Asymptotic Distribution of Negative Eigenvalues for Three Dimensional Pauli Operators with Nonconstant Magnetic Fields

By

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Abstract

We study the asymptotic distribution of negative eigenvalues of three dimensional Pauli operators with a two dimensional magnetic field and a three dimensional potential which decay to zero at infinity. For λ > 0 sufficiently small, we estimate the number of eigenvalues less than −λ of such Pauli operators.

§1. Introduction

In this paper, we study the asymptotic distribution of negative eigenvalues of three dimensional Pauli operators with a magnetic field and a potential which decay to zero at infinity. Pauli operator is the Hamiltonian of a quantum particle with spin in a magnetic field. The unperturbed Pauli operator is given by

\[ H_0 = (-i\nabla - A)^2 - \sigma \cdot B, \]

and it acts in \( L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \), where \( A: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is a vector potential, \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) is a vector of \( 2 \times 2 \) Pauli operators with components

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

and \( B = \nabla \times A \) is a magnetic field. Throughout this paper, we assume that the direction of the magnetic field is constant. We denote the elements of \( \mathbb{R}^3 \)

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by \((x, z) = (x_1, x_2, z)\). We may assume that the direction of the magnetic field is parallel to the positive \(z\) axis. Then we can show that magnetic field \(B\) is independent of \(z\), and that it has the form
\[
B(x) = (0, 0, b(x)).
\]

Let \(A(x) = (a_1(x), a_2(x), 0)\) be a vector potential associated with \(b(x)\). We assume that \(a_j \in C^1(\mathbb{R}^2)\) \((j = 1, 2)\) is a real valued function. Namely \(b(x) = \partial_1 a_2(x) - \partial_2 a_1(x)\), (where \(\partial_j = \partial/\partial x_j\)). The unperturbed Pauli operator has the form
\[
H_p = \begin{pmatrix}
H_+ - \partial_2^2 & 0 \\
0 & H_- - \partial_2^2
\end{pmatrix},
\]
where
\[
H_\pm = (-i\nabla_x - A)^2 \mp b = \Pi_1^2 + \Pi_2^2 \mp b, \quad \Pi_j = -i\partial_j - a_j \quad (j = 1, 2).
\]

Since \(b = i[\Pi_2, \Pi_1]\), we see
\[
H_\pm = (\Pi_1 + i\Pi_2)^{\ast} (\Pi_1 + i\Pi_2) \geq 0.
\]

Hereafter we discuss the asymptotic distribution of negative eigenvalues of following Pauli operators;
\[
(1.1) \quad H = H_+ - \partial_2^2 - V, \quad H_+ = (-i\nabla_x - A)^2 - b.
\]

We assume that the magnetic field \(b\) and the potential \(V\) satisfy the following Assumptions (b) and (V), respectively:

**Assumption (b).** \(b \in C^1(\mathbb{R}^2)\) and there exist constants \(0 \leq d < 2, C > 1\) such that
\[
(1.2) \quad \frac{1}{C}\langle x \rangle^{-d} \leq b(x) \leq C\langle x \rangle^{-d}, \quad |\nabla b(x)| \leq C\langle x \rangle^{-d-1}.
\]

**Assumption (V).** \(V \in C^1(\mathbb{R}^3)\) and there exist constants \(m > 0, C > 1\) such that
\[
(1.3) \quad \frac{1}{C}\langle x, z \rangle^{-m} \leq V(x, z) \leq C\langle x, z \rangle^{-m}, \quad |\nabla V(x, z)| \leq C\langle x, z \rangle^{-m-1}.
\]

Here we denote \(\langle x \rangle = (1 + |x|^2)^{1/2}, \langle x, z \rangle = (1 + |x|^2 + z^2)^{1/2}\).
Under these assumptions, the operator $H$ given by (1.1) is essentially self-adjoint, and the essential spectrum of $H_+ - \partial^2_z$ and $H$ are $[0, \infty)$.

For self-adjoint operator $T$ and $c \in \mathbb{R}$, we denote the number of eigenvalues less than and greater than $c$ of $T$ by $N(T < c)$, $N(T > c)$, respectively.

The purpose of this work is to estimate the order of $N(H < -\lambda)$ for small $\lambda$. The next theorem is our main result.

**Theorem 1.1.** Assume Assumptions (b) and (V). If $0 < d < m < 2$, $m/2 + d < 2$, then

$$N(H < -\lambda) = F(\lambda)(1 + o(1)), \quad \lambda \to 0,$$

where

$$F(\lambda) = 2(2\pi)^{-2} \int_{\{(x,z) \in \mathbb{R}^3: V(x,z) > \lambda\}} b(x)(V(x,z) - \lambda)^{\frac{1}{2}} \, dx \, dz.$$

In the remainder of this section, we recall several known results.

First, we consider the known results of two dimensional Pauli operators.

**Assumption (V').** $V \in C^1(\mathbb{R}^2)$ and there exist constants $m > 0$, $C > 0$ such that

$$|V(x)| \leq C|x|^{-m}, \quad |\nabla V(x)| \leq C|x|^{-m-1}.$$

Let

$$H' = H_+ - V, \quad H_+ = (-i\nabla_x - A)^2 - b,$$

and we assume that it acts in $L^2(\mathbb{R}^2)$. Following theorem is proved in [5], [6].

**Theorem A ([5], [6]).** Assume Assumptions (b) and (V'). Moreover suppose $V$ satisfies

$$\liminf_{\lambda \downarrow 0} \lambda^{2/d} \int_{V(x) > \lambda} dx > 0,$$

$$\limsup_{\lambda \downarrow 0} \lambda^{2/d} \int_{(1-\delta)\lambda < |V(x)| < (1+\delta)\lambda} \langle x \rangle^{-d} \, dx = o(1), \quad \delta \to 0.$$

Then

$$N(H' < -\lambda) = (2\pi)^{-1} \int_{V(x) > \lambda} b(x) \, dx(1 + o(1)), \quad \lambda \to 0.$$
Concerning three dimensional Pauli operators, following theorem is obtained in [5].

**Theorem B ([5]).** Assume Assumptions (b) and (V). If $d = 0$, $0 < m < 2$, then

$$N(H < -\lambda) = F(\lambda)(1 + o(1)), \quad \lambda \to 0,$$

where $F(\lambda)$ is given by (1.5).

§2. Preliminaries

In this section, we prepare lemmas for the proof of Theorem 1.1.

We first consider following unperturbed Pauli operators in $L^2(\mathbb{R}^2)$:

$$(2.1) \quad \tilde{H}_\pm = (-i\nabla_x - \tilde{A})^2 \mp \tilde{b} = \tilde{\Pi}^2_1 + \tilde{\Pi}^2_2 \pm \tilde{b}, \quad \tilde{\Pi}_j = -i\partial_j - \tilde{a}_j, \quad (j = 1, 2).$$

Assume that $\tilde{b} \in C^1(\mathbb{R}^2)$, and that there exist constants $c, C > 0$ such that

$$c \leq \tilde{b}(x) \leq C.$$

Then, it is known that $\tilde{H}_+$ has zero as an eigenvalue with infinite multiplicity, and that zero is an isolated point of the spectrum of $\tilde{H}_+$ ([1], [8]). As noted in Section 1, we have $\tilde{H}_+ \geq 0$. On the other hand, we see $\tilde{H}_- \geq c > 0$ by (2.1). It is known that the non-zero spectrum of $\tilde{H}_+$ and $\tilde{H}_-$ coincide ([4], Theorem 6.4). Hence $\tilde{H}_-$ has a spectral gap above zero, and the spectral gap is greater than or equal to $c > 0$. Let $P$ be the orthogonal projection on the zero-eigenspace, and let $Q = I - P$. Then we see $Q\tilde{H}_+Q \geq cQ > 0$.

