Cauchy Problems for Mixed-Type Operators

By

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Abstract

We give a general theory of the Cauchy problems for various types of operators, containing hyperbolic operators, elliptic operators, and mixed-type operators. We will give a necessary and sufficient condition for the Cauchy problems to be well-posed.

§ 1. Introduction

The aim of this paper is to give a general theory for Cauchy problems, applicable for hyperbolic operators, elliptic operators, and mixed-type operators. For example, it will turn out that we can consider Cauchy problems for

\[(1) \quad P = (D_1^2 - x_1 D_n^2)(D_2^2 - x_1^2 D_n^2)(D_1^2 + x_1^2 D_n^2) + \text{(fifth-order operator)}.\]

The general theory is as follows. Let \( P(x, D) \) be a microdifferential operator defined at \( x^* = (0; 0, \ldots, 0, \sqrt{-1}) \in \sqrt{-1} T^* \mathbb{R}^n \) of order \( m \geq 2 \), written in the form

\[(2) \quad P(x, D) = D_1^m + \sum_{0 \leq j \leq m-1} P_j(x, D') D_1^j, \quad \text{ord } P_j \leq m - j.\]

Here we have written \( D = \partial / \partial x \), and \( D' = (D_2, \ldots, D_n) \). We also write as \( D'' = (D_1, \ldots, D_{n-1}) \), \( D''' = (D_2, \ldots, D_{n-1}) \). Let \( \sigma_m(P)(x, \xi) \) be the principal symbol of \( P(x, D) \). We assume that

\[(3) \quad \begin{cases} \text{if } x_1 = 0, & \text{then } \sigma_m(P) = \xi_1^m; \\ \text{if } x_1 \neq 0, & \text{then the equation } \sigma_m(P) = 0 \text{ has } m \text{ distinctive roots} \\ \xi_1 = \varphi_1(x, \xi'), \ldots, \varphi_m(x, \xi'). \end{cases}\]

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We denote by \( \mathcal{O} \) the sheaf of holomorphic functions, and we define \( \mathcal{O}_{(j)} = \sum_{0 \leq k \leq j-1} x^{k/j} \mathcal{O} \) for \( j \in \mathbb{N} \). Without loss of generality, we may assume that for \( \exists q_j \in \mathbb{N}/m' \), \( \exists a_j(x, \xi') \in \mathcal{O}((m'), x) \) we have \( \varphi_j(x, \xi') = x^q_j a_j(x, \xi') \), \( a_j(x^*) \neq 0 \) \((1 \leq j \leq m)\). Here each \( \varphi_j(x, \xi') \) is homogeneous in \( \xi' \) of degree 1. We also assume that

\[
(4) \quad i \neq j \Rightarrow (q_i, a_i(x^*)) \neq (q_j, a_j(x^*)).
\]

We denote by \( \mathcal{E} \) (resp. \( \mathcal{D} \)) the sheaf of microfunctions (resp. micro-differential operators). Let us consider the Cauchy problem

\[
\begin{align*}
Pu &= 0, \\
D^{\alpha}_{1} u(0, x') &= v_j(x'), \quad 1 \leq j \leq m,
\end{align*}
\]

where \( u \in \mathcal{E}_{\mathbb{R}^{n-1}, x^*} \) and \( v_j \in \mathcal{E}_{\mathbb{R}^{n-1}, x^*} \) \((x^*)' = (0; 0, \ldots, 0, \sqrt{-1}) \in \sqrt{-1} T^* \mathbb{R}^{n-1} \). If \( P(x, D) \) is microhyperbolic, (5) is well-posed for arbitrary initial values, as is well-known (See [5]). Otherwise (5) may be solvable for some initial values (e.g., for \( v_1 = \cdots = v_m = 0 \)), but may be unsolvable for other initial values. Therefore there arises a problem to know for which initial values (5) becomes solvable.

To give the main theorem we need to give some preliminaries. Let \( A(x', D') \) be a both-side invertible \( m \times m \) matrix whose components \( A(\phi, \psi) \in \mathcal{E} \) are independent of \((x_1, D_1)\). Here we denote by \( \mathcal{E} \) the sheaf of holomorphic microlocal operators (c.f. [2,9]). We choose \( r \) rows of this matrix in an arbitrary way. To be clear, let \( 1 \leq j_1 < j_2 < \cdots < j_r \leq m \) and choose the \( j_1, \ldots, j_r \)-th rows of \( A \). Then we obtain an \( r \times m \) matrix \( A'(x', D') \) of holomorphic microlocal operators. Let us choose some \( r \) rows of some \( A \). We say that \( v_1(x'), \ldots, v_m(x') \in \mathcal{E}_{\mathbb{R}^{n-1}, x^*} \) satisfy an \( r \)-relation (defined by \( A' \)) if \( A'(x', D') \mathfrak{v}(x') = 0 \). Here \( \mathfrak{v} \) denotes \( (v_1, \ldots, v_m) \), and we denote the set of such vectors by \( (\mathcal{E}_{\mathbb{R}^{n-1}, x^*})^m \). We have \( A : (\mathcal{E}_{\mathbb{R}^{n-1}, x^*})^m \rightarrow (\mathcal{E}_{\mathbb{R}^{n-1}, x^*})^m \), and an \( r \)-relation means that \( r \) of the components of \( \mathfrak{v} \) disappear when it is sent to right-hand side. We note that even if \( v_1(x'), \ldots, v_m(x') \) satisfy an \( r \)-relation and another \( s \)-relation, it does not necessarily mean an \((r + s)\)-relation.

We next define a classification of the characteristic roots. Let \( \theta \in \{0, \pi\} \). Let \( \omega \subset \mathbb{R}^n \times \sqrt{-1} \mathbb{R}^{n-1} \) be a small neighborhood of \( x^* \), and let \( \omega_{\theta} = \{(x, \xi') \in \omega; x_1 \neq 0, \arg x_1 = \theta\} \). We define

\[
M = \{1, 2, \ldots, m\},
\]

\[
M_{0, \theta} = \{ \lambda \in M; \Re(x_1 \varphi_2(x, \xi')) = 0, \text{ if } (x, \xi') \in \omega_{\theta}\},
\]

\[
M_{\pm, \theta} = \{ \lambda \in M; \pm \Re(x_1 \varphi_2(x, \xi')) > 0, \text{ if } (x, \xi') \in \omega_{\theta}\},
\]

\[
M_{\theta} = M \setminus M_{0, \theta} \setminus M_{+, \theta} \setminus M_{-, \theta}.
\]
To give another expression of this definition, we consider the solutions 
\( \psi^{(\pm, \lambda)}(x, \xi') \) of the following phase equation:

\[
(6)_+ \quad \begin{cases} 
\partial_{x_1} \psi^{(\pm, \lambda)} - \varphi_2(x_1, \xi') + \partial_{\xi} \psi^{(\pm, \lambda)} = 0, \\
\psi^{(\pm, \lambda)}(0, x', \xi') = 0,
\end{cases}
\]

and define \( \psi^{(-, \lambda)}(x, \xi') \) by

\[
(6)_- \quad \begin{cases} 
\partial_{x_1} \psi^{(-, \lambda)} + \varphi_2(x_1, x' + \partial_{\xi} \psi^{(\lambda)}, \xi') = 0, \\
\psi^{(-, \lambda)}(0, x', \xi') = 0.
\end{cases}
\]

Then it is easy to see that

\[
M_{0,\theta} = \{ \lambda \in M; \Re \psi^{(\pm, \lambda)}(x, \xi') = 0 \},
\]

\[
= \{ \lambda \in M; \Re \psi^{(-, \lambda)}(x, \xi') = 0 \},
\]

\[
M_{\pm,\theta} = \{ \lambda \in M; \pm \Re \psi^{(\pm, \lambda)}(x, \xi') > 0 \},
\]

\[
= \{ \lambda \in M; \pm \Re \psi^{(-, \lambda)}(x, \xi') < 0 \}.
\]

Let \( m_{0,\theta}, m_{\pm,\theta} \) be the numbers of the elements belonging to \( M_{0,\theta}, M_{\pm,\theta} \), respectively. We always assume that

\[
(M'_{0} = M'_{\pi} = \emptyset).
\]

**Example.** If we have

\[
\begin{align*}
\varphi_1(x, \xi') &= x_1 \xi_n, \\
\varphi_2(x, \xi') &= \sqrt{-1} x_1 \xi_n, \\
\varphi_3(x, \xi') &= -\sqrt{-1} x_1 \xi_n, \\
\varphi_4(x, \xi') &= x_1^{1/2} \xi_n, \\
\varphi_5(x, \xi') &= x_1^{-1/2} \xi_n, \\
\varphi_6(x, \xi') &= x_1^2 (1 + \sqrt{-1} x_2) \xi_n,
\end{align*}
\]

then the above classification is as follows. We have \((x, \xi') \in \mathbb{R}^n \times \sqrt{-1} \mathbb{R}^{n-1}\), \(\Im \xi_n > 0\), and \(\arg x_1 = \theta\), \(\arg \xi_n = \pi/2\). Therefore we have \(\Re(x_1 \varphi_1(x, \xi')) = 0\), and \(1 \in M_{0,\theta}\). Similarly we have \(2 \in M_{-\theta}, 3 \in M_{+,\theta}\) (for each \(\theta \in \{0, \pi\}\)). We next note that \(\arg(x_1 \varphi_4(x, \xi')) = 3\theta/2 + \pi/2\). It follows that \(4 \in M_{0,\theta}, 4 \in M_{+,\pi}\). Similarly we have \(5 \in M_{0,0}, 5 \in M_{-,\pi}\). Finally we have

\[
\Re(x_1 \varphi_6(x, \xi')) = \Re(x_1^3 (1 + \sqrt{-1} x_2) \xi_n) = -\Re(x_1^3 x_2) \Im \xi_n,
\]

and it follows that \(6 \in M'_{0}\). This means that the types of \(\varphi_4(x, \xi'), \varphi_5(x, \xi')\) change from elliptic to hyperbolic as \(x_1\) varies, but that of \(\varphi_6(x, \xi')\) changes as \(x_1, x_2\) vary. Our assumption (7) means that we do not consider such a
complicated characteristic root as \( \varphi_6(x, \xi') \). Let us summarize the above calculation.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( M_{0, \theta} )</th>
<th>( M_{+, \theta} )</th>
<th>( M_{-, \theta} )</th>
<th>( M'_{0, \theta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{1, 4, 5}</td>
<td>{3}</td>
<td>{2}</td>
<td>{6}</td>
</tr>
<tr>
<td>( \pi )</td>
<td>{1}</td>
<td>{3, 4}</td>
<td>{2, 5}</td>
<td>{6}</td>
</tr>
</tbody>
</table>

Now we give the main result.

**Theorem 1.** We assume (2), (3), (4), and (7). Then there exist an \( m_{+, 0} \)-relation and an \( m_{+, \pi} \)-relation (which are defined only by \( P \)), such that the Cauchy problem (5) has a solution \( u \in C_{\mathbb{R}^n, x^*} \) if, and only if, \( v_1(x'), \ldots, v_m(x') \in C_{\mathbb{R}^{n-1}, x^*} \) satisfy both these relations.

We give some examples. At first we remind the reader of the well-known result for the operators of principal type.

**Example (Lewy-Mizohata operators).** If \( P_{\pm} = D_1 \pm \sqrt{-1} x_1 D_n \), then we have \( M_{\pm, 0} = \{1\} \) (\( = M \)), \( M_{\pm, \theta} = \emptyset \). This means that \( P_{-} u = 0, u(0, x') = v(x') \) is solvable for any \( v \in C_{\mathbb{R}^{n-1}, x^*} \) without any relations, which coincides with the well-known result. On the other hand, \( P_{+} u = 0, u(0, x') = v(x') \) is solvable only for the case when \( v(x') \) satisfies a one-relation. This means \( v = 0 \), and \( u = 0 \). It follows that \( P_{+} u = 0 \Rightarrow u = 0 \), i.e., \( P_{+} \) is hypo-elliptic (See [9]).

Lewy-Mizohata operators are the simplest case of our theory, and our theorem gives a similar result even for more complicated operators. The characteristic roots belonging to \( M_{+, \theta} \) cause obstruction, and correspondingly the Cauchy data must satisfy suitable relations. Let us see the case \( m = 2 \).

**Example (microhyperbolic operators).** Let \( P(x, D) = D_1^2 - x_1^2 D_n^2 + P'(x, D) \), ord \( P' \leq 1 \). Without loss of generality, we may assume that \( P' \) is a polynomial in \( D_1 \) of degree 1. In this case we have \( \varphi_1(x, \xi') = x_1 \xi_n, \varphi_2(x, \xi') = -x_1 \xi_n \). It follows that \( M_{0, \theta} = \{1, 2\}, M_{\pm, \theta} = \emptyset \) for \( \theta \in \{0, \pi\} \). This means that (5) is solvable for an arbitrary \( v_1(x'), v_2(x') \in C_{\mathbb{R}^{n-1}, x^*} \) without any relations (See [1, 7, 8, 10, 13]).

**Example (Tricomi operators).** Let \( P(x, D) = D_1^2 - x_1 D_n^2 + P'(x, D) \), ord \( P' \leq 1 \). We have \( \varphi_1(x, \xi') = \sqrt{-1} x_1 \xi_n, \varphi_2(x, \xi') = -\sqrt{-1} x_1 \xi_n \). It follows that \( M_{0, 0} = \{1, 2\}, m_{+, 0} = 0 \), and that \( M_{+, \pi} = \{1\}, M_{-, \pi} = \{2\}, m_{+, \pi} = 1 \). Therefore there exists a 1-relation, and (5) is solvable if, and only if, the Cauchy data satisfy this relation. This case was investigated by [7].

