Wave Operators for $-\Delta$ in a Domain with Non-Finite Boundary

Dedicated to Professor Atuo Komatu in honor of his 60th birthday

By

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§ 1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a domain (open connected set) exterior to obstacles such that the obstacles, not necessarily finite in number, form a closed set enclosed in a cylinder $S_{r_0} = \{ x = (x_1, \ldots, x_n) = (\bar{x}, x_n) \in \mathbb{R}^n : |x| < r_0, r_0 > 0 \}$. The complement of $S_{r_0}$ is, therefore, contained in $\Omega$. We consider the differential operator $-\Delta$ on $C_0^\infty(\Omega)$, which will be denoted by $A$. It is easy to see that $A$ is a well-defined, non-negative definite operator in the Hilbert space $L_2(\Omega)$, so that it has at least one self-adjoint extension. Let $H$ be any such extension. We are to compare $H$ with the operator $H_0$ in $L_2(\mathbb{R}^n)$ defined as follows: $D(H_0) = \{ u \in L_2(\mathbb{R}^n) : |\xi|^2 \hat{u}(\xi) \in L_2(\mathbb{R}^n) \}$, $(H_0 u)^{\wedge}(\xi) = |\xi|^2 \hat{u}(\xi)$ for $u \in D(H_0)$, where $\hat{u}$ denotes the Fourier transform of $u$, i.e.,

\[ \hat{u}(\xi) = (2\pi)^{-n/2} \text{lim. } \int e^{-i\xi \cdot x} u(x) dx. \]

$H_0$ is also known to be the unique self-adjoint extension of the negative Laplacian defined on $C_0^\infty(\mathbb{R}^n)$. Let $J$ be the bounded linear map: $L_2(\mathbb{R}^n) \to L_2(\Omega)$ defined by

\[ J u = \chi \hat{u}(\xi) \hat{\chi}(\xi) \text{ for } \chi \in C_0^\infty(\mathbb{R}^n), \]

\[ \int |\xi|^2 \hat{u}(\xi) \hat{\chi}(\xi) d\xi = \chi \hat{u}(\xi) \hat{\chi}(\xi) \text{ for } \chi \in C_0^\infty(\mathbb{R}^n). \]

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1) $C_0^\infty(\Omega)$ is the set of all infinitely differentiable functions with compact support in $\Omega$.
2) We may say that different boundary conditions give rise to different $H$.
3) $D(A)$ denotes the domain of $A$.
4) $\text{l.i.m. } \int_{|x| < R} = \text{limit in the mean for } R \to \infty \text{ of } \int_{|x| < R} dx.$
(1.2) \((fu)(x) = u(x)\), \(x \in \Omega\).

Then the wave operator \(W_{\pm} = \pm (H, H_0; J)\) for the pair \((H, H_0)\) and the identification operator \(J\) is defined to be the strong limit

\[(1.3) W_{\pm}(H, H_0; J) = s-lim_{t \to \pm \infty} e^{itH} Je^{-itH_0} \]

if it exists. Now we assert the following

**Theorem.** The wave operators \(W_{\pm}\) exist and are isometries.

The existence of the isometric wave operators \(W_{\pm}\) implies that there is a subspace \(M\) in \(L^2(\Omega)\) reducing \(H\) such that the part of \(H\) in \(M\) is unitarily equivalent with \(H_0\) (see Kato [2]). Consequently, the absolutely continuous spectrum of any self-adjoint extension of \(A\) is never empty and contains at least \([0, \infty)\), since \(H_0\) is known to have the absolutely continuous spectrum \([0, \infty)\). This property is thus independent of whatever (homogeneous) boundary condition may be attached to \(-\Delta\) in \(\Omega\).

In closing this Introduction we mention that the existence and some related properties of the wave operators have been obtained for a bounded (set of) obstacle(s) (see, e.g., Ikebe [1], Lax-Phillips [3] and Shenk [4]).

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§ 2. A Decay Principle

If \(\varphi(x)\) is a (measurable) function defined on \(\mathbb{R}^n\) or \(\Omega\), let us denote by \(\varphi\) the operator of multiplication by \(\varphi(x)\).