Throughout this section, we assume that the magnetic field $\tilde{b}$ satisfies Assumption (b) with $0 < d < m < 2$, $m/2 + d < 2$. We use a smooth partition of unity $\{\psi_1, \psi_2\}$ on $\mathbb{R}^2$ such that

$$(2.2) \quad \psi_1(x)^2 + \psi_2(x)^2 = 1, \quad x \in \mathbb{R}^2,$$

$$(2.3) \quad \psi_1(x) = 1 \; \text{if} \; |x| \leq 1; \quad \psi_1(x) = 0 \; \text{if} \; |x| \geq 2.$$

We choose $\alpha$ so that

$$(2.4) \quad \frac{1}{m} < \alpha < \frac{1}{d}.$$ 

By Proposition 4.1 of [6], there exists $\phi_0 \in C^2(\mathbb{R}^2)$ such that

$$\Delta \phi_0 = b, \quad |\phi_0(x)| \leq \text{const.} \langle x \rangle^{2-p}, \quad (\forall p < d).$$
Then we set a vector potential \( A(x) = (a_1(x), a_2(x)) \) associated with the magnetic field \( b \) as
\[
a_1(x) = -\partial_2 \phi_0(x), \quad a_2(x) = \partial_1 \phi_0(x).
\]

Let
\[
\phi_\lambda(x) = \phi_0(x) + \eta \lambda^{\alpha d} |x|^2 \psi_2(\lambda^\alpha x),
\]
\[
A_\lambda(x) = (-\partial_2 \phi_\lambda(x), \partial_1 \phi_\lambda(x)),
\]
\[
b_\lambda(x) = \Delta \phi_\lambda(x) = \nabla \times A_\lambda(x).
\]

By Assumption (b), we can choose \( \eta > 0 \) so small that
\[
b_\lambda(x) \geq c_\alpha \lambda^{\alpha d}, \quad c_\alpha > 0.
\]

We assume that a potential \( V \) satisfies Assumption (V). We consider a Pauli operator \( K_\lambda \) in \( L^2(\mathbb{R}^2) \) with the magnetic field \( b_\lambda \) and the potential \( V \):
\[
K_\lambda = K_{+\lambda} - \partial_z^2 - V, \quad K_{+\lambda} = (-i \nabla x - A_\lambda)^2 - b_\lambda.
\]

By (2.8), \( K_{+\lambda} \) has zero as an eigenvalue with infinite multiplicity, and zero is an isolated point of the spectrum of \( K_{+\lambda} \). Moreover it has a spectral gap above zero, and the spectral gap is greater than or equal to \( c_\alpha \lambda^{\alpha d} > 0 \). Let \( P_\lambda \) be the orthogonal projection on the zero-eigenspace, and let \( Q_\lambda = I - P_\lambda \). Then it follows that
\[
Q_\lambda K_{+\lambda} Q_\lambda \geq c_\alpha \lambda^{\alpha d} Q_\lambda > 0.
\]

**Lemma 2.1.** Assume Assumptions (b) and (V). Then for any \( \varepsilon > 0 \) small enough, there exists \( \lambda_\varepsilon > 0 \) such that
\[
N(K_\lambda < -(1 + \varepsilon) \lambda) \leq N(H < -\lambda) \leq N(K_\lambda < -(1 - \varepsilon) \lambda)
\]
for \( 0 < \lambda < \lambda_\varepsilon \).

**Proof.** Let \( \lambda > 0 \), and let \( \psi_1(x), \psi_2(x) \) be the partition of unity defined above. Let
\[
\psi_{\lambda,1}(x, z) = \psi_1(\lambda^\alpha x/2), \quad \psi_{\lambda,2}(x, z) = \psi_2(\lambda^\alpha x/2)
\]
for \( (x, z) \in \mathbb{R}^3 \). By the IMS localization formula ([4], Theorem 3.2), we have
\[
H = \psi_{\lambda,1}(H - \Psi_\lambda)\psi_{\lambda,1} + \psi_{\lambda,2}(H - \Psi_\lambda)\psi_{\lambda,2},
\]
where

\( \Psi_\lambda(x, z) = |\nabla \psi_\lambda,1(x, z)|^2 + |\nabla \psi_\lambda,2(x, z)|^2 = O(\lambda^{2\alpha}) = o(\lambda), \quad \lambda \to 0. \)

By the definition of \( A_\lambda \) and (2.3), \( A_\lambda(x) = A(x) \) for \( |x| < \lambda^{-\alpha} \). Hence

\( H = \psi_\lambda,1(K_\lambda - \Psi_\lambda)\psi_\lambda,1 + \psi_\lambda,2(H - \Psi_\lambda)\psi_\lambda,2. \)

By Assumption (V), \( V(x, z) = O(\lambda^{m\alpha}) = o(\lambda) \) \( \lambda \to 0 \) for \( x \in \text{supp} \psi_\lambda,2 \).

Combining (2.12) with this estimate, for any \( \varepsilon > 0 \) small enough, we learn that there exists \( \lambda_\varepsilon > 0 \) sufficiently small such that for \( 0 < \lambda < \lambda_\varepsilon \),

\[ \psi_\lambda,2(H - \Psi_\lambda)\psi_\lambda,2 = \psi_\lambda,2(H - V - \Psi_\lambda)\psi_\lambda,2 \geq -\varepsilon \lambda. \]

By (2.13), it follows

\[ H \geq \psi_\lambda,1(K_\lambda - \Psi_\lambda)\psi_\lambda,1 - \varepsilon \lambda. \]

Therefore for any \( \varepsilon > 0 \) small enough, there exists \( \lambda_\varepsilon > 0 \) sufficiently small such that for \( 0 < \lambda < \lambda_\varepsilon \),

\[ N(H < -\lambda) \leq N(K_\lambda, D < -(1 - \varepsilon)\lambda) \leq N(K_\lambda < -(1 - \varepsilon)\lambda), \]

where \( K_\lambda, D \) is the operator \( K_\lambda \) with the Dirichlet boundary condition on the domain \( \{ (x, z): |x| < \lambda^{-\alpha} \} \). Similarly, we obtain

\[ N(K_\lambda < -(1 + \varepsilon)\lambda) \leq N(H < -\lambda). \]

The order of \( F(\lambda) \) is computed as follows:

\textbf{Lemma 2.2.} \quad Assume Assumptions (b) and (V). Then for sufficiently small \( \lambda > 0 \),

\[ c\lambda^{\frac{1}{2} + \frac{d}{m} - \frac{m}{2}} \leq F(\lambda) \leq C\lambda^{\frac{1}{2} + \frac{d}{m} - \frac{m}{2}}, \]

where \( c, \ C > 0 \) is constants which is independent of \( \lambda \).

\textbf{Proof.} \quad By Assumptions (b) and (V),

\[
\begin{align*}
\int_{V(x, z) > \lambda} b(x)(V(x, z) - \lambda)^{1/2} \, dx \, dz & \leq \int_{(x, z) < \text{const.}} \lambda^{-1/m} b(x)V(x, z)^{1/2} \, dx \, dz \\
& \leq \text{const.} \int_{(x, z) < \text{const.}} \lambda^{-1/m} \langle x \rangle^{-d} \langle x, z \rangle^{-m/2} \, dx \, dz \\
& \leq \text{const.} \lambda^{-1/m} \int_{\langle x \rangle < \text{const.}} \lambda^{-1/m} \langle x \rangle^{-d} \langle x \rangle^{-m/2} \, dx.
\end{align*}
\]
By simple calculation, it follows that the right hand side is $O(\lambda^{1/2+d/m-3/m})$. Therefore we obtain the second inequality of (2.16).

On the other hand, since we have

$$\int_{V(x,z) > \lambda} b(x)(V(x,z) - \lambda)^{1/2} \, dx \, dz$$

$$\geq \int_{V(x,z) > 2\lambda} b(x)(V(x,z) - \lambda)^{1/2} \, dx \, dz$$

$$\geq \text{const.} \lambda^{1/2} \int_{|x| < |\text{const.} \lambda^{-1/m}} (x)^{-d} \, dx \, dz$$

$$\geq \text{const.} \lambda^{1/2} \cdot \lambda^{d/m} \cdot \lambda^{-3/m},$$

the first inequality of (2.16) follows. \hfill \Box

**Lemma 2.3.** To prove Theorem 1.1, it is sufficient to show that under the assumptions of Theorem 1.1,

$$\limsup_{\lambda \downarrow 0} \frac{N(K_{\lambda} < -\lambda)}{F(\lambda)} \leq 1,$$  \hfill (2.17)

$$\liminf_{\lambda \downarrow 0} \frac{N(K_{\lambda} < -\lambda)}{F(\lambda)} \geq 1.$$  \hfill (2.18)

**Proof.** Suppose (2.17) and (2.18). Then by Lemma 2.1, it follows that for any $\varepsilon > 0$ small enough,

$$\limsup_{\lambda \downarrow 0} \frac{N(H < -\lambda)}{F((1 - \varepsilon)\lambda)} \leq 1,$$  \hfill (2.19)

$$\liminf_{\lambda \downarrow 0} \frac{N(H < -\lambda)}{F((1 + \varepsilon)\lambda)} \geq 1.$$  \hfill (2.20)