**Example (hypoelliptic operators).** Let \( P(x, D) = D_1^2 + x_1^2 D_n^2 + P'(x, D) \), ord \( P' \leq 1 \). Since \( \varphi_1(x, \xi') = \sqrt{-1} x_1 \xi_n, \varphi_2(x, \xi') = -\sqrt{-1} x_1 \xi_n \), it is easy to see
that $M_{-, \theta} = \{1\}$, $M_{+, \theta} = \{2\}$, $m_{+, \theta} = 1$ for $\theta \in \{0, \pi\}$. It follows that (5) is solvable if, and only if, $v_1, v_2$ satisfy a 1-relation and another 1-relation. If these two 1-relations mean a 2-relation, (5) is solvable only in the case $v_1 = v_2 = 0$, and we have $u = 0$. In other words, $Pu = 0$ does not have non-trivial solutions. It is well-known that this is true if the principal symbol $\sigma_1(P')$ of the lower order term satisfies $\xi_n^{-1}\sigma_1(P') \notin \{\sqrt{-1}, 3\sqrt{-1}, 5\sqrt{-1}, \ldots\}$ (See [4]).

Of course our result applies for higher order operators. For instance, let us consider the Cauchy problem (5) for the sixth-order operator (1). Then the Cauchy data $v_1, \ldots, v_6$ must satisfy a 1-relation and a 2-relation. Roughly speaking, we can give three microfunctions arbitrarily among $v_1, \ldots, v_6$.

In order to prove Theorem 1, we shall show that fixing $\text{arg} x_1$, we can give a canonical representation of the elementary solution of $P(x, D)$. This result has its own interest. But to give its precise statement, we need to prepare a symbol theory. Therefore in the next section we shall give such a symbol theory, and we postpone the discussion of the canonical form until Section 3.

§ 2. An Operator Theory

In this section we give an operator theory, which is necessary for the proof of Theorem 1. We may assume that $q_1 \leq \cdots \leq q_m$, and we define $q = \max(q_1, \ldots, q_m)$ ($= q_m$).

§ 2.1. A Theory of Formal Operators

Let $C > 0$, $i \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. We define

$$
\Omega(C) = \{(x, \xi') \in \mathbb{C}^n \times \mathbb{C}^{n-1}; C|x| < 1,
C|\xi'|^m < \text{Im} \xi_n, C|\text{Re} \xi_n| < \text{Im} \xi_n, C^{\text{Re} \xi_n} < \text{Im} \xi_n\},$
$$

$$
\Omega_i(C) = \{(x, \xi') \in \Omega(C); C^{\text{Re} \xi_n}(i + 1) < \text{Im} \xi_n\}.
$$

Assume that for $\exists C > 0$, $\exists R \in (0, 1)$, $\forall \epsilon > 0$, $\exists C_\epsilon > 0$ a formal series $f = \sum_{i \in \mathbb{Z}_+} f_i(x, \xi')$ satisfies $f_i \in \mathcal{O}(\Omega_i(C))$ and

$$
|f_i(x, \xi')| \leq C_\epsilon R^i \exp((C|x|)^{1+1/m'}) + C(|\text{Re} \xi'|/\text{Im} \xi_n)^2 + \epsilon) \text{Im} \xi_n)
$$
on $\Omega_i(C)$. We denote by $\mathcal{F}(\Omega(C))$ the set of such formal series. If $f = \sum_i f_i$, we define a formal series $f^\# = \sum_i f_i^\#$ by $f_i^\# = \sum_{0 \leq j < i} f_j$, and $\mathcal{N}(\Omega(C))$ by

$$
\mathcal{N}(\Omega(C)) = \{f \in \mathcal{F}(\Omega(C)); f^\# \in \mathcal{F}(\Omega(C))\}.
$$
Finally we define
\[ \mathcal{T} = \lim_{C \to 0} \mathcal{T}(\Omega(C)), \quad \mathcal{N} = \lim_{C \to 0} \mathcal{N}(\Omega(C)). \]

**Remark.** (i) According to the useful method of Aoki [2], we can neglect the elements of \( \mathcal{N} \), as we shall see.

(ii) We write \( \sum f_i = \sum g_i \), if, and only if, \( f_i = g_i \) for any \( i \). Since this does not merely mean that the sums of these two series are the same, we sometimes write as \( \sum f_i = \sum g_i \).

(iii) We identify a function \( f_0 \) with a formal series \( f_0 + 0 + 0 + \cdots \in \mathcal{T}(\Omega(C)) \), if it satisfies (8) for \( i = 0 \). A function \( f_0 \) belongs to \( \mathcal{N}(\Omega(C)) \) if, and only if, it is exponentially decreasing.

(iv) For a formal series \( f = \sum f_i \) we define \( \partial_x f = \sum \partial_x f_i \).

Let \( f = \sum f_i \in \mathcal{T}(\Omega(C)) \). We next define
\[ \mathcal{F}(f)(x', y') = (2\pi\sqrt{-1})^{-n+1} \sum_{i} e^{x'(x'-y')} f_i(x, \xi')d\xi', \]
and show that it defines a formal operator. To be precise, we use an analytic partition of the unity. Let \( C' \gg C \). We consider the following linear transformation:
\[
\begin{cases}
\xi_j' = -C'\xi_j + \xi_n, & 2 \leq j \leq n - 1, \\
\xi_j' = C'(\xi_2 + \cdots + \xi_{n-1}) + \xi_n, & j = n.
\end{cases}
\]
Then \( \sqrt{-1}\mathbb{R}^{n-1} \in \xi' \mapsto \tilde{\xi}' \in \sqrt{-1}\mathbb{R}^{n-1} \) is an isomorphism, and the first octant \( A = \{ \tilde{\xi}' \in \mathbb{C}^{n-1}; \text{Im} \tilde{\xi}_j > C'|\text{Re} \tilde{\xi}_j|, 2 \leq j \leq n \} \) corresponds to a small neighborhood of \( \xi' = (0, \ldots, 0, \sqrt{-1}) \in \mathbb{C}^{n-1} \). We define the central region \( A^{ce} \) and the boundary region \( A^{bo} \) of \( A \) by \( A^{ce} = \{ \tilde{\xi}' \in A; n^2 \text{Im} \tilde{\xi}_j > \text{Im} \tilde{\xi}_k, \forall j, \forall k \} \) and \( A^{bo} = \{ \tilde{\xi}' \in A; C' \text{Im} \tilde{\xi}_j < \text{Im} \tilde{\xi}_k, 3j, 3k \} \), respectively. There exists \( e(\tilde{\xi}') \in \mathcal{O}(A) \) such that for \( \forall \epsilon > 0, \exists C_\epsilon > 0 \)

\[ |e(\tilde{\xi}')| \leq C_\epsilon \exp\left((C'|\text{Re} \tilde{\xi}'|^2(\text{Im} \tilde{\xi}_n)^2 + \epsilon) \text{Im} \tilde{\xi}_n \right) \quad \text{on} \quad A, \tag9 \]

\[ |e(\tilde{\xi}') - 1| \leq C'| \exp(-C'^{-1} \text{Im} \tilde{\xi}_n) \quad \text{on} \quad A^{ce}, \]

\[ |e(\tilde{\xi}')| \leq C' \exp(-C'^{-1} \text{Im} \tilde{\xi}_n) \quad \text{on} \quad A^{bo} \tag{10} \]

(See [11]).

Let \( f = \sum f_i(x, \xi') \in \mathcal{T}(\Omega(C)) \). From (9) it follows that \( ef = \sum e(\tilde{\xi}') \cdot f_i(x, \xi') \in \mathcal{T}(\Omega(C')) \) for some \( C' \). From (10) we have \( e(\tilde{\xi}') = 1, ef = f \) in \( \mathcal{T}/\mathcal{N} \).

We define \( \mathcal{F}(f)(x, y') \) by
(11)  \[ \mathcal{F}(f)(x, y') = (2\pi \sqrt{-1})^{-n+1} \sum_{\zeta, j} e^{\xi_j(x'-y')} \overline{f_j(x, \xi_j)} d\xi_j, \]

where \( A_j(C^n) = \{ \xi_j \in \mathbb{R}^{n-1}; \xi_j > C^n(j+1), 2 \leq j \leq n \}. \)

We define \( \zeta'^{(j)} = (0, \ldots, 0, 1) \in \mathbb{R}^{n-1}, Z' = \{ \zeta' \in \mathbb{R}^{n-1}; |\zeta'| = 1, -1 < \zeta_n < 0 \} \) and

\[
W(C, r) = \{ (x, y') \in C^n \times C^{n-1}; C|(x, y')| < 1, \Im(x_n - y_n) + C^{-1}|\Re(x' - y')|^2 > r|\Im(x'' - y'')| + C|x_1|^{1+(1/m')}, \}
\]

\[
W(C, \delta, \zeta') = \left\{ (x, y') \in C^n \times C^{n-1}; C|(x, y')| < 1, \Im(x' - y') > C|x_1|^{1+(1/m')}, \right\}
\]

\[
\frac{1}{|\Im(x' - y')|} \Im(x' - y') - \frac{1}{|\theta \zeta'^{(j)} + (1 - \theta)\zeta'|} (\theta \zeta'^{(j)} + (1 - \theta)\zeta') \left| < \delta, 0 \leq \theta \leq 1 \right\}
\]

for \( C, r, \delta > 0, \zeta' \in Z' \). Then we have the following

**Lemma 1.** (i) Let \( f = \sum f_j(x, \xi_j) \in \mathcal{F}(\Omega(C)). \mathcal{F}(f) \) is holomorphic on \( W(C^n, r) \) for \( \exists C'' > 0 \) and \( \exists r > 0. \)

(ii) Let \( C'' < \tilde{C}'' \). If we replace \( A_i(C^n) \) by \( A_i(\tilde{C}'' \) in (11), we obtain a different \( \mathcal{F}(f) \). Let \( \tilde{\mathcal{F}}(f) \) be the function thus obtained. Then we have \( \tilde{\mathcal{F}}(f) - \mathcal{F}(f) = \sum_{j, \text{finite}} \exists F_j(x, y'), \) where \( F_j(x, y') \in \mathcal{O}(W(C''\delta, \zeta'^{(j)})) \) with some \( C'' > 0, \delta > 0, \zeta'^{(j)} \in Z' \) for each \( j. \)

(iii) If \( f \in \mathcal{N}(\Omega(C)), \) then \( \mathcal{F}(f) \) is holomorphic in a neighborhood of the origin.

We can prove this lemma by an elementary calculation. Now let \( \mathcal{W} = \lim \mathcal{O}(W(C, r)), \) let \( \mathcal{W}_1 \) be the set of holomorphic functions defined on \( W(C, r) \cup W(C, \delta, \zeta') \) with some \( C, r, \delta, \zeta', \) and let \( \mathcal{W}_2 \) be the set of finite sums of the elements of \( \mathcal{W}_1 \). We have defined a map \( \mathcal{F}: \mathcal{F}/\mathcal{N} \to \mathcal{W}/\mathcal{W}_2. \) Similarly to [11], we can prove that this is an isomorphism. We next define a ring structure of \( \mathcal{W}/\mathcal{W}_2. \) Let \( u_1(x, y'), u_2(x, y') \in \mathcal{O}(W(C, r)). \) Let \( C' \gg C \) and let

\[
A(C', \varepsilon) = \{ z' \in \mathbb{R}^{n-2} \times C; |\Re z_k| \leq 2C'^{-1} \} \quad (2 \leq k \leq n),
\]

\[
\Im z_n = \max(-C'^{-3}, (-C'^{-1}|\Re z'|^2 + C'\{x_1|^{1+(1/m')} + \varepsilon\}_+))\}.
\]

where \( t_+ = \max(0, t). \) Then we have the following

**Lemma 2.** (i) Let \( 0 < C \ll C' \ll C'', 0 < \varepsilon < C''^{-1}. \) If \( (x, z') \in W(C'', r) \) and \( x' - y' \in A(C', \varepsilon), \) then we have \( (x, y'), (x_1, y_1, z') \in W(C, r). \)
(ii) If \( u_1(x, y'), u_2(x, y') \in \mathcal{O}(W(C, r)) \), then

\[
u_1 \ast u_2(x, z') = \int_{A(C', \varepsilon)} u_1(x, y')u_2(x_1, y', z')d(x' - y') \in \mathcal{O}(W(C'', r))
\]

is well-defined. Here \( u_1 \ast u_2 \) does not depend on \( \varepsilon \). Furthermore, if we replace \( C' \) by \( C'' \), then the difference is holomorphic in a neighborhood of the origin.

(iii) If at least one of \( u_1(x, y'), u_2(x, y') \) belongs to \( \mathcal{O}(W(C, \delta, \xi')) \), then \( u_1 \ast u_2 \) belongs to \( \mathcal{O}(W(C, \delta', \xi')) \) for some \( \delta > 0 \) and \( \xi' \in \mathbb{Z}' \).

Therefore we obtain a map \( \mathcal{W}/\mathcal{W}_2 \times \mathcal{W}/\mathcal{W}_2 \ni (u_1, u_2) \mapsto u_1 \ast u_2 \in \mathcal{W}/\mathcal{W}_2 \), and we can endow \( \mathcal{W}/\mathcal{W}_2 \) with a ring structure with the unit element \( \mathcal{F}(1) \).

Furthermore, we can easily prove the following

**Lemma 3.** Let \( \sum f_j, \sum g_j \in \mathcal{F}(\Omega(C)) \). We define \( h = \sum h_j(x, \xi') \) by

\[
h_j = \sum_{k+\ell+|\alpha| = j} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha} g_r.
\]

Then we have \( \sum h_j \in \mathcal{F}(\Omega(C')) \) for \( C' \gg C \), and as an element of \( \mathcal{W}/\mathcal{W}_2 \) we have \( \mathcal{F}(h) = \mathcal{F}(f) \ast \mathcal{F}(g) \). (We denote \( \sum h_j \) also by \( f \circ g \).