**Lemma 2.1.** Let \(\varphi(x)\) be a bounded function on \(\mathbb{R}^n\) such that \supp \((\varphi)^{7} \subset S_r\) for an \(r > 0\). Then for any \(u \in L^2(\mathbb{R}^n)\) we have

\[(2.1) \|\varphi e^{-itH_0}u\|_{L^2(\mathbb{R}^n)} \to 0 \quad (t \to \pm \infty). \]

**Proof.** In order to show (2.1) it is sufficient to prove that (2.1) holds for \(u\) in a fundamental set \(D\), since the operator norm of

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6) See footnote 2).
7) \(\text{supp}(f)\) = support of \(f(x)\).
8) The norm of a Hilbert space \(X\) is designated by \(\|x\|_X\).
\( \varphi \exp (-itH) \) is uniformly bounded in \( t \). Let \( D \) be the totality of such functions \( u \) that \( u(x) = f(x)g(x) \) with \( f \in C_c^\infty (\mathbb{R}^n) \) and \( g \in C_c^\infty (\mathbb{R}^l) \). For \( u = f \cdot g \in D \), we have

\[
(\exp(-itH)u)(\xi) = e^{-it||\xi||^2}f(\xi)\hat{g}(\xi),
\]

which implies

\[
e^{-itH_0}u(x) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{-it||\xi||^2}\hat{f}(\xi)\hat{g}(\xi) d\xi \times
\]

\[
\times \int_{\mathbb{R}^l} e^{ix_n\xi_n - it||\xi_n||^2}\hat{g}_n(\xi_n) d\xi_n.
\]

Fixing \( \xi \) and integrating with respect to \( x_n \) we get

\[
\int_{\mathbb{R}^1} |e^{-itH_0}u(\xi, x_n)|^2 dx_n \leq \text{const.} \varphi(\xi)\|g\|_{L^2(\mathbb{R}^l)}^2 F(\xi, t),
\]

where \( \varphi(\xi) = \sup \{ |\varphi(\xi, x_n)| : x_n \in \mathbb{R}^l \} \) and

\[
F(\xi, t) = \left| \int_{\mathbb{R}^n} e^{ix\cdot\xi - it||\xi||^2}\hat{f}(\xi) \hat{g}(\xi) d\xi \right|^2.
\]

Consequently, noting that \( \varphi(\xi) \) is bounded with compact support in \( \mathbb{R}^{n-1} \), we obtain

\[
||e^{-itH_0}u||_{L^2(\mathbb{R}^{n})} \leq \text{const.} \int_{\text{supp}(\varphi)} F(\xi, t) d\xi.
\]

By the Riemann-Lebesgue lemma \( F(\xi, t) \) tends to 0 as \( |t| \) goes to infinity, and this convergence is uniform in \( \xi \in \text{supp}(\varphi) \). Hence we have the right side of (2.6) tending to 0 in view of the bounded convergence theorem. Q. E. D.

§ 3. Proof of the Theorem

We shall consider \( W_+ \) alone, for \( W_- \) can be handled quite similarly.

Let \( \eta(x) \) be a smooth function on \( \mathbb{R}^n \) satisfying the following conditions: \( 0 \leq \eta(x) \leq 1 \); \( \eta(x) = 1 \) in a neighborhood of the boundary of \( \Omega \); \( \text{supp}(\eta) \subset S_r \) for a sufficiently large \( r \). Put \( \zeta(x) = 1 - \eta(x) \). Then \( W(t) = \exp (-itH)J \exp (-itH_0) \) can be written

\[
W(t) = W_+(t) + W_-(t)
\]
with

\( W_1(t) = e^{itH} \eta e^{-itH_0} \), \( W_2(t) = e^{itH} \zeta e^{-itH_0} \).

Since we have

\( \| W_1(t)u \|_{L^2(\Omega)} \leq \| \eta e^{-itH_0}u \|_{L^2(\Omega)} \leq \| \eta e^{-itH_0}u \|_{L^2(\mathbb{R}^n)} \),

it follows from Lemma 2.1 with \( \varphi = \eta \) that for \( u \in L_2(\mathbb{R}^n) \)

\( \| W_1(t)u \|_{L^2(\Omega)} \to 0 \quad (t \to \infty) \).

In order to show the strong convergence of \( W_2(t) \), we first differentiate \( W_2(t)u \), \( u \in D(H_0) \), obtaining

\( dW_2(t)u/dt = ie^{itH}(H\zeta - J\zeta H_0)e^{-itH_0}u \).

Now (3.5) makes sense. Indeed, \( \exp(-itH_0)u \in D(H_0) \) and \( \zeta(x) \) is smooth and bounded, and hence \( \zeta \exp(-itH_0)u(x) \) is twice strongly differentiable. Since in addition \( \zeta(x) \) vanishes identically near the boundary of \( \Omega \), the application of \( J \) to \( \zeta \exp(-itH_0)u \) does not affect the differentiability, and thus \( J\zeta \exp(-itH_0)u \in D(H) \). On the other hand, \( J\zeta H_0 \exp(-itH_0)u \) is meaningful, for \( J \) and \( \zeta \) are bounded operators. Thus (3.5) holds for \( u \in D(H_0) \). Now since

\( (HJ\zeta - J\zeta H_0)v = -2J(\text{grad } \zeta) \cdot (\text{grad } v) - J(\Delta \zeta) v \)

for \( v = \exp(-itH_0)u \in D(H_0) \), we have on integrating (3.5)

\( W_2(t)u - J\zeta u = -2i \int_0^t e^{isH}(\text{grad } \zeta) \cdot (\text{grad } e^{-isH_0}u)ds - \\
- i \int_0^t e^{isH}(\Delta \zeta)e^{-isH_0}uds \).