On the other hand, for any $\varepsilon > 0$ small enough,

$$F((1 - \varepsilon)\lambda) - F(\lambda)$$

$$= 2(2\pi)^{-2} \int_{V(x,z) > \lambda} b(x)\{(V(x,z) - (1 - \varepsilon)\lambda)^{1/2} - (V(x,z) - \lambda)^{1/2}\} \, dx \, dz$$

$$+ 2(2\pi)^{-2} \int_{(1 - \varepsilon)\lambda < V(x,z) < \lambda} b(x)(V(x,z) - (1 - \varepsilon)\lambda)^{1/2} \, dx \, dz$$

$$= \varepsilon^{1/2}O(\lambda^{1/2+d/m-3/m}),$$  \hfill (2.21)
by Assumptions (b) and (V). Therefore there exists $C > 0$ such that for any $\varepsilon > 0$ and $\lambda > 0$ small enough,

\begin{equation}
\frac{F((1 - \varepsilon)\lambda)}{F(\lambda)} \leq 1 + C\varepsilon^{1/2},
\end{equation}

(2.22)

by Lemma 2.2. Similarly, we can show that there exists $C > 0$ such that for any $\varepsilon > 0$ and $\lambda > 0$ small enough,

\begin{equation}
\frac{F((1 + \varepsilon)\lambda)}{F(\lambda)} \geq 1 - C\varepsilon^{1/2}.
\end{equation}

(2.23)

By (2.19) and (2.22), we obtain

\begin{equation}
\limsup_{\lambda \downarrow 0} \frac{N(H < -\lambda)}{F(\lambda)} \leq 1.
\end{equation}

(2.24)

By (2.20) and (2.23), we also obtain

\begin{equation}
\liminf_{\lambda \downarrow 0} \frac{N(H < -\lambda)}{F(\lambda)} \geq 1.
\end{equation}

(2.25)

These imply the conclusion of Theorem 1.1. 

\section*{3. Proof of Theorem 1.1}

Let $P_\lambda$ be the orthogonal projection on the zero-eigenspace, and let $Q_\lambda = I - P_\lambda$, defined in Section 2. Let $\alpha$ be the constant defined in (2.4). Hereafter we assume

\begin{equation}
0 < d < m < 2, \quad \frac{m}{2} + d < 2.
\end{equation}

(3.1)

To prove Theorem 1.1, we use the next proposition (see Lemma 3.3 and Section 10 of [6]).

**Proposition 3.1** ([6]). Assume that Assumption (b), and suppose that $U \in C^1(\mathbb{R}^2)$ satisfies

\begin{equation}
0 < U(x) \leq C|x|^{-m}, \quad |\nabla U(x)| \leq C|x|^{-m-1},
\end{equation}

(3.2)

where $C > 0$ is a constant independent of $x$. Then for any $\delta > 0$ small enough,
there exists $\lambda_\delta > 0$ such that

\begin{equation}
N(P\lambda UP > \lambda) \leq (2\pi)^{-1} \int_{\{x \in \mathbb{R}^2: U(x) > (1-\delta)\lambda\}} b(x) \, dx + \delta C \frac{2-d}{m} O(\lambda^{-\frac{2-d}{m}}),
\end{equation}

(3.3)

\begin{equation}
N(P\lambda UP > \lambda) \geq 2(2\pi)^{-1} \int_{\{x \in \mathbb{R}^2: U(x) > (1+\delta)\lambda\}} b(x) \, dx - (2\pi)^{-1} \int_{\{x \in \mathbb{R}^2: U(x) > (1-\delta)\lambda\}} b(x) \, dx - \delta C \frac{2-d}{m} O(\lambda^{-\frac{2-d}{m}}),
\end{equation}

(3.4)

for $0 < \lambda < \lambda_\delta$.

\section*{3.1. Proof of (2.17): Upper bound}

In this subsection, we show some lemmas for the upper bound of Theorem 1.1.

The next Propositions 3.2 through 3.4 are obtained in [6] of Lemmas 2.1, 3.1 and 3.2, respectively.

**Proposition 3.2** ([6]). Let $T_1, T_2$ be nonnegative compact self-adjoint operators, and let $\lambda > 0$. Then for any $\delta > 0$ small enough,

\[ N(T_1 + T_2 > \lambda) \leq N(T_1 > (1-\delta)\lambda) + N(T_2 > \delta \lambda). \]

**Proposition 3.3** ([6]). Assume Assumption (b), and suppose $0 < s < 1/m$. Assume that $U(x) = U(x, \lambda) \geq 0$ is a function on $\mathbb{R}^2$ which is uniformly bounded respect to $\lambda$, with support in $\{x \in \mathbb{R}^2: |x| < \lambda^{-s}\}$. Then for any $L > 0$,

\[ N(P\lambda UP > \lambda^L) = o(\lambda^{-\frac{2-d}{m}}), \quad \lambda \to 0. \]

**Proposition 3.4** ([6]). Assume Assumption (b), and suppose that $U \in C^1(\mathbb{R}^2)$ satisfies

\[ |U(x)| \leq \text{const.} \langle x \rangle^{-m}, \quad |\nabla U(x)| \leq \text{const.} \langle x \rangle^{-m-1}. \]

Then

\[ N(H_+ - U < -\lambda) = O(\lambda^{-\frac{2-d}{m}}), \quad \lambda \to 0. \]

Hereafter we identify the operator $P\lambda \otimes I$ acting in $L^2(\mathbb{R}^2_+, \mathbb{R}) = L^2(\mathbb{R}^2_+) \otimes L^2(\mathbb{R}_+)$ with the operator $P\lambda$ acting in $L^2(\mathbb{R}^2_+)$. 
Lemma 3.5. Assume Assumptions (b), (V), and let \( \lambda > 0 \). Then for any \( c > 0 \),

\[
N(K_\lambda < -\lambda) \leq N(-\partial_x^2 - P_\lambda(V + c\lambda^{-\alpha_d}V^2)P_\lambda < -\lambda) + N(Q_\lambda(K_{+,\lambda} - V - c^{-1}\lambda^{\alpha_d})Q_\lambda < -\lambda).
\]

Moreover if \( c > 0 \) is large enough,

\[
N(Q_\lambda(K_{+,\lambda} - V - c^{-1}\lambda^{\alpha_d})Q_\lambda < -\lambda) = o(\lambda^{-\frac{2-\alpha_d}{2m}}), \quad \lambda \to 0.
\]

Proof. It is easy to see

\[
-P_\lambda V Q_\lambda - Q_\lambda V P_\lambda \geq -c^{-1}\lambda^{\alpha_d}Q_\lambda - c\lambda^{-\alpha_d}P_\lambda V^2 P_\lambda,
\]

for any \( c > 0 \). From this, we obtain

\[
N(K_\lambda < -\lambda) \leq N(-\partial_x^2 - P_\lambda(V + c\lambda^{-\alpha_d}V^2)P_\lambda < -\lambda) + N(Q_\lambda(K_{+,\lambda} - V - c^{-1}\lambda^{\alpha_d})Q_\lambda < -\lambda).
\]

Therefore the first statement is proved.

Let \( u_m(x) = (x)^{-m} \). Then there exists \( \beta > 0 \) such that \( V(x, z) \leq \beta u_m(x) \).

By (2.10), we can choose \( c > 0 \) so that

\[
Q_\lambda(K_{+,\lambda} - V - c^{-1}\lambda^{\alpha_d})Q_\lambda \geq Q_\lambda(K_{+,\lambda} - \beta u_m - c^{-1}\lambda^{\alpha_d})Q_\lambda
\]

\[
\geq Q_\lambda \left( \frac{1}{2}K_{+,\lambda} - \beta u_m + c_2\lambda^{\alpha_d} \right) Q_\lambda,
\]

for some \( c_2 > 0 \). Hence

\[
N(Q_\lambda(K_{+,\lambda} - V - c^{-1}\lambda^{\alpha_d})Q_\lambda < -\lambda) \leq N(K_{+,\lambda} - 2\beta u_m < -2c_3\lambda^{\alpha_d}),
\]

for some \( c_3 > 0 \). On the other hand, as in the proof of (2.15), we obtain

\[
N(K_{+,\lambda} - 2\beta u_m < -2c_3\lambda^{\alpha_d}) \leq N(H_+ - 2\beta u_m < -c_3\lambda^{\alpha_d}).
\]

By Proposition 3.4, the right hand side of the above inequality is \( O(\lambda^{-\alpha_d(2-d)/m}) = o(\lambda^{-\alpha_d(2-d)/m}) \). (Note \( \alpha_d < 1 \) by (2.4)). From this and (3.5), the second statement follows.