Let us define formal operands corresponding to these operators. We define

\[
V(C, r) = \{ x \in \mathbb{C}^n; C|x| < 1, \text{Im} x_n > r, \text{Im} x_n + C|x_1|^{1+(1/m^s)} \},
\]

\[
V(C, \delta, \xi') = \left\{ x \in \mathbb{C}^n; C|x| < 1, |\text{Im} x'| > C|x_1|^{1+(1/m^s)} , \right\}
\]

\[
\left| \frac{\text{Im} x'}{|\text{Im} x'|} - \frac{1}{\theta \xi' + (1 - \theta) \xi'} \right| \left( \theta \xi'^2 + (1 - \theta) \xi' \right) < \delta,
\]

\[
0 \leq \theta \leq 1
\]

Let \( 0 < C \ll C' \ll C'' \). If \( u(x, y') \in \mathcal{O}(W(C, r)) \), \( f(x) \in \mathcal{O}(V(C, r)) \), then

\[
u \ast f(x) = \int_{A(C', \varepsilon)} u(x, y')f(x_1, y')d(x' - y') \in \mathcal{O}(V(C'', r))
\]

is well-defined. Let \( \mathcal{V} = \lim_{r \to C} \mathcal{O}(V(C, r)) \), let \( \mathcal{V}_1, \mathcal{V}_2 \) be defined similarly to \( \mathcal{W}_1, \mathcal{W}_2 \). We obtain a map \( \mathcal{W}/\mathcal{W}_2 \times \mathcal{V}/\mathcal{V}_2 \ni (u, f) \mapsto u \ast f \in \mathcal{V}/\mathcal{V}_2 \). In this way we can endow \( \mathcal{V}/\mathcal{V}_2 \) with the structure of a left \( \mathcal{W}/\mathcal{W}_2 \)-module. If \( a(x, \xi') \in \mathcal{V}/\mathcal{N} \), then \( u = \mathcal{F}(a) \) defines an integral operator \( \mathcal{V}/\mathcal{V}_2 \ni f \mapsto u \ast f \in \mathcal{V}/\mathcal{V}_2 \), which we denote by \( a(x, D') \).
Remark. (i) Let \( u(x, y') \in \mathcal{O}(W(C, r) \cup W(C, \delta, \zeta')) \), \( f(x) \in \mathcal{O}(V(C, r')) \) with \( 0 < r' \leq r \). Then \( u * f \in \mathcal{V}_2 \), and this means \( \int_{A(C', \epsilon)} u(x, y') f(x_1, y') \cdot d(x' - y') \in \bigcup_{k, \text{finite}} \mathcal{O}(V(3C^n, 3\delta, \zeta'(k))) \). However in the following special case this function is holomorphic in a full neighborhood of the origin. Assume that \( r' \) satisfies \( \delta_n + r' \leq 0 \) for every \( k \) (This is the case when \( r' \) is small enough). Let \( A'(C', \epsilon) = \lambda_1' \times \cdots \times \lambda_n' \), where \( \lambda_k' \) is the union of line segments joining \(-2C'_{r-1} - \sqrt{-1} \delta_{k,n}C'-3, -2C'_{r-1} + \sqrt{-1}R_{\delta_k}, 2C'_{r-1} + \sqrt{-1}R_{\delta_k}, \) and \( 2C'_{r-1} - \sqrt{-1} \delta_{k,n}C'-3 \), successively (\( 0 < R \ll 1 \)). We can easily prove that

\[
\int_{A(C', \epsilon)} u f d(x' - y') = \int_{A'(C', \epsilon)} u f d(x' - y').
\]

Furthermore, if \( x \in C^n, |x| \ll 1, x' - y' \in A'(C', \epsilon), \) then we have \( (x, y') \in W(C, r) \cup W(C, \delta, \zeta'), (x_1, y') \in V(C, r') \). This means

\[
\int_{A'(C', \epsilon)} u f d(x' - y') \in \mathcal{O}_C^n, 0,
\]

and therefore we have \( u * f \in \mathcal{O}_C^n, 0 \).

(ii) As we have said \( \mathcal{V}/\mathcal{V}_2 \) is a left \( \mathcal{W}/\mathcal{W}_2 \)-module. Therefore if \( u_1, u_2 \in \mathcal{W} \) and \( f \in \mathcal{V} \), then we have \( (u_1 * u_2) * f \equiv u_1 * (u_2 * f) \) modulo \( \mathcal{V}_2 \). Furthermore, by an elementary calculation we can prove \( (u_1 * u_2) * f \equiv u_1 * \) \( (u_2 * f) \) modulo \( \mathcal{O}_C^n, 0 \).

All the above discussions are formal. However, if \( u(x, y') \in \mathcal{W} \) (resp. \( f(x) \in \mathcal{V} \)), then \( u(0, x', y') \) (resp. \( f(0, x') \)) defines a microfunction, and restricting to \( \{x_1 = 0\} \), the above calculations are valid in the sense of microfunctions.

Remark. We sometimes consider functions with fractional powers in \( x_1 \). In this case we replace \( \mathcal{O} \) by \( \mathcal{O}_{(m')} \). We can generalize all the above arguments to this situation with trivial changes. For example let \( \mathcal{F}_{(m')} = \sum_{0 \leq k \leq m' - 1} x_1^{k/m'} \mathcal{F}, \) and \( N_{(m')}, \mathcal{W}_{(m'), \mathcal{W}_2, (m'), \mathcal{V}_{(m'), \mathcal{V}_2, (m')} \) be defined similarly. Then \( \mathcal{W}_{(m')}/\mathcal{W}_2, (m') \) is a ring, and \( \mathcal{V}_{(m')}/\mathcal{V}_2, (m') \) is a left \( \mathcal{W}_{(m')}/\mathcal{W}_2, (m') \)-module.

We sometimes differentiate \( f = \sum_{0 \leq k \leq m' - 1} x_1^{k/m'} f_k \in \mathcal{O}_{(m'), x} \) by \( x_1 \), where \( f_k \in \mathcal{O}_{x} \) \((0 \leq k \leq m' - 1) \). Since we do not necessarily have \( \partial_{x_1} f \in \mathcal{O}_{(m'), x} \), \( x \), we must assume \( f_k|_{x_1 = 0} = 0 \) for \( k \neq 0 \), in this case.

§ 2.2. A Theory of Real Operators

We next consider how we can make the above discussions valid in the sense of microfunctions on the half space \( \{x_1 \geq 0\} \). This is a special case of the
theory of mild microfunctions studied by [6]. Let \( C > 0, i \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \). Let \( \theta \in \{0, \pi\} \). We define

\[
\Omega_\theta(C) = \{(x, \xi') \in \mathbb{C}^n \times \mathbb{C}^{n-1}; C|x| < 1, \text{Re}(e^{\sqrt{-1}\theta}x_1) > 0, C|\xi'| < \text{Im} \xi_n, C|\text{Re} \xi_n| < \text{Im} \xi_n, C^{4mq} < \text{Im} \xi_n\},
\]

\[
\Omega_{\theta, i}(C) = \{(x, \xi') \in \Omega_\theta(C); C^{4mq}(i + 1) < \text{Im} \xi_n\}.
\]

Assume that for \( \exists C > 0, \exists R \in (0, 1), \forall \varepsilon > 0, \exists C_\varepsilon > 0 \), a formal series \( f = \sum_{i \in \mathbb{Z}_+} f_i(x, \xi') \in \mathcal{F}(\Omega(C)) \) satisfies

\[
|f_i(x, \xi')| \leq C_\varepsilon R^i \exp((C|x_1|^{1/m'}|\text{Im} x| + C|x_1|^{1/m'}|\text{Re} \xi'|/\text{Im} \xi_n
+ C(|\text{Re} \xi'|/\text{Im} \xi_n)^2 + \varepsilon)\text{Im} \xi_n)
\]
on \Omega_{\theta, i}(C) (Note that \( f_i \in \mathcal{O}(\Omega_i(C)) \) satisfies (8) on \( \Omega_i(C) \), and (13) on \( \Omega_{\theta, i}(C) \)). We denote by \( \mathcal{S}_\theta(\Omega(C)) \) the set of such formal series, and \( \mathcal{S}_\theta = \lim_{C \to 0} \mathcal{S}_\theta(\Omega(C)) \).

If \( f = \sum f_i(x, \xi') \in \mathcal{S}_\theta(\Omega(C)) \), we have \( ef = \sum e(\tilde{\xi}')f_i(x, \xi') \in \mathcal{S}_\theta(\Omega(C')) \) for some \( C'' \), and \( \mathcal{F}(f) \) is holomorphic on \( W(C'', r) \cup W_\theta(C'', r) \), where

\[
W_\theta(C'', r) = \{(x, y') \in \mathbb{C}^n \times \mathbb{C}^{n-1}; C''|\text{Im} x - y'| < 1, \text{Re}(e^{\sqrt{-1}\theta}x_1) > 0,\text{Im}(x_n - y_n) + C''^{-1}((|\text{Re}(x' - y')| - C''|x_1|^{1/m'})^2
> C''|x_1|^{1/m'}|\text{Im} x| + r|\text{Im}(x'' - y'')|}\}
\]

with \( \exists C''', \exists r \). Replacing the sheaf \( \mathcal{O} \) by \( \mathcal{O}(m') \), we can define \( \mathcal{S}_{\theta, (m')} \) similarly.

If \( f = \sum f_i(x, \xi') \in \mathcal{S}_{\theta, (m')}(\Omega(C')) \), then we have \( \mathcal{F}(f) \in \mathcal{O}(m')(W(C'', r)) \).

It is easy to see that if \( u(x, y') \in \mathcal{O}(m')(W(C, r) \cup W_\theta(C, r)) \), then it defines a microfunction \( \text{sp}u \) on \( \{\text{Re}(e^{\sqrt{-1}\theta}x_1) > 0\} \). Note that \( \text{sp}(u(0, x', y')) \) is also well-defined, and it is a microlocal operator.

Let us define the corresponding operands. We define

\[
V_\theta(C, r) = \{x \in \mathbb{C}^n; C|x| < 1, \text{Re}(e^{\sqrt{-1}\theta}x_1) > 0, \text{Im} x_n > r|\text{Im} x''|\},
\]

where \( C, r > 0 \). If \( \mathcal{O}(m')(V(C, r) \cup V_\theta(C, r)) \), then \( f \) defines a microfunction \( \text{sp} f \) on \( \{\text{Re}(e^{\sqrt{-1}\theta}x_1) > 0\} \). Moreover \( \text{sp} f \) is a mild microfunction in the sense of [6], and \( \text{sp}(f(0, x')) \) is well-defined.

If \( u(x, y') \in \mathcal{O}(m')(W(C, r) \cup W_\theta(C, r)), f(x) \in \mathcal{O}(m')(V(C, r) \cup V_\theta(C, r)) \), then

\[
u * f(x) = \int_{A(C', \varepsilon)} u(x, y')f(x_1, y')d(x' - y')
\]

\[
\in \mathcal{O}(m')(V(C'', r) \cup V_\theta(C'', r))
\]

with \( \exists C''', \exists r \). Replacing the sheaf \( \mathcal{O} \) by \( \mathcal{O}(m') \), we can define \( \mathcal{S}_{\theta, (m')} \) similarly.

If \( f = \sum f_i(x, \xi') \in \mathcal{S}_{\theta, (m')}(\Omega(C')) \), then we have \( \mathcal{F}(f) \in \mathcal{O}(m')(W(C'', r)) \).

It is easy to see that if \( u(x, y') \in \mathcal{O}(m')(W(C, r) \cup W_\theta(C, r)) \), then it defines a microfunction \( \text{sp}u \) on \( \{\text{Re}(e^{\sqrt{-1}\theta}x_1) > 0\} \). Note that \( \text{sp}(u(0, x', y')) \) is also well-defined, and it is a microlocal operator.

Let us define the corresponding operands. We define

\[
V_\theta(C, r) = \{x \in \mathbb{C}^n; C|x| < 1, \text{Re}(e^{\sqrt{-1}\theta}x_1) > 0, \text{Im} x_n > r|\text{Im} x''|\},
\]

where \( C, r > 0 \). If \( \mathcal{O}(m')(V(C, r) \cup V_\theta(C, r)) \), then \( f \) defines a microfunction \( \text{sp} f \) on \( \{\text{Re}(e^{\sqrt{-1}\theta}x_1) > 0\} \). Moreover \( \text{sp} f \) is a mild microfunction in the sense of [6], and \( \text{sp}(f(0, x')) \) is well-defined.

If \( u(x, y') \in \mathcal{O}(m')(W(C, r) \cup W_\theta(C, r)), f(x) \in \mathcal{O}(m')(V(C, r) \cup V_\theta(C, r)) \), then

\[
u * f(x) = \int_{A(C', \varepsilon)} u(x, y')f(x_1, y')d(x' - y')
\]

\[
\in \mathcal{O}(m')(V(C'', r) \cup V_\theta(C'', r))
\]
MIXED-TYPE OPERATORS

is well-defined \((C'' \gg C' \gg C)\). (14) coincides with the integration in the sense of microfunction on \(|\text{Re}(e^{\sqrt{-1}\theta} x_1)| > 0\}. Moreover, we can also restrict (14) to \(\{x_1 = 0\}\) in the sense of microfunction.

