 1/2 \).

The Hölder continuity of \( \left\{ \frac{1}{(2\pi i)} S(\lambda) \right\}^{1/2} \) has been shown by Kuroda in case \( n = 3 \) ([2]). He used the spherical coordinate representation
of $S(\lambda)$. His method is also valid for $n > 3$ if we take the $n$-dimensional spherical harmonics instead of the 3-dimensional.

Since $\lim_{\lambda \to \infty} Q(\lambda) = Q(0)$ holds, applying Theorem 1 of [5] to our case, we know that negative eigenvalues do not accumulate at zero. Another conclusions of our theorem follow from the result of Kuroda.

§ 3. Remarks

1. Outline of the proof of Theorem K.

By (K.1) we have the second resolvent equation:

$$R_i(z) = R_o(z) - R_i(z) (H_i - H_o) R_o(z).$$

Since $A$ is bounded ((a) of (K.2)), it holds that

$$R_o(z)A = R_o(z)A - R_i(z) (H_i - H_o) R_o(z)A.$$

Noticing that $\mathcal{D}(B) \cap \mathcal{D}(R_o(z)A) ((b) of (K.2))$, we have

$$(H_i - H_o) R_o(z)A = ABR_o(z)A \quad ((c) of (K.2))$$

$$= AQ(z) \quad ((K.3)).$$

Therefore the following identity is obtained.

$$(3.1) \quad R_i(z)A(1 + Q(z)) = R_o(z)A.$$

Define $G_0(z) \equiv 1 + Q(z)$ for $\text{Im} \, z \neq 0$. If $u \in \mathcal{D}$ satisfies that $u + Q(z)u = 0$ then $v = R_0(z) Au$ satisfies $H_i v = zw$. Since $\text{Im} \, z \neq 0$, self-adjointness of $H_i$ implies that $v = 0$. Hence $u = 0$. Moreover $Q(z)$ is completely continuous ((K.4)). Therefore there exists a bounded inverse of $(1 + Q(z))$. We write $G_i(z) \equiv (1 + Q(z))^{-1}$ for $\text{Im} \, z \neq 0$. Then we have from (3.1) identities

$$(3.2) \quad \begin{cases} R_o(z) AG_i(z) = R_i(z) A, \\ R_i(z) AG_0(z) = R_o(z) A. \end{cases}$$

Let $S_j(z) \equiv A^* \{ R_j(z) - R_j(\bar{z}) \} A$ for $j = 0, 1$ and $\text{Im} \, z \neq 0$. We have

$$A^* \{ R_j(z) - R_j(\bar{z}) \} A = (z - \bar{z}) A^* R_j(\bar{z}) R_j(z) A$$

$$= (z - \bar{z}) (R_j(z) A)^* R_j(z) A.$$
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\[ (z - \bar{z}) (R_k(z) AG_{j}(z))^* R_k(z) AG_j(z) \quad \text{(by (3.2))} \]

\[ = G_j(z)^* A^* \{ R_k(z) - R_k(\bar{z}) \} AG_j(z) \].

This relation is rewritten as

\[ (3.3) \quad S_j(z) = G_j(z)^* S_k(z) G_j(z) \]

where \((j, k) = (0, 1)\) or \((1, 0)\).

On the other hand it is well known that for \(\alpha\) and \(\beta\) with

\[-\infty \leq \alpha < \beta \leq \infty,\]

\[ \int_a^\beta dE_j(\lambda) = \text{s-lim}_{\epsilon_0} \frac{1}{2\pi i} \int_a^\beta (R_j(\lambda + i\epsilon) - R_j(\lambda - i\epsilon)) d\lambda. \]

From this identity we obtain that for \(u, v \in \mathcal{D},\)

\[ (3.4) \quad \int_a^\beta d(E_j(\lambda) A u, A v) d\lambda = \frac{1}{2\pi i} \text{s-lim}_{\epsilon_0} \int_a^\beta (S_j(\lambda + i\epsilon) u, v) d\lambda. \]

For \(j = 0\) condition (K.3) assures that

\[ \int_a^\beta d(E_0(\lambda) A u, A v) = \frac{1}{2\pi i} \int_a^\beta (S(\lambda) u, v) d\lambda. \]

By the continuity of \(S(\lambda),\) we can conclude that for any measurable set \(\omega\) of the real line

\[ A^* E_0(\omega) A = \frac{1}{2\pi i} \int_\omega S(\lambda) d\lambda \]

(the integration can be performed in the operator topology). This identity implies that \(\mathcal{R}(A)\) is contained in \(\mathcal{D}_{0, a_c}.\) But \(\mathcal{R}(A)\) is dense in \(\mathcal{D}\) and \(\mathcal{D}_{0, a_c}\) is closed. We know \(\mathcal{D}_{0, a_c} = \mathcal{D}.\) This means conclusion 1° of Theorem K.