Since for any \( c > 0 \) large enough,

\[
N(K_\lambda < -\lambda) \leq N(-\partial_x^2 - P_\lambda(V + c\lambda^{-\alpha_d}V^2)P_\lambda < -\lambda) + o(\lambda^{-\frac{2-\alpha_d}{m}}), \quad \lambda \to 0
\]
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(according to Lemma 3.5), it is sufficient to estimate $N(-\partial_z^2 - P_\lambda (V + c\lambda^{-\alpha}dV^2) \times P_\lambda < -\lambda)$. (Note Lemma 2.2 and $0 < d < m < 2$).

Since $m < 2$, we can choose constants $r$ and $\alpha$ such that they satisfy (2.4) and following relations:

(3.7) \hspace{1cm} mr - \alpha d = 0,

(3.8) \hspace{1cm} \frac{1}{2} < r < \frac{1}{m}.

(For example, we may get $\alpha = (1/2d)(m/2 + 1)$ and $r = (1/2)(1/2 + 1/m)$ if $d < m^2$; $\alpha = (1/2)(1/m + 1/d)$ and $r = (1/2m)(d/m + 1)$ if $d \geq m^2$). Hereafter we fix $\delta > 0$ small enough. Let $\{I_k\}_{k \in \mathbb{Z}}$ be a sequence of disjoint open intervals satisfying $|I_k| = \lambda^{-r}$, $\mathbb{R} = \bigcup_{k \in \mathbb{Z}} I_k$. Let $z_k$ be the center of $I_k$, and let

\[ J_k = \left\{ z \in \mathbb{R} : |z - z_k| < \frac{1 + \delta}{2} \lambda^{-r} \right\}. \]

Let $\{\varphi_k\}_{k \in \mathbb{Z}}$ be a smooth partition of unity which satisfies following properties:

(3.9) \hspace{1cm} \sum_{k \in \mathbb{Z}} \varphi_k(z)^2 = 1, \hspace{0.5cm} z \in \mathbb{R},

(3.10) \hspace{1cm} \text{supp} \varphi_k \subset J_k,

(3.11) \hspace{1cm} \left| \frac{d}{dz} \varphi_k(z) \right| \leq C\delta^{-1} \lambda^r.

Let

(3.12) \hspace{1cm} W_\lambda(x, z) = V(x, z) + c\lambda^{-\alpha}dV(x, z)^2.

Let $N_k(t)$ be the number of eigenvalues less than $-t$ of the operator $-\partial_z^2 - P_\lambda W_\lambda P_\lambda$ in $L^2(\mathbb{R}^2 \times J_k)$ with the Dirichlet boundary condition.

**Lemma 3.6.**

(3.13) \hspace{1cm} N(K_\lambda < -\lambda) \leq \sum_{k \in \mathbb{Z}} N_k((1 - \delta)\lambda) + o(\lambda^{-2d/m}), \hspace{0.5cm} \lambda \to 0.

**Proof.** According to (3.11) and (3.8),

\[ \sum_{k \in \mathbb{Z}} \left| \frac{d}{dz} \varphi_k(z) \right|^2 = O(\delta^{-2}) \lambda^{2r} \leq \delta \lambda \]
holds for $\lambda > 0$ small enough. Hence we have

$$
-\partial_z^2 - \lambda \varphi_k \left(-\partial_z^2 - \lambda \sum_{j \in \mathbb{Z}} \left| \frac{d}{dz} \varphi_j \right|^2 \right) \varphi_k \\
\geq \sum_{k \in \mathbb{Z}} \varphi_k (-\partial_z^2 - \lambda \varphi_k) \varphi_k - \delta \lambda,
$$

(3.14)

by the IMS localization formula. (3.13) follows from (3.6) and (3.14). 

To estimate $N_k((1 - \delta)\lambda)$ ($k \in \mathbb{Z}$), we decompose $R_z$ into three parts. According to Assumption (V) and the fact $\omega_0 < 1$, we can choose $M > 0$ so large that if $|z_k| > M\lambda^{-1/m}$,

$$
\sup_{x \in \mathbb{R}^2} W_\lambda(x, z) \leq \frac{\lambda}{2},
$$

uniformly in $z \in J_k$. Then let

$$
\Omega_{1, \lambda} = \{k \in \mathbb{Z} : |z_k| \leq \delta^{-1} \lambda^{-r} \}, \quad \Omega_{2, \lambda} = \{k \in \mathbb{Z} : \delta^{-1} \lambda^{-r} < |z_k| < M\lambda^{-\frac{1}{m}} \},
\quad \Omega_{3, \lambda} = \{k \in \mathbb{Z} : |z_k| \geq M\lambda^{-\frac{1}{m}} \}.
$$

**Lemma 3.7.** Assume Assumption (b), and suppose that $U_1$, $U_2 \in C^1(\mathbb{R}^2)$ satisfy

$$
|U_1(x)| \leq C|x|^{-m}, \quad |\nabla U_1(x)| \leq C|x|^{-m-1},
$$

$$
|U_2(x)| \leq C|x|^{-2m}, \quad |\nabla U_2(x)| \leq C|x|^{-2m-1},
$$

(3.17) (3.18)

where $C > 0$ is a constant independent of $x$. Then for any $\delta > 0$ small enough, there exists $\lambda_\delta > 0$ such that for $0 < \lambda < \lambda_\delta$

$$
N(P_\lambda(U_1 + c\lambda^{-\alpha d}U_2)P_\lambda > \lambda)
\leq (2\pi)^{-1} \int_{\{x \in \mathbb{R}^2 : U_1(x) > (1 - \delta)\lambda\}} b(x) \, dx + \delta C^{\frac{2-ad}{m}} O(\lambda^{-\frac{2-ad}{m}}), \quad (c > 0).
$$

**Proof.** We choose $s$ such that $(\alpha d + 1)/2m < s < 1/m$. By the assumption on $U_2$, we have

$$
\lambda^{-\alpha d} U_2(x) = O(\lambda^{-\alpha d + 2ms}) = o(\lambda), \quad \lambda \to 0
$$

for $|x| > \lambda^{-s}$. Applying Proposition 3.3 to $\chi_{\{|x| \leq \lambda^{-s}\}} U_2$, we learn

$$
N(P_\lambda(c\lambda^{-\alpha d}U_2)P_\lambda > \lambda) = N(P_\lambda U_2 P_\lambda > c^{-1}\lambda^{1+\alpha d}) = o(\lambda^{-\frac{2-ad}{m}}), \quad \lambda \to 0,
$$

(3.19)
where \( \chi_{\{|x| \leq \lambda^{-s}\}} \) is the characteristic function of the set \( \{|x| \leq \lambda^{-s}\} \). Since \( P_\lambda U_1 P_\lambda \) and \( P_\lambda U_2 P_\lambda \) are compact operators, by Proposition 3.2, we see

\[
N(P_\lambda (U_1 + c\lambda^{-\alpha d}U_2)P_\lambda > \lambda) \leq N(P_\lambda U_1 P_\lambda > (1 - \delta)\lambda) + N(P_\lambda (c\lambda^{-\alpha d}U_2)P_\lambda > \delta \lambda)
\]

for \( \delta > 0 \) small enough. By (3.19), the second term of the right hand side is \( o(\lambda^{-(2-d)/m}) \). Applying Proposition 3.1 (3.3) to the first term, we complete the proof.

We begin with the cases \( k \in \Omega_{1,\lambda} \) and \( k \in \Omega_{3,\lambda} \).

**Lemma 3.8.**

\[
\sum_{k \in \Omega_{1,\lambda}} N_k((1 - \delta)\lambda) = o(\lambda^{1/2 + \frac{d}{2m} - \frac{1}{m}}), \quad (\lambda \to 0),
\]

\[
\sum_{k \in \Omega_{3,\lambda}} N_k((1 - \delta)\lambda) = 0.
\]

**Proof.** For \( k \in \Omega_{3,\lambda} \), we have by (3.15),

\[-\partial_z^2 - P_\lambda W_\lambda P_\lambda \geq -\frac{\lambda}{2} > -(1 - \delta)\lambda.\]

These operators are considered in \( L^2(\mathbb{R}^2_z \times J_k) \) with the Dirichlet boundary condition. From this, we learn \( N_k((1 - \delta)\lambda) = 0 \), and hence (3.21) follows.