§ 2.3. A Theory of Annihilating Operators

Let \(\theta \in \{0, \pi\}\). Assume that for \(\exists C > 0, \exists R \in (0,1), \forall \varepsilon > 0, \exists C_\varepsilon > 0\), a formal series \(f = \sum_{i \in \mathbb{Z}} f_i(x, \xi') \in \mathcal{F}(\Omega(C))\) satisfies

\[
|f_i(x, \xi')| \leq C e^{Ri} \exp((C|x_1|^{1/m'}|\text{Im} x| + C|x_1|^{1/m'}|\text{Re} \xi'|/\text{Im} \xi_n
+ C(|\text{Re} \xi'|/\text{Im} \xi_n)^2 - C^{-1}|x_1|^{1+(1/m')} + \varepsilon) \text{Im} \xi_n)
\]
on \Omega_\theta,i(C). Then \(\mathcal{F}(f)\) is holomorphic on \(W(C'', r) \cup W_\theta(C'', r)\) with \(\exists C'', \exists r\), where

\[
W_\theta(C'', r) = \{(x, y') \in \mathbb{C}^n \times \mathbb{C}^{n-1}; C''| (x, y')| < 1, \text{Re}(e^{\sqrt{-1}\theta} x_1) > 0, \text{Im}(x_n - y_n) + C''| x_n|^{1+(1/m')} > C''| x_n|^{1/m'}|\text{Im} x| + r|\text{Im}(x'' - y'')|\}.
\]

Let

\[
V_\theta(C, r) = \{x \in \mathbb{C}^n; C|x| < 1, \text{Re}(e^{\sqrt{-1}\theta} x_1) > 0, \text{Im} x_n + C^{-1}|x_n|^{1+(1/m')} > r|\text{Im} x''|\}.
\]

If \(u \in \mathcal{O}(W_\theta'(C, r))\) (resp. \(f \in \mathcal{O}(V_\theta'(C, r))\)), then \(spu = 0\) (resp. \(sp f = 0\)) on \(\{\text{Re}(e^{\sqrt{-1}\theta} x_1) > 0\}\). If \(u \in \lim_{C, r} \mathcal{O}(W_\theta'(C, r))\) or \(f \in \lim_{C, r} \mathcal{O}(V_\theta'(C, r))\), then we have \(u \ast f \in \lim_{C, r} \mathcal{O}(V_\theta(C, r))\). Of course we can consider the case of fractional powers in \(x_1\).

§ 2.4. Other Symbol Classes

Sometimes it is important to consider a formal series defined only for the case when \(x_1\) belongs to a sector. Let \(\theta \in [0, 2\pi]\) = \(\{t \in \mathbb{R}; 0 \leq t \leq 2\pi\}\), and let

\[
\Omega(C, \theta) = \{(x, \xi') \in \Omega(C); x_1 \neq 0, C |\text{arg} x_1 - \theta| < 1\},
\]

\[
\Omega_i(C, \theta) = \Omega(C, \theta) \cap \Omega_i(C)
\]
(Here we do not restrict \(\theta\) to \(\{0, \pi\}\). We define \(\mathcal{F}^\theta(\Omega(C, \theta))\) (resp. \(\mathcal{S}^\theta(\Omega(C, \theta))\)) as the set of formal series \(f = \sum_{i \in \mathbb{Z}} f_i(x, \xi')\) satisfying (8) (resp. (13)) on \(\Omega_i(C, \theta)\), instead of \(\Omega_i(C)\). We define \(\mathcal{A}^\theta(\Omega(C, \theta))\) as the set of formal
series $\sum f_i \in \mathcal{F}(\Omega(C))$ such that for $\exists R \in (0,1)$, $\forall \epsilon > 0$, $\exists C_\epsilon > 0$ we have $|f_i(x, \xi')| \leq C_\epsilon R^I \exp(\epsilon \text{Im} \xi_n)$ on $\Omega_i(C, \theta)$. We define $\mathcal{F} = \lim_{C \to 0} \mathcal{F}(\Omega(C, \theta))$, and similarly we define $\mathcal{F}_\theta$. 

**Lemma 4.** Let $\theta \in \{0, \pi\}$. We have $\mathcal{T} \cap \mathcal{F}_\theta \subset \mathcal{F}_\theta$ and $\mathcal{T}(m') \cap \mathcal{F}_\theta \subset \mathcal{F}_\theta(m')$.

**Proof.** Let $\sum f_i \in \mathcal{T} \cap \mathcal{F}_\theta$. Let $\epsilon$ be an arbitrary number. If $(x, \xi') \in \Omega_{\theta, i}(C)$ satisfies $|\text{Im} x_1| \ll |\text{Re} x_1|$, we have

$$|f_i(x, \xi')| \leq \exists C_\epsilon R^I \exp(C|\text{Im} x_1|^{1/m'}|\text{Re} x_1|^{1/m'} + C|\text{Re} \xi'|^2 |\text{Im} \xi_n|^{-1} + \epsilon) \text{Im} \xi_n \}. $$

If $(x, \xi') \in \Omega_{\theta, i}(C)$ does not satisfy $|\text{Im} x_1| \ll |\text{Re} x_1|$, we have

$$|f_i(x, \xi')| \leq \exists C_\epsilon R^I \exp((C|\text{Im} x_1|^{1+(1/m')} + C|\text{Re} \xi'|/|\text{Im} \xi_n| + \epsilon) \text{Im} \xi_n \}.$$ 

This means $\sum f_i \in \mathcal{F}_\theta$. The proof of the latter statement is the same. Q.E.D.

### §3. Transformation by Holomorphic Microlocal Operators

In this section, we given a canonical representation of the elementary solution of the Cauchy problem (5). We first rewrite the equation using matrices. Let $L(x, D)$ be an $m \times m$ matrix defined by $L(x, D) = D_1 I_m + L(x, D')$, where $I_m$ is the unit matrix and $L$ is defined by

$$
L = \begin{pmatrix}
0, & -1, & 0 \\
\ldots & \ldots & \ldots \\
0, & 0, & -1, \\
P_0(x, D'), & P_1(x, D'), & \ldots, & P_{m-1}(x, D')
\end{pmatrix}.
$$

Let $\tilde{u}(x) = (u(D_1 u, \ldots, D_{m-1} u)) \in (\mathcal{C}_{R^+, x'})^m$ and $\tilde{v}(x') = (v_1, \ldots, v_m) \in (\mathcal{C}_{R^+, x'})^m$. Then (5) is equivalent to

$$L \tilde{u} = 0, \quad \tilde{u}(0, x') = \tilde{v}(x').$$

We grade the complete symbol of $L$ as follows. If $A(x, D) = \sum_{j \leq \ell} A_j(x, D) \in \mathcal{F}_x$, is a microdifferential operator of order at most $\ell \in \mathbb{Z}$, and each $A_j$ is homogeneous in $D$ of degree $j$, we denote $A_j(x, \xi)$ by $\sigma_j(A)$ (This notation depends on the choice of the symplectic coordinate system, except for the principal symbol). Now we define
for \( j \in \mathbb{Z}_+ \). If \( Y \) is a ring, we denote by \( Y^{m \times m} \) the ring of \( m \times m \) matrices whose components belong to \( Y \). It is easy to see that we can inductively define \( E_i^{(\pm)}(x, \xi) \in (\mathcal{O}(\Omega(C)))^{m \times m}, i \in \mathbb{Z}_+, \ C \gg 1, \) by

\[
\begin{aligned}
\frac{\partial}{\partial x_1} E_i^{(+)}(x, \xi') + \tilde{L}(x, \xi') \circ E_i^{(+)}(x, \xi') &= 0, & E_i^{(+)}(0, x', \xi') &= 1, \\
\frac{\partial}{\partial x_1} E_i^{(-)}(x, \xi') - E_i^{(-)}(x, \xi') \circ \tilde{L}(x, \xi') &= 0, & E_i^{(-)}(0, x', \xi') &= 1,
\end{aligned}
\]

where \( E^{(\pm)} = \sum_i E_i^{(\pm)} \). In fact, if \( E_i^{(\pm)} \) are already calculated for \( 0 \leq i' \leq i - 1 \), (15) is an ordinary differential equation for \( E_i^{(\pm)} \), which is easy to solve.

For each number \( \lambda \in \{1, \ldots, m\} \), let \( X^{(\pm, \lambda)}(x, D') \) be a Fourier integral operator (maybe with a complex phase function) satisfying

\[
\begin{aligned}
\frac{\partial}{\partial x_1} X^{(+, \lambda)}(x, \xi') - \varphi_\lambda(x, \xi') \circ X^{(+, \lambda)}(x, \xi') &= 0, & X^{(+, \lambda)}(0, x', \xi') &= 1, \\
\frac{\partial}{\partial x_1} X^{(-, \lambda)}(x, \xi') + X^{(-, \lambda)}(x, \xi') \circ \varphi_\lambda(x, \xi') &= 0, & X^{(-, \lambda)}(0, x', \xi') &= 1,
\end{aligned}
\]

respectively.

**Remark.** Here we are considering holomorphic functions in \((x_1^{1/m'}, x', \xi')\). If \( f = \sum_{0 \leq k \leq m' - 1} \chi_k^{m'} f_k \in \mathcal{O}(m'), x' \) with \( f_k \in \mathcal{O}(x') \), then we define \( f|_{x_1 = 0} = f_0|_{x_1 = 0} \). We consider the above equations in this sense.

In fact we can calculate the complete symbol \( X^{(\pm, \lambda)}(x, \xi') \) of the corresponding operator in the form \( X^{(\pm, \lambda)}(x, \xi') \equiv \exp(\psi^{(\pm, \lambda)}(x, \xi')) \sum_i X_i^{(\pm, \lambda)}(x, \xi') \) modulo \( \mathcal{N}(m')(\Omega(C)) \) with some elliptic amplitude function \( \sum_i X_i^{(\pm, \lambda)} \) of order 0. Here we denote by \( \psi^{(\pm, \lambda)}(x, \xi') \) the phase function defined by (6)\(_\pm\). Let \( X^{(\pm)}(x, \xi') \) be the matrix defined by

\[
X^{(\pm)}(x, \xi') = \begin{pmatrix}
X^{(\pm, 1)} & 0 \\
0 & \ddots \\
& & X^{(\pm, m)}
\end{pmatrix} \in \mathcal{F}^{m \times m}_{(m')}. 
\]

Now we have the following

**Theorem 2.** We assume (2), (3), (4), and (7). We have \( E^{(\pm)}(x, \xi') \in \mathcal{F}^{m \times m}, \) and for each \( \theta \in [0, 2\pi] \) there exist \( F^{(\pm, \theta)}(x, \xi'), G^{(\pm, \theta)}(x', \xi') \in (\mathcal{F}_{(m')} \cap \mathcal{R}^0)^{m \times m} \) such that
Remark. (i) Note that \( \theta \) is an arbitrary real number belonging to \([0, 2\pi]\) this time.

(ii) \( F^{(\pm, \theta)} \) is defined in a full neighborhood of \( x_1 = 0 \), belongs to \( \mathcal{T}^{m \times m} \) there, and belongs to \( (\mathcal{B}^{\theta})^{m \times m} \) in a sector around \( x_1 = 0 \). Therefore \( F^{(\pm, \theta)}(0, x', \xi') = I_m \).

(iii) Since \( G^{(\pm, \theta)} \) does not depend on \( x_1 \), it follows that its \((\mu, \nu)\)-element \( G_{i,i}^{(\pm, \theta)} \) is holomorphic on \( \mathcal{O}(\Omega_i(C)) \), and \( \forall R \in (0, 1), \forall \epsilon > 0, \exists C_\epsilon > 0 \) we have \( |G_{i,i}^{(\pm, \theta)}(x', \xi')| \leq C_\epsilon R^i \exp(\epsilon \mathrm{Im} \xi_n) \) on \( \Omega_i(C) \). According to [2], it is a symbol of \( G_{i,i}^{(\pm, \theta)}(x', D') \in \mathcal{E}_x^{*} \), and we have \( G^{(\pm, \theta)}(x', D') G^{(\mp, \theta)}(x', D') = I_m \).

(iv) \( E^{(\pm)}(x, D'), X^{(\pm)}(x, D'), F^{(\pm, \theta)}(x, D'), G^{(\pm, \theta)}(x', D') \) are operators acting on \( \mathcal{E}_x^{(m)} \). Some of these operators (and their composites) do not contain fractional powers in \( x_1 \).

(v) Let \( \theta \in \{0, \pi\} \). Then \( F^{(\pm, \theta)}(x, D') \) is a map of \( \mathcal{O}(V(C, r) \cup V_\theta(C, r)) \) into \( \mathcal{O}(V(C', r) \cup V_\theta(C', r)) \) with \( C' \gg C \).

(vi) Let \( \theta \in \{0, \pi\} \). If \( \mu \in M_{0,0} \cup M_{-\theta} \) (resp. \( \mu \in M_{0,0} \cup M_{+\theta} \)), then \( X_{(\mu,\mu)}^{(\pm)} \) (resp. \( X_{(\mu,\mu)}^{(-)} \)) is a map of \( \mathcal{O}(V(C, r) \cup V_\theta(C, r)) \) into \( \mathcal{O}(V(C', r) \cup V_\theta(C', r)) \) with \( C' \gg C \). This operator is well-defined in the sense of microfunctions on \( \{e^{\sqrt{-1} \theta} x_1 > 0\} \).

(vii) Let \( \theta \in \{0, \pi\} \). If \( \mu \in M_{-\theta} \) (resp. \( \mu \in M_{+\theta} \)), then \( X_{(\mu,\mu)}^{(\pm)} \) (resp. \( X_{(\mu,\mu)}^{(-)} \)) is a map of \( \mathcal{O}(V(C, r) \cup V_\theta(C, r)) \) into \( \mathcal{O}(V(C', r) \cup V_\theta(C', r)) \) with \( C' \gg C \). This operator annihilates all the microfunctions on \( \{e^{\sqrt{-1} \theta} x_1 > 0\} \).