If we can show that

\( \int_0^\infty \| (\text{grad } \zeta) \cdot (\text{grad } e^{-itH_0}u) \|dt < \infty \),

(3.9) \( \int_0^\infty \| (\Delta \zeta)e^{-itH_0}u \|dt < \infty \)

for \( u \) in a fundamental set \( D \subset D(H_0) \), then the existence of the strong limit \( W_+ \) will be concluded in virtue of the uniform boundedness in \( t \) of the operator norm of \( W_2(t) \), and of (3.4), (3.1).

As \( D \) we take all functions \( u_a(x) \) for which
(3.10) \[ \tilde{u}_a(\xi) = (\prod_{i=1}^{\infty} \xi_i) \exp \left( -|\xi|^2 - i\xi \cdot a \right), \quad a \in \mathbb{R}^n. \]

Obviously \( D \subset D(H_0) \). That \( D \) is fundamental follows from a theorem of Wiener [5] in view of the fact that \( u_a(x) = u(x - a) \), where \( u(x) \) is a constant multiple of \( \prod_{i=1}^{\infty} x_i \exp (-|x|^2/4) \) which is \( \pm 0 \) almost everywhere. If we put

(3.11) \[ v(x, t; a) = \exp \left[ -|x-a|^2/(4+4it) \right], \]

we have

(3.12) \[ e^{-itH_0} u_a(x) = \text{const.} (1 + it)^{-3n/2} \prod_{i=1}^{\infty} (x_i - a_i) v(x, t; a), \]

(3.13) \[ (\text{grad } e^{-itH_0} u_a(x))_j = \text{const.} (1 + it)^{-3n/2} \prod_{i=j}^{\infty} (x_i - a_i) v(x, t; a) - (2 + 2it)^{-1} (x_j - a_j) v(x, t; a). \]

A straightforward computation shows that

(3.14) \[ |(\Delta \zeta)e^{-itH_0} u_a(x)| \leq \text{const.} \left| 1 + it \right|^{-3(n-1)/2} (\Delta \zeta)^{-1}(\bar{x} - \bar{a}) \left| x - a \right|^{-n-3} \]

(3.15) \[ |(\text{grad } \zeta) \cdot (\text{grad } e^{-itH_0} u_a)(x)| \leq \text{const.} \left| 1 + it \right|^{-3(n-1)/2} (\text{grad } \zeta)^{-1}(\bar{x} - \bar{a}) \left| x - a \right|^{-n-1} \times \]

\[ \times \left| 1 + it \right|^{-3/2} (1 + |1 + it|^{-1} |\bar{x} - \bar{a}|) (|\bar{x} - \bar{a}| + |x_n - a_n|) v(x_n, t; a_n) \].

Noting the inequality

(3.16) \[ \int_{-\infty}^{\infty} \left| 1 + it \right|^{-2p} |x_n|^{2m} |v(x_n, t; a_n)|^2 dx_n \leq \text{const.}, \quad m \geq 0, \]

where \( p \geq m + 1/2 \) and the constant is independent of \( t \), we obtain (3.8) and (3.9) in view of the facts that \( \text{supp } (\Delta \zeta)^{-1} \) and \( \text{supp } (\text{grad } \zeta)^{-1} \) are bounded in \( \mathbb{R}^{n-1} \), and that we have the factor \( |1 + it|^{-3(n-1)/2} \) on the right-hand side of both (3.14) and (3.15). This completes the proof of the existence of \( W_+ \).

It remains to verify the isometry of \( W_+ \). Let \( \chi_s(x) \) denote the characteristic function of \( S \), and let \( CS \) be the complement of \( S \). Then

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9) For the definition of the \( \sim \) operation see just below (2.4).
(3.17) \[ ||W(t)u||_{L^2(\Omega)}^2 = ||J e^{-itH_0}u||_{L^2(\Omega)}^2 = ||\chi_{\Omega} e^{-itH_0}u||_{L^2(\mathbb{R}^n)}^2 \]
\[ = ||u||_{L^2(\mathbb{R}^n)}^2 - ||\chi_{\Omega} e^{-itH_0}u||_{L^2(\mathbb{R}^n)}^2. \]

The last term tends to 0 as \( t \to \infty \) by Lemma 2.1, which proves the desired isometry. Q. E. D.

References