Next we consider the case \( k \in \Omega_{1,\lambda} \). Let \( u_m(x) = \langle x \rangle^{-m} \). Since

\[W_\lambda(x, z) \leq \beta(u_m(x) + c\lambda^{-\alpha d}u_m(x)^2)\]

for \( z \in J_k \), it follows that

\[
-\partial_z^2 - P_\lambda W_\lambda P_\lambda \geq -\partial_z^2 - \beta P_\lambda (u_m + c\lambda^{-\alpha d}u_m^2)P_\lambda,
\]

in \( L^2(\mathbb{R}^2_z \times J_k) \) with the Dirichlet boundary condition. Let \( \mu_j^{(\lambda)} \) be the \( j \)-th eigenvalue of the operator \( P_\lambda (u_m + c\lambda^{-\alpha d}u_m^2)P_\lambda \in B(L^2(\mathbb{R}^2_z)) \), where \( B(L^2(\mathbb{R}^2_z)) \) is the set of all bounded operators acting in \( L^2(\mathbb{R}^2_z) \). Then the eigenvalues of the operator in the right hand side of (3.22) are

\[
\frac{\ell^2\pi^2}{|J_k|^2} - \beta \mu_j^{(\lambda)}, \quad l \in \mathbb{N}.
\]
Let $\mu > 1/2\beta$. Applying Proposition 3.7 with $U_1 = u_m/\mu$, $U_2 = u_m^2/\mu$, for any $\varepsilon > 0$, we learn that there exists $\lambda > 0$ so small that

\[ N(P_\lambda(u_m + c\lambda^{-\alpha_d}u_m^2)P_\lambda > \mu \lambda) \leq (2\pi)^{-1} \int_{\{x \in \mathbb{R}^2 : u_m(x) > (1-\varepsilon)\mu \lambda\}} b(x) \, dx + \varepsilon(\mu \lambda)^{-\frac{2-\alpha_d}{m}} \leq \text{const.} / (\mu \lambda)^{-\frac{2-\alpha_d}{m}}, \quad (c > 0). \]

Since, this implies that $\mu_j^{(\lambda)} \leq \text{const.} j^{-m/(2-d)}$, there exists $p > 0$ such that

\[ \sum_{j \geq p} \lambda_j \leq \text{const.} \lambda_j^{-\frac{2-\alpha_d}{m}}. \]

(3.23)

Next we consider the case $k \in \Omega_{2,\lambda}$. Let $v_k(x) = V(x, z_k)$. By Assumption (V) (3.1), and the relation $mr - \alpha_d = 0$, it follows that

\[ W_\lambda(x, z) = v_k(x)(1 + O(\delta)) \]

for $z \in J_k$, uniformly in $x \in \mathbb{R}^2$. Thus there exists $\beta > 0$ such that

\[ -\partial_z^2 P_\lambda W_\lambda P_\lambda \geq -\partial_z^2 - (1 + \beta \delta)P_\lambda v_k P_\lambda, \]

in $L^2(\mathbb{R}_2^2 \times J_k)$ with the Dirichlet boundary condition.

Lemma 3.9.

(3.25)
where

\[(3.26) \quad G_{k,1}(\lambda) = \frac{|J_k|}{\pi} \int_{\eta_k(\lambda)}^{R(\nu-1)}((1+\beta\delta)\nu-(1-\delta)\lambda)^{1/2} dg_k((1-\delta)\nu),\]

\[(3.27) \quad \eta_k(\lambda) = \frac{1-\delta}{1+\beta\delta},\]

\[(3.28) \quad g_k(\nu) = (2\pi)^{-1} \int_{v_k(x)>(1-\delta)\nu} b(x) dx.\]

\[\text{Proof. Let } \nu_{k,j}^{(\lambda)} \text{ be the j-th eigenvalue of the operator } P_\lambda v_k P_\lambda \in B(L^2(\mathbb{R}^2)). \text{ Then the eigenvalues of the right hand side of (3.24) are}\]

\[l^2 \frac{\pi^2}{|J_k|^2} - (1 + \beta\delta)\nu_{k,j}^{(\lambda)}, \; l \in \mathbb{N}.\]

Let \(\nu > 1/(\beta + 2)\). We apply (3.3) in Proposition 3.1 to \(v_k/\nu\). Then for any \(\varepsilon > 0\), we can choose \(\lambda_\varepsilon > 0\) such that for \(0 < \lambda < \lambda_\varepsilon\),

\[(3.29) \quad N(P_\lambda v_k P_\lambda > \nu \lambda) \leq (2\pi)^{-1} \int_{\nu_k(x) > (1-\varepsilon)\nu \lambda} b(x) dx + \varepsilon O((\nu \lambda)^{-\frac{2-d}{m}})\]

Thus we see that \(\nu_{k,j}^{(\lambda)} \leq \text{const.} j^{-m/(2-d)}\), and that there exists \(p > 0\) such that

\[(3.30) \quad \frac{\ell^2 \pi^2}{|J_k|^2} - (1 + \beta\delta)\nu_{k,j}^{(\lambda)} > -(1-\delta)\lambda\]

for \(j > p\lambda^{-2/(2-d)/m}\). Therefore

\[N_k((1-\delta)\lambda) \leq \sum_{j=1}^{p\lambda^{-\frac{2-d}{m}}} \left( \frac{|J_k|}{\pi}((1+\beta\delta)\nu_{k,j}^{(\lambda)} - (1-\delta)\lambda)^{1/2} + 1 \right).\]

Let \(m_k^{(\lambda)}(\nu) = N(P_\lambda v_k P_\lambda > \nu \lambda)\). Since, by Assumption (V), there exists \(R > 0\) large enough such that \(m_k^{(\lambda)}(\nu) = 0\) for all \(\nu > R\lambda^{-m(1-m)}\), we see that

\[(3.31) \quad N_k((1-\delta)\lambda) \leq -\frac{|J_k|}{\pi} \int_{\zeta_\delta}^{R(\nu-1)}((1+\beta\delta)\nu \lambda - (1-\delta)\lambda)^{1/2} dm_k^{(\lambda)}(\nu) + O(\lambda^{-\frac{2-d}{m}}),\]

where

\[(3.32) \quad \zeta_\delta = \frac{1-\delta}{1+\beta\delta} = 1 + O(\delta).\]
Let
\[ G_k(\lambda) = -\frac{|J_k|}{\pi} \int_{\zeta_k}^{R\lambda^{-(1-r)m}} ((1 + \beta\delta)\nu \lambda - (1 - \delta)\lambda)^{1/2} \, d\nu_k^{(\lambda)}(\nu). \]

Since \( \Omega_{2,\lambda} = O(\lambda^{-(1/m)+r}) \) and \( r > 1/2 \), we see
\[ \sum_{k \in \Omega_{2,\lambda}} N_k((1 - \delta)\lambda) \leq \sum_{k \in \Omega_{2,\lambda}} G_k(\lambda) + O(\lambda^{\frac{1}{m} + \frac{d}{2} + r}) \]
\[ = \sum_{k \in \Omega_{2,\lambda}} G_k(\lambda) + o(\lambda^{\frac{1}{m} + \frac{d}{2} + r}), \]
by (3.31). It follows that
\[ m_k^{(\lambda)}(\nu) \leq g_k((1 - \delta)\lambda\nu) + \delta O((\nu\lambda)^{-\frac{2-d}{m}}), \]
by (3.29). Let
\[ f(\nu) = -\frac{|J_k|}{\pi}((1 + \beta\delta)\nu \lambda - (1 - \delta)\lambda)^{1/2}. \]
By integration by parts, we see
\[ G_k(\lambda) \leq -\frac{|J_k|}{\pi} \int_{\zeta_k}^{R\lambda^{-(1-r)m}} ((1 + \beta\delta)\nu \lambda - (1 - \delta)\lambda)^{1/2} \, d\nu_k^{(\lambda)}((1 - \delta)\lambda\nu) \]
\[ - O(\delta)\lambda^{-\frac{1}{2}d} \int_{\zeta_k}^{R\lambda^{-(1-r)m}} \nu^{-\frac{1}{m}d} \, df(\nu) \]
\[ \leq G_{k,1}(\lambda) - O(\delta)\lambda^{-\frac{1}{2}d} \lambda^{(1-r)m}\int_{\zeta_k}^{R\lambda^{-(1-r)m}} f(\nu) \, d\nu^{-\frac{1}{m}d} \]
\[ + O(\delta)\lambda^{-\frac{1}{2}d} \int_{\zeta_k}^{R\lambda^{-(1-r)m}} f(\nu) \, d\nu^{-\frac{1}{m}d} \]
\[ = I + II + III, \]
from (3.35). Since \( m/2 + d < 2 \) and \( r < 1/m \),
\[ II = O(\delta)\lambda^{\frac{1}{2}d + \frac{d}{m} - \frac{2}{m}} \lambda^{-(2-d-\frac{d}{m})r + \frac{2-d}{m}} \]
\[ = \delta O(\lambda^{\frac{1}{2}d + \frac{d}{m} - \frac{2}{m} - r}), \]
\[ III = O(\delta)\lambda^{\frac{1}{2}d + \frac{d}{m} - \frac{2}{m} - r} \int_{\zeta_k}^{R\lambda^{-(1-r)m}} \nu^{\frac{3}{2}} \nu^{-\frac{1}{m}d - 1} \, d\nu \]
\[ = \delta O(\lambda^{\frac{1}{2}d + \frac{d}{m} - \frac{2}{m} - r}). \]
Hence we obtain
\[ G_k(\lambda) \leq G_{k,1}(\lambda) + \delta O(\lambda^{\frac{1}{2}d + \frac{d}{m} - \frac{2}{m} - r}). \]
Since \( \Omega_{2,\lambda} = O(\lambda^{-(1/m)+r}) \), (3.25) follows from (3.34) and (3.37). \( \square \)
Lemma 3.10.