Admitting Theorem 2, we can prove Theorem 1 as follows. We want to prove that the Cauchy problem is solvable if, and only if, \( v_1(x'), \ldots, v_m(x') \in \mathcal{E}_{x'}^{m-1} \) satisfy the following \( m_{+0} \)-relation:

\[
(16)_0 \quad (G^{(-,0)}(x', D') \bar{v}(x'))_{\mu} = 0, \quad \mu \in M_{+0},
\]

and the following \( m_{+\pi} \)-relation:

\[
(16)_{\pi} \quad (G^{(-,\pi)}(x', D') \bar{v}(x'))_{\mu} = 0, \quad \mu \in M_{+\pi}.
\]
Let us prove the sufficiency. We want to show that if $\tilde{v}$ satisfies (16)$_0$ and (16)$_1$, then $E^{(+)}(x, D')\tilde{v}$ is well-defined as a microfunction and its first component $(E^{(+)}(x, D')\tilde{v})$ satisfies (5).

Let $\theta \in \{0, \pi\}$, and let $\tilde{w}^{(\theta)} = G^{(-, \theta)}(x', D')\tilde{v}(x')$. We assume (16)$_{\theta}$. Therefore we have $\tilde{w}^{(\theta)} = 0$ for $\mu \in M_{+, \theta}$, and $\tilde{w}(x') = G^{(+, \theta)}(x', D')\tilde{w}^{(\theta)}(x')$. We have

$$v \notin M_{+, \theta} \Rightarrow \left\{ \begin{array}{l}
(E^{(+)}(x, \xi'') \circ G^{(+, \theta)}(x', \xi'))_{(\mu, \nu)} \in \mathcal{F} \cap S^\theta \subset S^\theta, \\
(\mathcal{F}(E^{(+)} \circ G^{(+, \theta)}))_{(\mu, \nu)}(x, y') \in \mathcal{C}(W(C, r) \cup W_0(C, r)), \quad \exists C, \exists r
\end{array} \right.$$

Let $\tilde{v} = \{u^-_1, \ldots, u^-_m\} \in V^m$ by $\tilde{v} = \mathcal{F}(E^{(+)})* \tilde{v}$. Then we have

$$u^-_\mu(x) = (\mathcal{F}(E^{(+)})* \tilde{v}^-)_\mu \equiv \sum_{1 \leq \kappa \leq m} (\mathcal{F}(E^{(+)})* \mathcal{F}(G^{(+, \theta)}))_{(\mu, \kappa)} \ast \tilde{w}^{(\theta)}_{\kappa} \ast \tilde{v}^- (x')$$

for any $\mu$. It follows that $\tilde{u}^-(x) \in \mathcal{C}(V(C, r) \cup V_0(C, r) \cup V_{\pi}(C, r))^m$. We have $\mathcal{F}(L) \ast \tilde{u}^- (x) \equiv \mathcal{F}(L) \ast \mathcal{F}(E^{(+)})* \tilde{v}^- (x') \equiv \mathcal{F}(E^{(+)})* (D_1 \tilde{v}^- (x')) = \tilde{0}$ modulo $\mathcal{C}^m_{C^*, 0}$. This means $\tilde{u} \in (\mathcal{F}_{R^*, x^*})^m$, $L(x, D)u = \tilde{0}$, $\tilde{u}(0, x') = \tilde{v}(x')$, and $u_1$ satisfies (5).

We next prove the necessity. Let $\theta \in \{0, \pi\}$. We want to show that if $u \in M_{+, \theta}$, then $(G^{(-, \theta)}(x', D')E^{(-)}(x, D')\tilde{v})$ is a microfunction which does not depend on $x_1$, coincides with $(G^{(-, \theta)}(x', D')\tilde{v})$ if $x_1 = 0$, and vanishes if $e^\sqrt{-\theta}x_1 > 0$ (Note that this means (16)$_{\theta}$, (16)$_{\theta}$). We assume that $u$ is a solution of (5). We define $\tilde{u} = \{u_1, \ldots, u_m\}$ by $u_j = D_1^{j-1} u$, and therefore $\tilde{u}$ satisfies $L(x, D)\tilde{u} = \tilde{0}$, $\tilde{u}(0, x') = \tilde{v}(x') = \{v_1, \ldots, v_m\}$. We define

$$\tilde{U} = \left\{ \begin{array}{l}
\tilde{u}, \quad |x'| < R, |\text{Im } \xi'| < R \text{ Im } \xi_n, \\
\tilde{0}, \quad |\text{Im } \xi'| > R \text{ Im } \xi_n, \\
\tilde{0}, \quad |x'| > R,
\end{array} \right.$$

where $R > 0$ is small enough. Let $\tilde{V}(x') = \tilde{U}(0, x')$. Then we have supp$(L(x, D)\tilde{U}) \subset \{(x', \xi'); |x'| = R\} \cup \{|\text{Im } \xi'| = R \text{ Im } \xi_n\}$, and $\tilde{V}(x') = \tilde{v}(x')$ on $\{(x', \xi'); |x'| < R, |\text{Im } \xi'| < R \text{ Im } \xi_n\}$, and supp$(\tilde{V}(x')) \subset \{(x', \xi'); |x'| \leq R, |\text{Im } \xi'| \leq R \text{ Im } \xi_n\}$. Let $K = (K_{(\mu, \nu)}) \in \mathbb{C}^{m \times m}$ be the diagonal matrix defined by $K_{(\mu, \mu)} = 0$ if $\mu \notin M_{+, \theta}$, and $K_{(\mu, \mu)} = 1$ if $\mu \in M_{+, \theta}$. We may assume that $\tilde{U}$ has a defining function $\tilde{U}^- \in (\mathcal{C}(V(C, r))^m$ with $0 < r \ll 1$. Let $\tilde{W}^{(\theta)} = (KG^{(-, \theta})* E^{(-)})* \tilde{v}^-$, and let $\tilde{V}(x') = \tilde{U}^- (0, x')$. We have
\[ D_1 \tilde{W}^{(\theta)}(x) \equiv D_1(KG^{(-\theta)} \ast E^{-\theta}) \ast \tilde{U}^{-\theta} \]
\[ \equiv (KG^{(-\theta)} \ast E^{-\theta}) \ast (\mathcal{F}(L) \ast \tilde{U}^{-\theta}) \quad \text{modulo } \mathcal{O}^{m}_{Cm,0}. \]

Here the right-hand side belongs to \( \gamma_2^{m} \). Furthermore, it defines a hyperfunction, which is microanalytic at \( dxn_{\infty} \) on \( \{e^{\sqrt{-1}\theta}x_1 > 0\} \). We also have
\[ \tilde{W}^{(\theta)}(0, x') \equiv KG^{(-\theta)} \ast \tilde{U}^{-\theta}(0, x') \equiv KG^{(-\theta)} \ast \tilde{V}^{-\theta}(x') \quad \text{modulo } \mathcal{O}^{m}_{Cm,0}. \]

It follows that \( \tilde{W}^{(\theta)}(x) \equiv \tilde{W}^{(\theta)}(0, x') \) modulo \( \gamma_2^{m} \). We have \( KG^{(-\theta)} \ast E^{-\theta} \in \mathcal{O}(W'_{\theta}(C', r'))^{m \times m} \) with some \( \exists C' > 0, \exists r' > 0, \) and \( \tilde{W}^{(\theta)} \in \mathcal{O}(V'_{\theta}(C'', r''))^{m} \) with some \( \exists C'' > 0, \exists r'' > 0, \) and \( \tilde{W}^{(\theta)}(x_1, x') \) is analytic at \( x' = 0 \), and we obtain \( \tilde{W}^{(\theta)}(0, x') \in \gamma_2^{m} \). Taking the singularity spectrum we obtain \( (16)_{\theta} \).

Therefore we next need to prove Theorem 2, and the plan is as follows. We define \( \tilde{M}(x, \xi') \) by
\[ \tilde{M}(x, \xi') = \begin{pmatrix} -\varphi_1(x, \xi') & 0 \\ 0 & -\varphi_m(x, \xi') \end{pmatrix}. \]

It suffices to find \( F^{(\pm, \theta)}(x, \xi') \in (\mathcal{F}(m') \cap \mathcal{H})^{m \times m} \) satisfying
\[
\begin{cases}
\partial_{x_1} F^{(\pm, \theta)} + \bar{L} \circ F^{(\pm, \theta)} - F^{(\pm, \theta)} \circ \tilde{M} = O, \\
\partial_{x_1} F^{(-, \theta)} + \bar{M} \circ F^{(-, \theta)} - F^{(-, \theta)} \circ L = O, \\
F^{(\pm, \theta)} \circ F^{(\mp, \theta)} = I_m.
\end{cases}
\]

For this purpose we need to consider the case \( |x_1 \xi_n| \gg 1 \) and the contrary case separately. Precisely speaking, let \( C \gg 1, \theta \in [0, 2\pi], \) and we define
\[ \mathcal{E}(C) = \{ (x, \xi') \in C^n \times C^{n-1}; C|x| < 1, C|\xi''| < |\text{Im} \xi_n, \}
\]
\[ C|\text{Re} \xi_n| < |\text{Im} \xi_n, C^{4m(q+1)} < |\xi_n \}
\]
\[ \mathcal{E}'(C) = \{ (x, \xi') \in \mathcal{E}(C); C(|\text{Im} \xi_n|^{1/(mq+m)}) < |x_1| \}
\]
\[ \mathcal{E}'(C, \theta) = \{ (x, \xi') \in \mathcal{E}'(C); C|\text{arg} x_1 - \theta| < 1 \}
\]
\[ \mathcal{E}''(C) = \{ (x, \xi') \in \mathcal{E}(C); |x_1| < C^{-1/2}(|\text{Im} \xi_n|^{1/(2mq+2m)}) \}
\]

Note that \( \mathcal{E}'(C) \cup \mathcal{E}''(C) = \mathcal{E}(C) \).

At first we shall calculate \( F^{(\pm, \theta)} \) on \( \mathcal{E}'(C) \). This part is divided into several steps. Section 4 is an auxiliary step. The most important part is Section 6, and in Section 5 we shall prepare an estimate of the phase functions, which will be necessary in Section 6. Since such a phase calculation is valid when \( x_1 \) belongs to a sector with its vertex at the origin, we shall discuss on \( \mathcal{E}'(C, \theta) \).
After that we shall calculate $F^{(\pm, \theta)}$ on $\Xi''(C)$ in Section 7, and also on the whole $\Xi(C)$ (We do not divide $\Xi''(C)$ into sectors). Let us illustrate our plan by a special case without proof.

Example. For the sake of simplicity we assume that $m = 2$ and let $L = D_1 I_2 + L(x_1, D_n)$, where

$$L(x_1, D_n) = \begin{pmatrix} 0 & -1 \\ -x_1^{2q} D_n^2 + a D_n & b \end{pmatrix}, \quad a, b \in \mathbb{C}$$

We need to solve

\[
\begin{align*}
\partial_x E^{(+)}(x_1, \xi_n) E^{(+)}(x_1, \xi_n) &= 0, \\
\partial_x E^{(-)}(x_1, \xi_n) E^{(-)}(x_1, \xi_n) &= 0,
\end{align*}
\]

Let

$$X^{(\pm)} = \begin{pmatrix} \exp \left( \pm \frac{1}{q+1} x_1^{q+1} \xi_n \right) & 0 \\ 0 & \exp \left( \pm \frac{1}{q+1} x_1^{q+1} \xi_n \right) \end{pmatrix}.$$ 

We can proceed as follows:

(i) Neglecting the initial value for the moment, we have solutions $\tilde{E}^{(+, \theta)} = F^{(+, \theta)} X^{(+)}$ and $\tilde{E}^{(-, \theta)} = X^{(-)} F^{(-, \theta)}$ of the above equations on $\Xi'(C, \theta)$. Here $F^{(\pm, \theta)}(x_1, \xi_n)$ are infra-exponential functions of $\xi_n$, and we have $F^{(\pm, \theta)} F^{(\mp, \theta)} = I_2$.

(ii) We next extend $F^{(\pm, \theta)}$ to $\Xi''(C)$. We can extend $F^{(\pm, \theta)}$ to a infra-exponential function on this region.