By (K.5) there exists

\[ G_0(\lambda \pm i\epsilon) = \text{lim}_{\epsilon \uparrow 0} G_0(\lambda \pm i\epsilon) = 1 + Q(\lambda \pm i\epsilon). \]

Since \(G_0(z) - 1 = Q(z)\) is completely continuous for \(\text{Im } z \neq 0\) ((K.4)), \(G_0(\lambda \pm i\epsilon) - 1\) is also completely continuous. And \(G_1(z) = G_0(z)^{-1}\) for \(\text{Im } z \neq 0.\) Therefore lemma 6.2 of [3] (or lemma 5.2 of [2]) assures that there exists a bounded inverse of \(G_0(\lambda \pm i0), G_1(\lambda \pm i0),\) for almost every \(\lambda\) of the real axis. We denote by \(e\) the subset of the real line which consists of points \(\lambda\) such that \(G_0(\lambda \pm i0)\) can not be
invertible. Since $G_0(\lambda \pm i0)$ is continuous in $\lambda$ of the real axis, the complement of $e, e'$ is open. Moreover $G_1(z)$ is continuous on $e'^+$ and $e'^-$ respectively, where $e'^+(e'^-)=\{z : \text{Re } z \in e', \text{ Im } z \geq 0 \}$ (Im $z \leq 0$). Therefore we have for $\lambda \in e$

$$G_1(\lambda \pm i0) = \lim_{\varepsilon \to 0} G_1(\lambda \pm i\varepsilon) \text{ in } \mathcal{B}.$$ 

Define $S_1(\lambda \pm i0)\equiv \pm G_1(\lambda \pm i0)\* S(\lambda) G_1(\lambda \pm i0)$ for $\lambda \in e$. By (3.3) it is obtained that for $\lambda \in e$

$$S_1(\lambda \pm i0) = \lim_{\varepsilon \to 0} S_1(\lambda \pm i\varepsilon) \text{ in } \mathcal{B}.$$ 

If $(\alpha, \beta) \cap e=\phi$, the equality (3.4) implies that for $u, v \in \mathcal{S}$

$$\int_a^b d(E_1(\lambda) Au, Av) = \pm \frac{1}{2\pi i} \int_a^b (S_1(\lambda \pm i0) u, v) d\lambda.$$ 

From this identity we conclude that for any closed set $\omega$ such that $\omega \cap e=\phi$, it holds that

$$A^* E_1(\omega) A = \pm \frac{1}{2\pi i} \int_\omega S_1(\lambda \pm i0) d\lambda.$$ 

Using this relation we can conclude that $\sigma_{ac}(H_1)=\sigma(H_0)$, and that $\sigma_s(H_1)\subset e$. This means conclusion 2 of Theorem K. We can also deduce conclusion 3° of Theorem K as in the proof of Theorems 5.1 to 5.4 of [3] (or Theorem 4.1 of [2]). Finally conclusion 4° of Theorem K follows from Theorem 7.1 of [3] (Professor Kuroda kindly informed the author that the condition of Hölder continuity of $Q(z)$ was missing in [3]).

2. Eigenfunction expansions.

Under conditions (K.1) to (K.5), we have more concrete knowledge concerning the spectral representation of $H_{1,ac}$ by Kuroda's criterion. Consider the following integral equation:

$$\phi^\pm(x, \xi) = e^{ist^2} \int_{R^n} \phi^\pm(y, \xi)$$

$$\times \left\{ 2i \sum_{j=1}^n (b_j(y) \frac{\partial}{\partial y_j} - \frac{\partial b_j(y)}{\partial y_j} + b_j(y)) + q(y) \right\}$$

$$\times R_s(|x-y|; |\xi|^2 \mp i0) dy$$
If $|\xi|^2 \in \sigma_s(H_i)$, we can find the unique $(x, \xi)$-measurable solution of $\varphi^\pm(x, \xi)$ with $\int_{\mathbb{R}^n} (1 + |x|)^{-2\alpha} |\varphi^\pm(x, \xi)|^2\,dx < \infty$. Define $(F^\pm f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) \varphi^\pm(x, \xi)\,dx$ for $f$ with $(1 + |x|)^{\alpha} f \in L^2(\mathbb{R}^n)$. Kuroda's abstract theorem (shown in [2]) asserts that $F$ can be extended to an isometric operator from $\mathcal{B}_1(\mathcal{E})$ onto $L^2(\mathbb{R}^n)$, satisfying $(F^\pm P, E(\lambda)|f)(\xi) = \chi_\lambda(\xi)(F^\pm P, f)(\xi)$ where $\chi_\lambda(\xi)$ is the characteristic function of $\{\xi : |\xi|^2 \leq \lambda\}$. Moreover if it holds that $(1 + |x|)^{\alpha} H_i f \in L^2$ for $f \in \mathcal{B}$, then we have for $|\xi|^2 \in \sigma_s(H_i)$, 
\[ \sum_{j=1}^n \left( \frac{\partial}{\partial x_j} b_j(x) \right)^2 + q(x) \varphi^\pm(x, \xi) = |\xi|^2 \varphi^\pm(x, \xi) \] in the sense of distribution. Some additional information of regularity of $\varphi^\pm(x, \xi)$ will be obtained under appropriate conditions on $b_j$ and $q$.

3. After this work was completed, the author heard the work of Ikebe-Tayoshi ([8]) from Professor Ikebe. They treated a 3-dimensional Schrödinger operator $L = -\Delta + \sum_{j,k=1}^3 \alpha_{j,k}(x) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{i=1}^3 \beta_k \frac{\partial}{\partial x_k} + \gamma$. They have established the unitary equivalence of $L$ and $-\Delta$ chiefly using the condition of $\alpha_{jk}, \beta_k$ and $\gamma \in L^1(\mathbb{R}^3)$. Their method is based on the fact that if $R_1(z) - R_2(z)$ is an operator of the trace class, then the part (a) of our Theorem follows from Kato's criterion (Chapt. X of [1]).

**References**