\begin{equation}
\sum_{k \in \Omega_{2,\lambda}} G_{k,1}(\lambda) \leq F(\lambda)(1 + O(\delta^{1/2})).
\end{equation}

**Proof.** By (3.28), (3.26) and the definition of Stieltjes integral, we see

\begin{equation}
G_{k,1}(\lambda) = 2(2\pi)^{-2} |J_k| c_\delta \int_{v_k(x) > (1-\delta)\eta_{\delta}(\lambda)} b(x)(v_k(x) - (1-\delta)\eta_{\delta}(\lambda))^{1/2} \, dx,
\end{equation}

where

\begin{equation}
c_\delta = \left( \frac{1 + \beta\delta}{1 - \delta} \right)^{1/2} = 1 + O(\delta).
\end{equation}

To estimate the integral in the right hand side of (3.39), we decompose the integral as follows;

\begin{equation}
\int_{v_k(x) > (1-\delta)\eta_{\delta}(\lambda)} b(x)(v_k(x) - (1-\delta)\eta_{\delta}(\lambda))^{1/2} \, dx
\end{equation}

\begin{align*}
&= \int_{v_k(x) > \lambda} b(x)(v_k(x) - \lambda)^{1/2} \, dx \\
&\quad + \int_{v_k(x) > \lambda} b(x) \left\{ (v_k(x) - (1-\delta)\eta_{\delta}(\lambda))^{1/2} - (v_k(x) - \lambda)^{1/2} \right\} \, dx \\
&\quad + \int_{(1-\delta)\eta_{\delta}(\lambda) < v_k(x) \leq \lambda} b(x)(v_k(x) - (1-\delta)\eta_{\delta}(\lambda))^{1/2} \, dx \\
&= I' + II' + III'.
\end{align*}

By (3.27),

\begin{align*}
II' &\leq \int_{v_k(x) > \lambda} b(x) \left\{ (v_k(x) - (1-C\delta)\lambda)^{1/2} - (v_k(x) - \lambda)^{1/2} \right\} \, dx \\
&= \int_{v_k(x) > \lambda} b(x) \left( \int_0^1 \frac{C\delta \lambda}{2(v_k(x) - (1-C\delta t)\lambda)^{1/2}} \, dt \right) \, dx \\
&\leq C\delta \lambda \int_{v_k(x) > \lambda} b(x) \left( \int_0^1 \frac{1}{2(C\delta \lambda t)^{1/2}} \, dt \right) \, dx \\
&= O(\delta^{1/2}) \lambda^{1/2} \int_{v_k(x) > \lambda} b(x) \, dx \\
&= \delta^{1/2} O(\lambda^{1/2 + \delta} - \delta^{1/2} \lambda).
\end{align*}
We also have
\[
III' \leq \int_{(1-C\delta)\lambda < v_k(x) \leq \lambda} b(x)(v_k(x) - (1 - C\delta)\lambda)^{1/2} \, dx \\
= O(\delta^{1/2})\lambda^{1/2} \int_{(1-C\delta)\lambda < v_k(x) \leq \lambda} b(x) \, dx \\
= \delta^{1/2}O(\lambda^{\frac{1}{2} + \frac{d}{2} - \frac{d}{2}}).
\]

Here we used the estimate
\[
\int_{v_k(x) > \lambda} b(x) \, dx = O(\lambda^{-\frac{d-2}{2}}). \tag{3.42}
\]
which follows from Assumptions (b) and (V). Hence we see
\[
(3.41) \leq \int_{v_k(x) > \lambda} b(x)(v_k(x) - \lambda)^{1/2} \, dx + \delta^{1/2}O(\lambda^{\frac{1}{2} + \frac{d}{2} - \frac{d}{2} - r}).
\]

Since \(|J_k| = \lambda^{-r}(1 + O(\delta))\), we obtain
\[
G_{k,1}(\lambda) \leq 2(2\pi)^{-2}|I_k| \int_{v_k(x) > \lambda} b(x)(v_k(x) - \lambda)^{1/2} \, dx + \delta^{1/2}O(\lambda^{\frac{1}{2} + \frac{d}{2} - \frac{d}{2} - r}) \tag{3.43}
\]
from (3.39) and (3.40).

Next we estimate the right hand side of (3.43). It follows from Assumption (V) that
\[v_k(x) = V(x, z)(1 + O(\delta))\]
for \(z \in I_k\). Thus we see
\[
|I_k| \int_{v_k(x) > \lambda} b(x)(v_k(x) - \lambda)^{1/2} \, dx \\
= \int_{I_k} \int_{v_k(x) > \lambda} b(x)(v_k(x) - \lambda)^{1/2} \, dx \, dz \\
\leq \int_{\{V(x, z) > \lambda, z \in I_k\}} b(x)((1 + C\delta)V(x, z) - \lambda)^{1/2} \, dx \, dz \\
= \int_{\{V(x, z) > \lambda, z \in I_k\}} b(x)(V(x, z) - \lambda)^{1/2} \, dx \, dz \\
+ \int_{\{V(x, z) > \lambda, z \in I_k\}} b(x)\left\{((1 + C\delta)V(x, z) - \lambda)^{1/2} - (V(x, z) - \lambda)^{1/2}\right\} \, dx \, dz \\
+ \int_{\{V(x, z) \leq \lambda, z \in I_k\}} b(x)((1 + C\delta)V(x, z) - \lambda)^{1/2} \, dx \, dz.
\]
Since

\( \int_{V(x,z) > \lambda} b(x)V(x,z)^{1/2} \, dx = O(\lambda^{\frac{1}{2} + \frac{d}{m} - \frac{r}{m}}) \)

uniformly in \( z \in I_k \) (this is easily seen from Assumptions (b) and (V)), the second and the third term in the right hand side of (3.44) are bounded by the same computation as in the estimate of the second and the third term in the right hand side of (3.41). Therefore

\[
\text{(the LHS of (3.44))} \leq \iint \{V(x,z) > \lambda, z \in I_k\} b(x)(V(x,z) - \lambda)^{1/2} \, dx \, dz + \delta^{1/2}O(\lambda^{\frac{1}{2} + \frac{d}{m} - \frac{r}{m}}) .
\]

Combining (3.43) with this estimate, we obtain

\[
G_{k,1}(\lambda) \leq 2(2\pi)^{-2} \iint \{V(x,z) > \lambda, z \in I_k\} b(x)(V(x,z) - \lambda)^{1/2} \, dx \, dz + \delta^{1/2}O(\lambda^{\frac{1}{2} + \frac{d}{m} - \frac{r}{m}}) .
\]  

Since \( 2\Omega_{2,\lambda} = O(\lambda^{-1/m + r}) \), we see

\[
\sum_{k \in \Omega_{2,\lambda}} G_{k,1}(\lambda) \leq \sum_{k \in \Omega_{2,\lambda}} 2(2\pi)^{-2} \iint \{V(x,z) > \lambda, z \in I_k\} b(x)(V(x,z) - \lambda)^{1/2} \, dx \, dz + \delta^{1/2}O(\lambda^{\frac{1}{2} + \frac{d}{m} - \frac{r}{m}}) .
\]

On the other hand, it is easily seen

\[
\sum_{k \in \Omega_{1,\lambda}} \iint \{V(x,z) > \lambda, z \in I_k\} b(x)(V(x,z) - \lambda)^{1/2} \, dx \, dz = o(\lambda^{\frac{1}{2} + \frac{d}{m} - \frac{r}{m}}) ,
\]

\[
\sum_{k \in \Omega_{3,\lambda}} \iint \{V(x,z) > \lambda, z \in I_k\} b(x)(V(x,z) - \lambda)^{1/2} \, dx \, dz = 0 .
\]

Therefore from (3.47) and Lemma 2.2, it follows

\[
\sum_{k \in \Omega_{2,\lambda}} G_{k,1}(\lambda) \leq F(\lambda) + \delta^{1/2}O(\lambda^{\frac{1}{2} + \frac{d}{m} - \frac{r}{m}}) \leq F(\lambda)(1 + O(\delta^{1/2})).
\]

The next lemma follows immediately from Lemmas 3.9 and 3.10.
Lemma 3.11.
\[
\sum_{k \in \Omega_{2, \lambda}} N_k((1 - \delta)\lambda) \leq F(\lambda)(1 + O(\delta^{1/2}))
\]  

Proof of the upper bound (2.17). It follows from Lemmas 3.6, 3.8 and 3.11 that
\[
N(K_\lambda < -\lambda) \leq F(\lambda)(1 + O(\delta^{1/2})) + o(\lambda^{-\frac{2d}{m}}), \quad \lambda \to 0.
\]  
Since \(\delta\) is arbitrary, (2.17) follows from Lemma 2.2.