(iii) We finally adjust the initial value. Let $\Xi(C, 0) = \Xi(C) \cap \{x_1 = 0\}$. $\Xi''(C)$ is not a usual conical neighborhood of $\Xi(C, 0)$, but it is a neighborhood of $\Xi(C, 0)$ in the topological sense. Therefore we can define $G^{(\pm, \theta)} = F^{(\mp, \theta)} |_{x_1 = 0}$ on $\Xi(C, 0)$. Then we have $G^{(\pm, \theta)} G^{(\mp, \theta)} = I_2$, and we define $H^{(\pm, \theta)}$ by

$$H^{(+, \theta)}(x_1, \xi_n) = \tilde{E}^{(+, \theta)}(x_1, \xi_n) (G^{(+, \theta)}(x_1, \xi_n))^{-1}$$

$$= \tilde{E}^{(+, \theta)}(x_1, \xi_n) G^{(-, \theta)}(x_1, \xi_n)$$

$$= F^{(+, \theta)}(x_1, \xi_n) X^{(+)}(x_1, \xi_n) G^{(-, \theta)}(x_1, \xi_n),$$

$$H^{(-, \theta)}(x_1, \xi_n) = G^{(+, \theta)}(x_1, \xi_n) X^{(-)}(x_1, \xi_n) F^{(-, \theta)}(x_1, \xi_n).$$

This means that $H^{(\pm, \theta)}$ are the solutions of the above Cauchy problem, therefore we have $H^{(\pm, \theta)} = E^{(\pm)}$. After that, we can study these symbols on the whole $\Xi(C)$. 
§ 4. Diagonalization of the Principal Symbol

We assume (2), (3), (4), and (7). Let \( C \gg 1 \). We define

\[
\Omega^0(C) = \{ (x, \xi') \in \Omega(C) ; x_1 \neq 0 \}. 
\]

We will diagonalize \( \tilde{L}(x, \xi) \) in several steps. Let us consider the following Vandermonde’s matrix on \( \Omega^0(C) \):

\[
A(x, \xi') = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\varphi_1(x, \xi') & \varphi_2(x, \xi') & \cdots & \varphi_m(x, \xi') \\
\vdots & \vdots & \ddots & \vdots \\
(\varphi_1(x, \xi'))^{m-1} & (\varphi_2(x, \xi'))^{m-1} & \cdots & (\varphi_m(x, \xi'))^{m-1}
\end{pmatrix}.
\]

It is easy to see that the \((\mu, \nu)\)-component \( A_{\mu, \nu}^{-1}(x, \xi') \) of the inverse matrix \( A^{-1} \) (in the sense of the usual multiplication, and not in the sense of the previous composition \( A \circ A^{-1} \)) satisfies

\[
|A_{\mu, \nu}^{-1}(x, \xi')| \leq \exists \text{ constant}(|x_1|^{q_{\nu} \text{ Im } \xi_n})^{-\nu+1}.
\]

Now let us calculate the inverse \( A^{-1}(x, \xi') \) of \( A(x, \xi') \) in the sense of our composition this time. Let \( f = \sum_{0 \leq j \leq i} f_{i,j} \) be a formal sum. We write \( f = g \) or \( f = g \), if \( f_{i,j} = g_{i,j} \) for every \( i \) and \( j \). On the other hand, if \( \sum f_{i,j} = \sum g_{i,j} \) for every \( i \), we write \( f = g \). Note that \( \sum f_{i,j} \) is a finite sum for each \( i \). Finally we define \( f \circ g = \sum h_{i,j} \) by

\[
h_{i,j} = \frac{1}{x_1 |x_1|} \partial_x^{\xi'} f_{i,j} \partial_x^{\xi'} g_{i,j}.
\]

**Lemma 5.** Let \( C \gg a \gg 1 \). We define \( A_{i,j}(x, \xi') = \delta_{i,j} \delta \partial \partial \partial A(x, \xi') \). Then there exist \( A_{i,j}^-(x, \xi') \in \mathcal{O}(\Omega^0(C))^{m \times m} \), \( 0 \leq j \leq i \), which satisfy \( A_{i,j}^+(x, \xi') \circ A_{i,j}^-(x, \xi') = I_m \), and we have

\[
|A_{i,j}^+(x, \xi')| \leq a(\text{Im } \xi_n)^{\mu-1},
\]

\[
|A_{i,j}^-(x, \xi')| \leq a^{2i+1}(i-j)!(i-j)! |x_1|^{-q(m-1)(j+1)}(\text{Im } \xi_n)^{-\nu+1-i}
\]

on \( \Omega^0(C) \).

**Proof.** Let \( A' = A \circ A^{-1} \). In other words we define \( A_{i,j}^{'}(x, \xi') = \sum_{1 \leq k \leq m} \frac{1}{x_1^{\alpha_k}} \partial_x^{\xi'} A \partial_x^{\xi'} A^{-1} \). Then we have \( A_{i,j}^{'} = I_m \), and

\[
|A_{i,j}^{'}(x, \xi')| \leq 3 \delta_{i,j} (i-j)! |x_1|^{q(1-v)}(\text{Im } \xi_n)^{\mu-\nu-1}.
\]
We define
\[ A_{i,j}' = \begin{cases} \delta_{j0} A_{0}', & i = 0, \\ \delta_{j1} A_{1}', & i \geq 1, \end{cases} \]
and \( A_{i,j}'' = \delta_{j0} \delta_{j0} I_m - \sum_{i' + i'' = |\alpha'| = i} \frac{1}{\alpha'!} \partial^{\alpha'} x_j \partial^{\alpha'} x_i A_{i',j}'' \partial^{\alpha'} x_i A_{i',j}'' \) by induction on \( i \). By a direct calculation we can prove
\[ |\partial^{\alpha'} x_j A_{i,j}(x, \xi')| \leq 3a^{2l+|\alpha'|+1} (i - j + |\alpha'|)! |x_1|^{-q(m-1)} (\text{Im} \xi_n)^{\mu - \nu - i}. \]
Defining \( A^- = A^{-1} \circ A'' \) we obtain Lemma 5. Q.E.D.

Let us define \( L'(x, \xi) = \xi_1 I_m + \bar{L}'(x, \xi') \), where
\[ \bar{L}'(x, \xi') = A^{-}(x, \xi') \circ \bar{L}(x, \xi') \circ A^+ + A^{-}(x, \xi') \circ \partial_x A^+. \]
Therefore we have \( L(x, D) A^+(x, D') = A^+(x, D') \bar{L}(x, D) \) formally. Here we remind the reader that we have defined \( \bar{L} = \sum_i I_i = \sum_{0 \leq j \leq i} \bar{L}_{i,j}, \) where \( \bar{L}_{i,j} = \delta_{j,0} \bar{L}_i \). Therefore we have \( \bar{L}' = \sum_{0 \leq j \leq i} \bar{L}_{i,j}' \) naturally. We have obtained the following

**Lemma 6.** If \( C \gg a \gg 1 \), then we have
\[ \partial_x A^+ + \bar{L} \circ A^+ - A^+ \circ \bar{L}' = O, \]
\[ |\bar{L}'_{0,(\mu, \nu)} + \partial_{\mu} \varphi_\mu(x, \xi')| \leq a |x_1|^{-q(m-1)-1}, \]
\[ |\bar{L}'_{i,j,(\mu, \nu)}| \leq a^{i+1}(i - j)! |x_1|^{-q(m-1)(i+1)} (\text{Im} \xi_n)^{-i+1}, \quad i \geq 1 \]
on \( \Omega^o \).

**§ 5. Miscellanea**

**§ 5.1. Formal Norms**

The next step is the most important part, and will be discussed in the next section. Here we give some preliminaries. At first we define formal norms similar to [3]. Let \( \omega \subset \mathbb{C}^n \times \mathbb{C}^{n-1} \) be an open set. Let \( f = \sum_j f_j(x, \xi') \) be a formal sum where \( f_j(x, \xi') \in \mathcal{O}(\omega) \). We define \( N(f, \omega) = N(f, \omega, t, x, \xi') \) by
\[ N(f, \omega) = \sum_{j \in \mathbb{Z}^n} \frac{2(2n)^{-j} j!}{(j + |\alpha'|)!(j + |\beta'|)!} |\partial^{\alpha'} x_j \partial^{\beta'} x_i f_j(x, \xi')| |\xi'|^{j+|\beta'|} t^{2j+|\alpha'|+|\beta'|}, \]
where \((x, \xi', \xi^\prime) \in \omega\). This is a formal series in \(t\). In place of \(|\partial_x^{\alpha'} \partial_{\xi'}^{\beta'} f_j(x, \xi^\prime)| \cdot |\xi'|^{j+|\beta'|}\) in the above definition, [3] considered its supremum. This means that our formal norm (resp. the formal norm of [3]) is a formal series in \(t\), whose coefficients are continuous functions on \(\omega\) (resp. constants). Let \(f(x, D') \in \mathcal{E}^R(\omega)\). Let \(\varepsilon > 0\) be an arbitrary number. Shrinking \(\omega\) if necessary, we can choose its formal symbol \(f(x, \xi')\) such that \(N(f, \omega)\) is convergent and \(N(f, \omega) \leq \exists C_\varepsilon \exp(\varepsilon \operatorname{Im} \xi_n)\) for \(0 < t \ll 1\), \((x, \xi') \in \omega\). Conversely, such a formal series defines a holomorphic microlocal operator on \(\omega\).

If \(f = \sum_j f_j(x, \xi')\) satisfies \(f_j = 0\) for \(0 \leq j \leq j_0 - 1\), then we define \(N_{j_0}(f, \omega) = N_{j_0}(f, \omega, t, x, \xi')\) by

\[
N_{j_0}(f, \omega) = \sum_{j \geq 0} 2(2n)^{-j} \frac{1}{(j + |\alpha'|)!(j + |\beta'|)!} |\partial_x^{\alpha'} \partial_{\xi'}^{\beta'} f_{j_0}(x, \xi')||\xi'|^{j+|\beta'|} t^{2j+|\alpha'|+|\beta'|}.
\]

If \(f = \sum f_j(x, \xi')\) and \(h_0(x, \xi') \in \mathcal{C}(\omega)\), then we define \(h_0 f = \sum h_0 f_j\). If \(a = \sum a_j t^j\) and \(b = \sum b_j t^j\) (\(a_j, b_j \in \mathbb{R}\)) satisfy \(a_j \leq b_j\) for every \(j\), then we write \(a \leq b\). As in [3] we have the following

**Lemma 7.** If \(f = \sum f_j, g = \sum g_j, \text{ and } h_0 \in \mathcal{C}(\omega)\), then we have

\[
N(f \circ g, \omega) \leq N(f, \omega)N(g, \omega), \quad N(h_0 f, \omega) \leq N(f, \omega)N(h_0, \omega)
\]
on \(\omega\). If \(f = \sum f_j, g = \sum g_j, \text{ and } h_0 \in \mathcal{C}(\omega)\), then we have

\[
N_{j_0+k_0}(f \circ g, \omega) \leq N_{j_0}(f, \omega)N_{k_0}(g, \omega), \quad N_{j_0}(h_0 f, \omega) \leq N_{j_0}(f, \omega)N(h_0, \omega)
\]
on \(\omega\).

We finally consider a formal series \(f = \sum_{0 \leq j \leq i} f_{i,j}(x, \xi')\), and define \(f^j = \sum f_{i,j}\) for each \(j\) (On the other hand, we define \(f_i = \sum j f_{i,j}\) for each \(i\)). For such a formal series with double indices, we define

\[
\mathcal{N}(f, \omega) = \mathcal{N}(f, \omega, t_1, t_2, x, \xi') = \sum_j N_j(f^j, \omega, t_1, x, \xi') t_2^j.
\]

If \(f = \sum_{0 \leq j \leq i} f_{i,j}(x, \xi')\) and \(g = \sum_{0 \leq j \leq i} g_{i,j}(x, \xi')\), then we define \(f \circ g = \sum h_{i,j}\), where \(h_{i,j}\) is defined by

\[
h_{i,j} = \sum_{i'+i'' + |\alpha'| = i} \frac{1}{\alpha'!} \partial_{\xi'}^{\alpha'} f_{i',j'} \partial_{\xi'^j} g_{i''} j^\nu.
\]
as before, and if \( h_{0,0}(x, \xi') \in \mathcal{O}(\omega) \) then we define \( h_{0,0}f = \sum_{0 \leq j \leq i} h_{0,0}f_{i,j} \). If \( a = \sum a_{i,j}t_1^j t_2^j \) and \( b = \sum b_{i,j}t_1^j t_2^j \) \((a_{i,j}, b_{i,j} \in \mathbb{R})\) satisfy \( a_{i,j} \leq b_{i,j} \) for every \( i \) and \( j \), then we write \( a \preceq b \). It is easy to see the following:

**Lemma 8.** If \( f = \sum_{0 \leq j \leq i} f_{i,j}(x, \xi') \), \( g = \sum_{0 \leq j \leq i} g_{i,j}(x, \xi') \), and \( h_{0,0}(x, \xi') \in \mathcal{O}(\omega) \), then we have

\[
\mathcal{N}(f \circ g, \omega) \preceq \mathcal{N}(f, \omega) \mathcal{N}(g, \omega), \quad \mathcal{N}(h_{0,0}f, \omega) \preceq \mathcal{N}(f, \omega) \mathcal{N}(h_{0,0}, \omega)
\]
on \( \omega \).

§ 5.2. Phase Functions

We next give a geometric discussion.

**Lemma 9.** Let \( \theta \in [0, 2\pi] \) be an arbitrary number. We can choose some numbers \( \theta_k', \theta_k'' \) \((1 \leq k \leq m)\) satisfying the following conditions:

(i) \( \theta \in (\theta_m', \theta_m'') \subset (\theta_{m-1}', \theta_{m-1}'') \subset \cdots \subset (\theta_1', \theta_1'') \),

(ii) \( (q_i + 1)\theta_i' + \arg a_i(x^*) \notin \frac{\pi}{2} \mathbb{Z}_+ \), \( \theta_i'' - \theta_i' = \pi/(q_i + 1) \),

(iii) If \( q_i = q_j \), \( i \neq j \), then we have

\[
(q_i + 1)\theta_i' + \arg (a_i(x^*) - a_j(x^*)) \notin \frac{\pi}{2} \mathbb{Z}_+, \quad \theta_i' = \theta_j', \quad \theta_i'' = \theta_j''.
\]

**Proof.** We first remind the reader that \( q_1 \leq q_2 \leq \cdots \leq q_m \). We can define \( \theta_i' \) such that

\[
(q_j + 1)\theta_i' + \arg a_j(x^*) \notin \frac{\pi}{2} \mathbb{Z}_+, \quad \text{if} \quad q_i = q_j,
\]

\[
(q_j + 1)\theta_i' + \arg (a_j(x^*) - a_k(x^*)) \notin \frac{\pi}{2} \mathbb{Z}_+, \quad \text{if} \quad q_i = q_j = q_k, j \neq k,
\]

\[
\theta \in (\theta_i', \theta_i' + \pi/(q_i + 1)),
\]

and \( \theta_i'' = \theta_i' + \pi/(q_i + 1) \). Assume that \( 2 \leq i \leq m \) and that we have already chosen \( \theta_k' \) satisfying \( \theta_k' < \theta < \theta_k' + \pi/(q_k + 1) \) for \( 1 \leq k \leq i - 1 \), and let us define \( \theta_i' \). In case of \( q_i = q_{i-1} \), we define \( \theta_i' = \theta_{i-1}' \). In case of \( q_i > q_{i-1} \) then we choose \( \theta_i' \) satisfying

\[
(q_i + 1)\theta_i' + \arg a_j(x^*) \notin \frac{\pi}{2} \mathbb{Z}_+, \quad \text{if} \quad q_i = q_j,
\]
\[(q_i + 1)\theta'_i + \arg(a_j(x^*) - a_k(x^*)) \notin \frac{\pi}{2} \mathbb{Z}_+ \quad \text{if} \quad q_i = q_j = q_k, j \neq k,\]

\[\theta \in (\theta'_i, \theta'_i + \pi/(q_i + 1)) \subset (\theta'_{i-1}, \theta'_{i-1} + \pi/(q_{i-1} + 1)).\]

Since \(\pi/(q_i + 1) < \pi/(q_{i-1} + 1)\), we can certainly choose such a \(\theta'_i\). In both cases we define \(\theta''_i = \theta'_i + \pi/(q_i + 1)\). It is easy to see (i)–(iii). Q.E.D.