§3.2. Proof of (2.18): Lower bound

In this subsection, we prove the lower bound in a similar way as the upper bound. The proof of the lower bound is simpler than the upper bound.

Let \(\delta > 0\) be fixed. Let \(r\) be a constant and let \(\{I_k\}\) be a sequence of open intervals defined in the proof of upper bound.

By Assumption (V), we can choose \(M > 0\) so large that
\[
\sup_{x \in \mathbb{R}^2} V(x, z) \leq \frac{\lambda}{2}
\]
for \(|z| > M\lambda^{-1/m}\), uniformly in \(z \in I_k\). Then let
\[
\Omega_{1, \lambda} = \{k \in \mathbb{Z}: |z_k| \leq \delta^{-1} \lambda^{-r}\}, \quad \Omega_{2, \lambda} = \{k \in \mathbb{Z}: \delta^{-1} \lambda^{-r} < |z_k| < M\lambda^{-\frac{r}{2}}\},
\]
\[
\Omega_{3, \lambda} = \{k \in \mathbb{Z}: |z_k| \geq M\lambda^{-\frac{r}{2}}\}.
\]

We note
\[
N(K_\lambda < -\lambda) \geq N(P_\lambda K_\lambda P_\lambda < -\lambda) = N(P_\lambda(-\partial_z^2 - V)P_\lambda < -\lambda) = N(-\partial_z^2 - P_\lambda VP_\lambda < -\lambda).
\]
Let \(N_k(t)\) be the number of eigenvalues less than \(-t\) of the operator \(-\partial_z^2 - P_\lambda VP_\lambda\) in \(L^2(\mathbb{R}^2_x \times I_k)\) with the Dirichlet boundary condition.

Lemma 3.12.
\[
N(K_\lambda < -\lambda) \geq \sum_{k \in \Omega_{2, \lambda}} N_k(\lambda).
\]
Let \( v_k(x) = V(x, z_k) \). Then we have

**Lemma 3.13.**

\[
\sum_{k \in \Omega_{2, \lambda}} N_k(\lambda) \geq 2 \sum_{k \in \Omega_{2, \lambda}} G_{k, 0}(\lambda) - F(\lambda) - \delta^{1/2} O(\lambda^{1/2} + \frac{1}{\lambda} - \frac{\lambda}{\pi}),
\]

where

\[
G_{k, 0}(\lambda) = -\frac{|I_k|}{\pi} \int_{\xi_k(\lambda)}^{\partial \mathcal{X}^m} ((1 - \beta \delta) \nu - \lambda)^{1/2} d g_k((1 + \delta) \nu),
\]

\[
\xi_k(\lambda) = \frac{1}{1 - \beta \delta} \lambda = \lambda(1 + O(\delta)),
\]

\[
g_k(\nu) = (2\pi)^{-1} \int_{v_k(x) > \nu} b(x) dx.
\]

**Proof.** Let \( k \in \Omega_{2, \lambda} \). By Assumption (V), it is easily seen that

\[ V(x, z) = v_k(x)(1 + O(\delta)) \]

for \( z \in I_k \), uniformly in \( x \in \mathbb{R}^2 \). Thus there exists \( \beta > 0 \) such that

\[
-\partial_z^2 - P_\lambda V P_\lambda \leq -\partial_z^2 - (1 - \beta \delta) P_\lambda v_k P_\lambda,
\]

in \( L^2(\mathbb{R}^2_x \times I_k) \) with the Dirichlet boundary condition. Let \( \nu_{k,j}^{(\lambda)} \) be the \( j \)-th eigenvalue of the operator \( P_\lambda v_k P_\lambda \in B(L^2(\mathbb{R}^2_x)) \). Then the eigenvalues of the right hand side of (3.57) are

\[
l^2 \frac{\pi^2}{|I_k|^2} - (1 - \beta \delta) \nu_{k,j}^{(\lambda)}, \quad l \in \mathbb{N}.
\]

Applying (3.29) to \( v_k/\nu \), we see that \( \nu_{k,j}^{(\lambda)} \leq \text{const.} j^{-m/(2-d)} \) and that there exists \( p > 0 \) such that

\[
\frac{l^2 \pi^2}{|I_k|^2} - (1 - \beta \delta) \nu_{k,j}^{(\lambda)} > -\lambda
\]

for \( j > p \lambda^{-(2-d)/m} \). Hence we have

\[
N_k(\lambda) \geq \sum_{j=1}^{p \lambda^{-2/(m-d)}} \left( \frac{|I_k|}{\pi} ((1 - \beta \delta) \nu_{k,j}^{(\lambda)} - \lambda)^{1/2} - 1 \right).
Let \( m_k^{(\lambda)}(\nu) = N(P_{\nu}v_kP_\lambda > \lambda \nu) \). By Assumption (V), there exists \( R > 0 \) large enough such that \( m_k^{(\lambda)}(\nu) = 0 \) for all \( \nu > R\lambda^{-(1-rm)} \). Hence we obtain

\[
(3.59) \quad N_k(\lambda) \geq -\frac{|I_k|}{\pi} \int_{\theta_\delta}^{R\lambda^{-(1-rm)}} ((1-\beta\delta)\nu - \lambda)^{1/2} dm_k^{(\lambda)}(\nu) - O(\lambda^{-\frac{2-r}{2m}}),
\]

where

\[
(3.60) \quad \theta_\delta = \frac{1}{1-\beta\delta} = 1 + O(\delta).
\]

Let

\[
(3.61) \quad G_k(\lambda) = -\frac{|I_k|}{\pi} \int_{\theta_\delta}^{R\lambda^{-(1-rm)}} ((1-\beta\delta)\nu - \lambda)^{1/2} dm_k^{(\lambda)}(\nu).
\]

Since \( \Omega_{2,\lambda} = O(\lambda^{-1/m+r}) \) and \( r > 1/2 \), we see that

\[
(3.62) \quad \sum_{k \in \Omega_{2,\lambda}} N_k(\lambda) \geq \sum_{k \in \Omega_{2,\lambda}} G_k(\lambda) - O(\lambda^{-\frac{2-r}{2m}} - \frac{2}{m} + r)
\]

by (3.59). Let \( \nu > 1 \). We apply (3.4) in Proposition 3.1 to \( \nu_k/\nu \) (\( \nu > 1 \)). Then for any \( \delta > 0 \), we can choose \( \lambda_\delta > 0 \) such that for \( 0 < \lambda < \lambda_\delta \),

\[
(3.63) \quad m_k^{(\lambda)}(\nu) \geq 2g_k((1+\delta)\lambda\nu) - g_k((1-\delta)\lambda\nu) - \delta O((\nu\lambda)^{-\frac{2-r}{2m}}).
\]

Therefore by integration by parts and the same computation as in (3.36), we obtain

\[
(3.64) \quad G_k(\lambda) \geq -\frac{2|I_k|}{\pi} \int_{\theta_\delta}^{R\lambda^{-(1-rm)}} ((1-\beta\delta)\nu - \lambda)^{1/2} dg_k((1+\delta)\lambda\nu)
\]

\[
+ \frac{|I_k|}{\pi} \int_{\theta_\delta}^{R\lambda^{-(1-rm)}} ((1-\beta\delta)\nu - \lambda)^{1/2} dg_k((1-\delta)\lambda\nu)
\]

\[
- \delta\lambda^{-\frac{2-r}{2m}} \int_{\theta_\delta}^{R\lambda^{-(1-rm)}} \nu^{-\frac{2m-r}{2m}} df(\nu)
\]

\[
\geq 2G_{k,0}(\lambda) - G_{k,1}(\lambda) - \delta O(\lambda^{\frac{1}{2}+\frac{1}{2m} - \frac{r}{2m}}),
\]

where

\[
(3.65) \quad G_{k,1}(\lambda) = -\frac{|I_k|}{\pi} \int_{\xi_k(\lambda)}^{R \lambda rm} ((1-\beta\delta)\nu - \lambda)^{1/2} dg_k((1-\delta)\nu),
\]

\[
(3.66) \quad f(\nu) = -\frac{|I_k|}{\pi} ((1-\beta\delta)\nu - \lambda)^{1/2}.
\]
Asymptotic Distribution of Eigenvalues

By the same computation in the proof of upper bound, we learn

\[ G_{k,1}(\lambda) \leq 2(2\pi)^{-1} \int_{\{V(x,z) > \lambda, z \in I_k\}} b(x)(V(x, z) - \lambda)^{1/2} \, dx \, dz 
+ \delta^{1/2} O(\lambda^{\frac{1}{2} + \frac{d}{m} - \frac{d}{m} - r}). \]

Therefore, since \$\sharp\Omega_{2,\lambda} = O(\lambda^{-1/m+r})\$, (3.53) follows from (3.62) and (3.64).