We define \(\theta''''_i = (\theta''_i + \theta''''_i)/2\), and formally let \(q_0 = q_1, \theta'_0 = \theta'_1, \theta''_0 = \theta''_1, \theta''''_0 = \theta''''_1\). We define \(x_1(\lambda) = (e^{-\sqrt{-1}\theta''''_1}x_1)^{q_1+1}\). Let 1 \(\ll a \ll C\). We define

\[\tilde{\Omega}'_1(C) = \{(x, \xi') \in \mathbb{C}^n \times \mathbb{C}^{n-1}; \mathbb{C}^{1/10}|x'| < 1, C^{1/10}|\xi'| < \Im \xi_n, C^{1/10}|\Re \xi_n| < \Im \xi_n,\]

\[2^{-1}C^4(\Im \xi_n)^{(q_1+1)/(mq+m)} < \Re(x_1(\lambda)) + a^{-1}|\Im(x_1(\lambda))| < C^{-q_1-1},\]

\[\theta'_1 < \arg x_1 < \theta''''_1\]

\[\tilde{\Omega}'_1(C) = \{x, \xi') \in \mathbb{C}^n \times \mathbb{C}^{n-1}; C|x'| < 1, C|\xi''''| < \Im \xi_n, C|\Re \xi_n| < \Im \xi_n,\]

\[C^4(\Im \xi_n)^{(q_1+1)/(mq+m)} < \Re(x_1(\lambda)) + a^{-1}|\Im(x_1(\lambda))| < C^{-q_1-1},\]

\[\theta'_1 < \arg x_1 < \theta''''_1\]

(See the figure below). We have the following

**Lemma 10.**

(i) If \(C < C'\), then we have

\[\tilde{\Omega}'_1(C) = \tilde{\Omega}'_1(C')\]

\[\tilde{\Omega}'_1(C) = \tilde{\Omega}'_1(C').\]

(ii) If \((x, \xi') \in \tilde{\Omega}'_1(C), (y, \eta') \in \mathbb{C}^n \times \mathbb{C}^{n-1}, \text{and} \quad |y_j| < C^{-1/5}, \quad |\eta_j|/\Im \xi_n < C^{-1/5} \quad \text{for} \quad 2 \leq j \leq n, \text{then we have} \quad (x_1, x'+ y', \xi' + \eta') \in \tilde{\Omega}'_1(C).\]

(iii) If \((x, \xi') \in \tilde{\Omega}'_1(C^2), (y, \eta') \in \mathbb{C}^n \times \mathbb{C}^{n-1}, \text{and} \quad |y_j| < C^{-2}, \quad |\eta_j|/\Im \xi_n < C^{-2} \quad \text{for} \quad 2 \leq j \leq n, \text{then we have} \quad (x_1, x'+ y', \xi' + \eta') \in \tilde{\Omega}'_1(C).\]

(iv) Let 1 \(\leq \lambda < \mu \leq m\). We define \(t^\pm_{\lambda, \mu} \in \mathbb{C}\) as follows. If \(q_\lambda = q_\mu\), then \(t^+_{\lambda, \mu} \in \mathbb{C}\) is the point at which \(\pm \Re((-a_\lambda(x^*) - a_\lambda(x^*))x_\lambda^{q_1+1})\) takes its maximum when \((x, \xi')\) belongs to the closure of \(\tilde{\Omega}'_1(C)\). If \(q_\lambda \neq q_\mu\), then it is the point at which \(\pm \Re(a_\lambda(x^*)z_\lambda^{q_1+1})\) takes its maximum. Then for any \(\lambda\) and \(\mu\), \((t^+_{\lambda, \mu})^{q_1+1}\) is one of \(\{A_{\lambda}^1, A_{\mu}^1\}\), and \((t^-_{\lambda, \mu})^{q_1+1}\) is the other. Here \(A_{\lambda}^1, A_{\lambda}^0, A_{\mu}^0\) are the points indicated in the following figure.
Proof. (i)–(iii) are direct consequences of a simple calculation, and we prove (iv). We first consider the case \( q_\lambda = q_\mu \). From (iii) of Lemma 9 it follows that in the above figure the line segment combining \((a_\mu(x^*) - a_\lambda(x^*))A''_\lambda\) and \((a_\mu(x^*) - a_\lambda(x^*))A'''_\lambda\) is parallel with neither the real axis nor the imaginary axis. Since \( a > 0 \) is large enough (compared with the slopes of these line segments), we may assume that we have either

\[
\text{Re}((a_\mu(x^*) - a_\lambda(x^*))A'_\lambda) < \text{Re}((a_\mu(x^*) - a_\lambda(x^*))A''_\lambda) < \text{Re}((a_\mu(x^*) - a_\lambda(x^*))A'''_\lambda)
\]
or

\[
\text{Re}((a_\mu(x^*) - a_\lambda(x^*))A'_\lambda) > \text{Re}((a_\mu(x^*) - a_\lambda(x^*))A''_\lambda) > \text{Re}((a_\mu(x^*) - a_\lambda(x^*))A'''_\lambda).
\]

Accordingly, if \((x, \xi^*)\) belongs to the closure of \( \tilde{\Omega}'_\lambda(C) \), we have either

\[
\text{Re}((a_\mu(x^*) - a_\lambda(x^*))A'_\lambda) \leq \text{Re}((a_\mu(x^*) - a_\lambda(x^*))x_1^{q_\lambda+1}) \leq \text{Re}((a_\mu(x^*) - a_\lambda(x^*))A''_\lambda)
\]
or

\[ \text{Re}(a_\mu(x^*) - a_\lambda(x^*)) A_\lambda^+(x^*_i) \geq \text{Re}(A_\mu(x^*) - a_\lambda(x^*) x_\lambda^{q_i+1}) \]

\[ \geq \text{Re}(a_\mu(x^*) - a_\lambda(x^*) A_\lambda^+(x^*_i)). \]

This proves (iv) for the case \( q_\lambda = q_\mu \). We can prove the case \( q_\lambda \neq q_\mu \) similarly. Q.E.D.

Let \( 1 < \lambda < \mu \leq m \). If \((x, \xi') \in \tilde{\Omega}_\lambda^+(C)\), then we define a continuous curve \( \gamma_{\lambda, \mu}^+(x_1) \) from \((t_{\lambda, \mu}^+)^{q_i+1}\) to \(x_\lambda^{q_i+1}\) as follows. Let us consider the case \( t_{\lambda, \mu}^+ = A_\lambda^+ \) (resp. \( t_{\lambda, \mu}^- = A_\lambda^- \)). If \( \arg x_1 \leq \theta_\mu^- \) (resp. \( \arg x_1 \geq \theta_\mu^+ \)), then \( \gamma_{\lambda, \mu}^+(x_1) \) is a line segment from \((t_{\lambda, \mu}^+)^{q_i+1}\) to \(x_\lambda^{q_i+1}\). If \( \arg x_1 > \theta_\lambda^+ \) (resp. \( \arg x_1 < \theta_\mu^- \)), then \( \gamma_{\lambda, \mu}^+(x_1) \) is the union of two line segments: one from \((t_{\lambda, \mu}^+)^{q_i+1}\) to \(y_\lambda^{q_i+1}\), and the other from \(y_\lambda^{q_i+1}\) to \(x_{\lambda}'^{q_i+1}\). Here \( y_1 \) is the point defined by \( \text{Re}(y_1(\lambda)) + a^{-1} |\text{Im} y_1(\lambda)| = \text{Re}(x_1(\lambda)) + a^{-1} |\text{Im} x_1(\lambda)| \), \( \arg y_1 = \theta_\lambda^+ \). Finally we define \( \delta_{\lambda, \mu}^+(x_1) = \{ t \in \mathbb{C}; t^{q_i+1} \in \gamma_{\lambda, \mu}^+(x_1) \} \), and denote by \( \rho_{\lambda, \mu}^+(x_1) \) the length of \( \delta_{\lambda, \mu}^+(x_1) \).

Note that we have

\[
(x, \xi') \in \tilde{\Omega}_\lambda^+(C), \ t \in \delta_{\lambda, \mu}^+(x_1) \setminus \{ t_{\lambda, \mu}^\pm \} \Rightarrow (t, x', \xi') \in \tilde{\Omega}_\lambda^+(C),
\]

\[
(x, \xi') \in \Omega_\lambda^+(C), \ t \in \delta_{\lambda, \mu}^-(x_1) \setminus \{ t_{\lambda, \mu}^\pm \} \Rightarrow (t, x', \xi') \in \Omega_\lambda^+(C).
\]

We illustrate the case \( t_{\lambda, \mu}^\pm = A_\lambda^\pm, \arg x_1 \leq \theta_\lambda^- \) in the following figure.
Let $1 \ll a \ll C_0 \ll C_1 \ll \cdots \ll C_m$, and let $\tilde{Q}_m' = \tilde{Q}'_*(C_*, \Omega'_*= \Omega_*'_*(C_*)$. Therefore we have

\[
\tilde{Q}_0' \supset \cdots \supset \tilde{Q}_m' \\
\cup \\
\Omega_0' \supset \cdots \supset \Omega_m'.
\]

The next lemma will be important in the next section.

**Lemma 11.** Let $1 \leq \lambda < \mu \leq m$, $(x, \xi') \in \tilde{Q}_\lambda$, $t \in \delta_{\lambda, \mu}(x_1)$, and let $\varphi_{\lambda, \mu}(x, t, \xi') = -\int_{x_1}^{x_1} (\tilde{L}_{0, (\lambda, \mu)}(s, x', \xi') - \tilde{L}_{0, (\mu, \mu)}(s, x', \xi')) ds$. Then we have

\[
|a_\mu(\xi')| \leq \frac{1}{a_\lambda(x_1)} t^{q_\lambda + 1} \frac{|\mathbf{Im} \xi_n|}{a_\lambda(x_1)}.
\]

**Proof.** Let

\[
\theta_0 = \begin{cases} 
\arg(a_\mu(x^*) - a_\lambda(x^*)), & q_\mu = q_\lambda, \\
\arg(-a_\lambda(x^*)), & q_\mu \neq q_\lambda.
\end{cases}
\]

We denote

\[
L_{0, (\nu, \nu)}(x, \xi') = x_1^{q_\nu} a_\nu(x, \xi') + b_\nu(x, \xi'),
\]

\[
a_\nu(x, \xi') = a_\nu(0, x', \xi') + x_1^{1/m} a_\nu'(x, \xi').
\]

By Lemma 6 we may assume $|a_\mu'(x, \xi')| \leq a_\lambda(x_1) |\mathbf{Im} \xi_n|$, $|b_\nu(x, \xi')| \leq a_\lambda(x_1)^{-q(m-1)-1}$. We have $\pm \mathbf{Re} \varphi_{\lambda, \mu}(x, t, \xi') = I + II + III + IV$, where

\[
I = \pm \mathbf{Re}(e^{-\sqrt{-1} \theta_0} a_\lambda(0, x', \xi')) \mathbf{Re}(e^{\sqrt{-1} \theta_0} (x_1^{q_\lambda} - t^{q_\lambda + 1}))/ (q_\lambda + 1)
\]

\[
+ \pm \mathbf{Re}(e^{-\sqrt{-1} \theta_0} a_\mu(0, x', \xi')) \mathbf{Re}(e^{\sqrt{-1} \theta_0} (x_1^{q_\mu} - t^{q_\mu + 1}))/ (q_\mu + 1),
\]

\[
II = \pm \mathbf{Im}(e^{-\sqrt{-1} \theta_0} a_\lambda(0, x', \xi')) \mathbf{Im}(e^{\sqrt{-1} \theta_0} (x_1^{q_\lambda} - t^{q_\lambda + 1}))/ (q_\lambda + 1)
\]

\[
+ \pm \mathbf{Im}(e^{-\sqrt{-1} \theta_0} a_\mu(0, x', \xi')) \mathbf{Im}(e^{\sqrt{-1} \theta_0} (x_1^{q_\mu} - t^{q_\mu + 1}))/ (q_\mu + 1),
\]

\[
III = \pm \mathbf{Re}\left(\int_{x_1}^{x_1} (s^{q_\mu + (1/m)} a_\mu'(s, x', \xi') - s^{q_\mu + (1/m)} a_\mu'(s, x, \xi')) ds\right),
\]

\[
IV = \pm \mathbf{Re}\left(\int_{x_1}^{x_1} (b_\mu(s, x', \xi') - b_\mu(s, x, \xi')) ds\right).
\]

We can prove

\[
I \leq \frac{1}{2a_\lambda^3(q_\lambda + 1)} |x_1^{q_\lambda} - t^{q_\lambda} + 1| \mathbf{Im} \xi_n
\]
as follows. We first consider the case $q_\lambda = q_\mu$. Since we have

\begin{equation}
|\text{arg}(e^{-\sqrt{-1}Q_0(\mu, 0, \xi) - a_\xi(0, \xi'))}| < a^{-10}
\end{equation}

it follows that \(\text{Re}(e^{-\sqrt{-1}Q_0(\mu, 0, \xi) - a_\xi(0, \xi'))}) \geq a^{-1/2} \text{Im} \xi_n\). From Lemma 10 it follows that \(\pm \text{Re}(e^{-\sqrt{-1}Q_0(x^{q_i+1} - t^{q_i+1}))} \leq -a^{-5/2}|x^{q_i+1} - t^{q_i+1}|\).