**Lemma 3.14.**

(3.67) \[ \sum_{k \in \Omega_{2,\lambda}} G_{k,0}(\lambda) \geq F(\lambda)(1 - O(\delta^{1/2})). \]

**Proof.** By (3.56), (3.54) and definition of Stieltjes integral, we have

(3.68) \[ G_{k,0}(\lambda) = 2(2\pi)^{-2} |I_k| c_\delta \int_{v_k(x) > (1+\delta)\xi_\delta(\lambda)} b(x)(v_k(x) - (1 + \delta)\xi_\delta(\lambda))^{1/2} \, dx, \]

where

(3.69) \[ c_\delta = \left( \frac{1 - \beta \delta}{1 + \delta} \right)^{1/2} = 1 + O(\delta). \]

In order to estimate the integral in the right hand side of (3.68) from below, we decompose it as follows:

(3.70) \[
\begin{align*}
& \int_{v_k(x) > (1+\delta)\xi_\delta(\lambda)} b(x)(v_k(x) - (1 + \delta)\xi_\delta(\lambda))^{1/2} \, dx \\
= & \int_{v_k(x) > \lambda} b(x)(v_k(x) - \lambda)^{1/2} \, dx \\
- & \int_{v_k(x) > (1+\delta)\xi_\delta(\lambda)} b(x) \left\{(v_k(x) - \lambda)^{1/2} - (v_k(x) - (1 + \delta)\xi_\delta(\lambda))^{1/2}\right\} \, dx \\
- & \int_{\lambda \leq v_k(x) < (1+\delta)\xi_\delta(\lambda)} b(x)(v_k(x) - \lambda)^{1/2} \, dx \\
= & I - II - III.
\end{align*}
\]
Recalling (3.55), we see that

\[ II \leq \int_{v_k(x) > (1+C\delta)\lambda} b(x) \left\{ (v_k(x) - \lambda)^{1/2} - (v_k(x) - (1-C\lambda)^{1/2} \right\} \, dx \]

\[ = \int_{v_k(x) > (1+C\delta)\lambda} b(x) \left( \int_0^1 \frac{C\delta \lambda}{2(v_k(x) - (1+C\delta t)\lambda)^{1/2}} \, dt \right) \, dx \]

\[ \leq C\delta \lambda \int_{v_k(x) > (1+C\delta)\lambda} b(x) \left( \int_0^1 \frac{1}{2(C\delta \lambda(1-t))^{1/2}} \, dt \right) \, dx \]

\[ = O(\delta^{1/2}) \lambda^{1/2} \int_{v_k(x) > (1+C\delta)\lambda} b(x) \, dx \]

\[ = \delta^{1/2} O(\lambda^{\frac{1}{2} + \frac{d}{2m} - \frac{3}{2}}), \]

and we also have

\[ III \leq \int_{\lambda \leq v_k(x) < (1+C\delta)\lambda} b(x)(v_k(x) - \lambda)^{1/2} \, dx \]

\[ = O(\delta^{1/2}) \lambda^{1/2} \int_{\lambda \leq v_k(x) < (1+C\delta)\lambda} b(x) \, dx \]

\[ = \delta^{1/2} O(\lambda^{\frac{1}{2} + \frac{d}{2m} - \frac{5}{2}}), \]

where we have used (3.42). Hence we obtain

\[ (3.70) \geq \int_{v_k(x) > \lambda} b(x)(v_k(x) - \lambda)^{1/2} \, dx - \delta^{1/2} O(\lambda^{\frac{1}{2} + \frac{d}{2m} - \frac{1}{2}}). \]

Noting \(|I_k| = \lambda^{-r}\), (3.42) and (3.69), we obtain

\[ (3.71) \]

\[ G_{k,0}(\lambda) \geq 2(2\pi)^{-2}|I_k| \int_{v_k(x) > \lambda} b(x)(v_k(x) - \lambda)^{1/2} \, dx - \delta^{1/2} O(\lambda^{\frac{1}{2} + \frac{d}{2m} - \frac{1}{2} - r}). \]

Next we estimate the right hand side of (3.71). It follows from Assumption (V) that

\[ V(x, z) = v_k(x)(1 + O(\delta)) \]
for $z \in I_k$. Thus we see that

$$|I_k| \int_{v_k(x) > \lambda} b(x)(v_k(x) - \lambda)^{1/2} \, dx$$

$$= \int \int_{v_k(x) > \lambda} b(x)(v_k(x) - \lambda)^{1/2} \, dx \, dz$$

$$\geq \int \left\{ V(x, z) > (1 + C\delta)\lambda, z \in I_k \right\} b(x)((1 + C\delta)^{-1}V(x, z) - \lambda)^{1/2} \, dx \, dz$$

(3.72)

$$= \int \left\{ V(x, z) > \lambda, z \in I_k \right\} b(x)(V(x, z) - \lambda)^{1/2} \, dx \, dz$$

$$- \int \left\{ V(x, z) > (1 + C\delta)\lambda, z \in I_k \right\} \times b(x) \left\{ (V(x, z) - \lambda)^{1/2} - ((1 + C\delta)^{-1}V(x, z) - \lambda)^{1/2} \right\} \, dx \, dz$$

$$- \int \left\{ \lambda \leq V(x, z) < (1 + C\delta)\lambda, z \in I_k \right\} b(x)(V(x, z) - \lambda)^{1/2} \, dx \, dz.$$

Now we recall (3.45). Then the second and the third term in the right hand side of (3.72) are bounded $\delta^{1/2}O(\lambda^{1/2+d/m-2/m-r})$ from above, by the same computation as in the estimate of (3.70). Therefore

$$(\text{the LHS of (3.72)}) \geq \int \left\{ V(x, z) > \lambda, z \in I_k \right\} \times b(x)(V(x, z) - \lambda)^{1/2} \, dx \, dz - \delta^{1/2}O(\lambda^{1/2+d/m-2/m-r}).$$

Combining this estimate with (3.71), we obtain

(3.73) \quad \#_{k,0} (\lambda) \geq 2(2\pi)^{-2} \int \left\{ V(x, z) > \lambda, z \in I_k \right\} \times b(x)(V(x, z) - \lambda)^{1/2} \, dx \, dz - \delta^{1/2}O(\lambda^{1/2+d/m-2/m-r}).$$

Since $\#_{k,2} = O(\lambda^{-1/m+r})$, we see that

(3.74) \quad \sum_{k \in \Omega_{2,\lambda}} G_{k,0}(\lambda) \geq \sum_{k \in \Omega_{2,\lambda}} 2(2\pi)^{-2} \int \left\{ V(x, z) > \lambda, z \in I_k \right\} \times b(x)(V(x, z) - \lambda)^{1/2} \, dx \, dz - \delta^{1/2}O(\lambda^{1/2+d/m-2/m-r}).$$

Therefore it follows from (3.48), (3.74) and Lemma 2.2 that

$$\sum_{k \in \Omega_{2,\lambda}} G_{k,0}(\lambda) \geq F(\lambda) - \delta^{1/2}O(\lambda^{\frac{1}{2} + \frac{d}{m} - \frac{1}{2} + \frac{m}{m}})$$

$$= F(\lambda)(1 - O(\delta^{1/2}))$$

$\square$
The next lemma follows immediately from Lemmas 3.13 and 3.14.

Lemma 3.15.

$$\sum_{k \in \Omega_{2, \lambda}} N_k(\lambda) \geq F(\lambda)(1 - O(\delta^{1/2})).$$

(3.75)

Proof of the lower bound (2.18). It follows from Lemmas 3.12 and 3.15 that

$$N(K_{\lambda} < -\lambda) \geq F(\lambda)(1 - O(\delta^{1/2})), \quad \lambda \to 0.$$  

(3.76)

Since $\delta > 0$ is arbitrary, (2.18) follows from Lemma 2.2.

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References