Therefore we obtain

\begin{align*}
I = \pm \text{Re}(e^{-\sqrt{-1}Q_0(a_\xi(0, \xi') - a_\xi(0, \xi'))}) \text{Re}(e^{-\sqrt{-1}Q_0(x^{q_i+1} - t^{q_i+1}))}/(q_\lambda + 1) \\
\leq -\frac{1}{a^3(q_\lambda + 1)}|x^{q_i+1} - t^{q_i+1}| \text{Im} \xi_n.
\end{align*}

We next consider the case $q_\lambda \neq q_\mu$. We can similarly prove

\begin{align*}
\pm \text{Re}(e^{-\sqrt{-1}Q_0(a_\xi(0, \xi') - a_\xi(0, \xi'))}) \text{Re}(e^{-\sqrt{-1}Q_0(x^{q_i+1} - t^{q_i+1}))}/(q_\lambda + 1) \\
\geq \frac{1}{a^3(q_\lambda + 1)}|x^{q_i+1} - t^{q_i+1}| \text{Im} \xi_n.
\end{align*}

If $\ell \in \mathbb{Z} + m'$ satisfies $0 \leq \ell \leq (m' - 1)/m'$, then we have

\begin{equation}
\arg(x_1^{1/2}t^{1/2} - t^{1/2}) \in \left[\left(q_\lambda + \frac{m' - 1}{m'}\right)\theta_\lambda', \left(q_\lambda + \frac{m' - 1}{m'}\right)\theta_\lambda''\right],
\end{equation}

and thus

\begin{equation}
\text{Re}(e^{-\theta^{1/2}Q_0(x_1^{1/2}t^{1/2} - t^{1/2})}) \geq \sin \left(\frac{\pi}{2m'(q_\lambda + 1)}\right)|x_1^{1/2}t^{1/2} - t^{1/2})^{1/2}((m' - 1)/m')\right).
\end{equation}

It follows that

\begin{align*}
|x^{q_i+1} - t^{q_i+1}| &\geq a^{-1} \sum_{0 \leq \ell \leq q_i+1((m'-1)/m')} |x_1^{1/m'-1}t^{1/m'-1}||x_1^{1/m'} - t^{1/m'}||(m'q_i + m' - 1) \\
&\geq a^{-2}|x_1^{1/m'} - t^{1/m'}||(m'q_i + m' - 1) \\
&\geq a^{-2}|x_1^{1/m'} - t^{1/m'}||(m'q_i + m' - 1) \\
\end{align*}

Since we have $q_\mu \geq q_\lambda + 1$, it follows that

\begin{equation}
|x^{q_i+1} - t^{q_i+1}| \leq a|x_1^{1/m'} - t^{1/m'}||(m'q_i + m' - 1) \\
\leq a^{-7}|x_1^{1/m'} - t^{1/m'}||(m'q_i + m' - 1) \\
\leq a^{-5}|x_1^{q_i+1} - t^{q_i+1}|.
\end{equation}

Therefore we have
\[ I \leq - (a^3(q_\lambda + 1))^{-1} |x_1^{q_\lambda + 1} - t^{q_\lambda + 1}| \text{Im} \xi_n + a|x_1^{q_\mu + 1} - t^{q_\mu + 1}| \text{Im} \xi_n \]
\[ \leq -(2a^3(q_\lambda + 1))^{-1} |x_1^{q_\lambda + 1} - t^{q_\lambda + 1}| \text{Im} \xi_n, \]

and we obtain (17).

We next prove
\[ \text{(20)} \]
\[ |\mathbb{II}|, |\mathbb{III}|, |\mathbb{IV}| \leq 2a^{-7} |x_1^{q_\lambda + 1} - t^{q_\lambda + 1}| \text{Im} \xi_n. \]

Let \( q_\lambda = q_\mu \). In this case (20) for \( \mathbb{II} \) is a direct consequence of (18). Let \( q_\lambda \neq q_\mu \). In this case from (18) and (19) we obtain
\[ |\mathbb{II}| \leq a^{-7} |x_1^{q_\lambda + 1} - t^{q_\lambda + 1}| \text{Im} \xi_n + a|x_1^{q_\mu + 1} - t^{q_\mu + 1}| \text{Im} \xi_n \]
\[ \leq 2a^{-7} |x_1^{q_\lambda + 1} - t^{q_\lambda + 1}| \text{Im} \xi_n. \]

We next prove (20) for \( \mathbb{III} \). We have
\[ |\mathbb{III}| \leq \int_{\mathbb{E}_{\lambda, \mu}(x_1) \setminus \mathbb{E}_{h}(x_1)} |s|^{1/(m'q_\lambda + m')} \text{Im} \xi_n |ds| \leq 2a^{-7} |x_1^{q_\lambda + 1} - t^{q_\lambda + 1}| \text{Im} \xi_n, \]

and we obtain (20) for \( \mathbb{III} \). The proof for \( \mathbb{IV} \) is similar. We obtain Lemma 11 from (17) and (20). Q.E.D.

From (ii) of Lemma 10 and Lemma 11 we obtain the following

**Corollary.** Let \( 1 \leq \lambda < \mu \leq m, (x, \xi') \in \mathbb{Q}_\lambda', t \in \mathbb{E}_{\lambda, \mu}(x_1) \). Then we have
\[ |\partial_{x'} \xi^{\beta'} \exp(\pm \text{Re} \varphi_{\lambda, \mu}(x, t, \xi'))| \]
\[ \leq C_{\lambda}^{(\lambda + \beta')/2} \xi^{(\lambda')/2} (|\text{Im} \xi_n|^{-|\beta'|} \text{exp}(-a^{-5} |t^{q_\lambda + 1} - x_1^{q_\lambda + 1}| \text{Im} \xi_n)). \]

\section{Diagonalization of the Complete Symbol}

**§ 6.** Calculation of Some Matrices

Let
\[ \bar{L}_{i,j}'' = \begin{cases} \bar{L}_{i,j}', & i = 0, \\ 0, & i \neq 0, j = 0, \\ \bar{L}_{i,j-1}', & i \neq 0, j \neq 0. \end{cases} \]

We have \( \sum_{i,j} \bar{L}_{i,j} = \sum_{i,j} \bar{L}_{i,j}'', \) and these two formal series are essentially the same.

We define \( \bar{M}(x, \xi') \) \( (= \bar{M}_{0,0}(x, \xi') \) by
We study the following equation:

$$\partial_{x_1} V(x, \xi') + \bar{L}''(x, \xi') \circ V(x, \xi') - V(x, \xi') \circ \bar{M}(x, \xi') = O.$$ 

The essential part is the following

**Proposition 1.** There exist matrices $U^{(\pm)}_{ij}(x, \xi') = \sum_{0 \leq j \leq i} U^{(\pm)}_{ij}(x, \xi')$, and $\bar{M}^{(\pm)}_{ij}(x, \xi') = \sum_{0 \leq j \leq i} \bar{M}^{(\pm)}_{ij}(x, \xi')$ of formal series satisfying the following conditions:

(i) All the components $U^{(\pm)}_{ij}$, $\bar{M}^{(\pm)}_{ij}$ are holomorphic on $Q_m'$.

(ii) $\bar{M}^{(\pm)}_{ij} \circ U^{(\mp)}_{ij} = I_m$.

(iii) $\bar{M}^{(\pm)}_{ij} = \delta_{ij} \bar{M}^{(\pm)}_{i,j, (\mu,v)}$, and $\bar{M}^{(\pm)}_{i,j, (\mu,v)}$ are diagonal matrices, and we have

$$\partial_{x_1} U^{(\pm)}_{ij} + \bar{L}'' \circ U^{(\pm)}_{ij} - U^{(\pm)}_{ij} \circ \bar{M}^{(\pm)}_{ij} = O, \quad 0 < t_1, t_2 < 1.$$
(Here we define $\tilde{M}^{(\lambda-1)} = \tilde{L}^n$ if $\lambda = 0$). In other words, we have

$$\tilde{M}^{(\lambda)} = \begin{pmatrix} * & \cdot & \cdots & \cdot \cr \cdot & \cdot & \cdots & \cdot \\
0 & \cdot & \cdots & \cdot \\
0 & \cdot & \cdots & \cdot \cr \end{pmatrix}_{m-\lambda \rightarrow m}^{\lambda \rightarrow \lambda},$$

and

$$U^{(+, \lambda)} = I_m + \begin{pmatrix} * \\
0 \\
\end{pmatrix}^{(\lambda)}.$$
220 KEISUKE UCHIKOSHI

\[
\begin{align*}
M^{(l-1)}_{(\mu, v)} - M^{(l)}_{(\mu, v)} & = \begin{cases}
M^{(l-1)}_{(\mu, v)} - M^{(l-1)}_{(\mu, \lambda)} \circ U^{(l)}_{(\lambda, v)}, & \text{if } \min(\mu, v) \leq \lambda - 1, \\
M^{(l-1)}_{(\mu, v)} + \sum_{\kappa \geq \lambda + 1} M^{(l-1)}_{(\mu, \kappa)} \circ U^{(l)}_{(\kappa, v)}, & \text{if } \min(\mu, v) \geq \lambda + 1, \\
0, & \text{if } \min(\mu, v) = \lambda \text{ and } \mu = v \end{cases}
\]

This means that if we obtain \( U^{(l)} \), then we must define

\[
(24) \quad M^{(l)}_{(\mu, v)} = \begin{cases}
M^{(l-1)}_{(\mu, v)}, & \text{if } \min(\mu, v) \leq \lambda - 1, \\
M^{(l-1)}_{(\mu, v)} + M^{(l-1)}_{(\mu, \lambda)} \circ U^{(l)}_{(\lambda, v)}, & \text{if } \min(\mu, v) \geq \lambda + 1, \\
M^{(l-1)}_{(\mu, v)} + \sum_{\kappa \geq \lambda + 1} M^{(l-1)}_{(\mu, \kappa)} \circ U^{(l)}_{(\kappa, v)}, & \text{if } \min(\mu, v) = \lambda \text{ and } \mu = v, \\
0, & \text{if } \min(\mu, v) = \lambda \text{ and } \mu \neq v.
\]

On the other hand, we must define \( U^{(l)} \) by

\[
(25) \quad \partial_{x_l} U^{(l)}_{(\mu, v)} + \sum_{\kappa \geq \lambda + 1} M^{(l-1)}_{(\mu, \kappa)} \circ U^{(l)}_{(\kappa, v)} - U^{(l)}_{(\mu, \kappa)} \circ M^{(l)}_{(\kappa, v)} = 0
\]

for \( \min(\mu, v) = \lambda, \mu \neq v \). Substituting (24) into (25), we can eliminate \( M^{(l)}_{(\mu, v)} \) from (25). We define \( A(\mu, v) \) and \( B(\mu, v) \) by

\[
A(\mu, v) = \{ \lambda \}, \quad B(\mu, v) = \{(i, \kappa); i = \lambda, \lambda + 1 \leq \kappa \leq m\}
\]

if \( \mu > v = \lambda \), and

\[
A(\mu, v) = \{ \lambda + 1, \ldots, m\}, \quad B(\mu, v) = \{(i, \kappa); \lambda + 1 \leq i \leq m, \kappa = \lambda\}
\]

if \( v > \mu = \lambda \). Then we obtain

\[
(26) \quad \partial_{x_l} U^{(l)}_{(\mu, v)} + \sum_{\kappa \geq \lambda} U^{(l-1)}_{(\mu, \kappa)} \circ U^{(l)}_{(\kappa, v)} - \sum_{(i, \kappa) \in B(\mu, v)} U^{(l)}_{(\mu, i)} \circ M^{(l-1)}_{(\kappa, v)} + U^{(l)}_{(\mu, \kappa)} \circ U^{(l)}_{(\kappa, v)} = 0
\]

for \( \min(\mu, v) = \lambda, \mu \neq v \). Therefore we need to solve (26) for \( U^{(l)}_{(\mu, v)} \) at first, and define \( M^{(l)}_{(\mu, v)} \) by (24), after that. We will do this in the next section.
§ 6.2. A Study of Ordinary Differential Equations

We solve (26) by the following successive approximation. Let \( \min(\mu, v) = \lambda, \mu \neq v \). We consider

\[
\partial_{x_l} U_{\iota, j, (\mu, v)}^{(\lambda, k)} + (\bar{L}_0^{(\mu, \mu)} - \bar{L}_0^{(v, v)}) U_{\iota, j, (\mu, v)}^{(\lambda, k)} = F_{\iota, j, (\mu, v)}^{(\lambda, k)},
\]

where \( F_{\iota, j, (\mu, v)}^{(\lambda, k)} = \sum_{0 \leq t \leq 3} F_{\iota, j, (\mu, v)}^{(\lambda, k, t)} \) and

\[
F_{\iota, j, (\mu, v)}^{(\lambda, k, 0)} = \frac{\partial^k}{\partial x^k} \bar{M}_{\iota, j, (\mu, \mu)}^{(\lambda, 1)} - \delta_{\mu v} \bar{L}_0^{(\mu, \mu)} U_{\iota, j, (\mu, v)}^{(\lambda, k-1)},
\]

\[
F_{\iota, j, (\mu, v)}^{(\lambda, k, 1)} = \sum_{\kappa \geq \lambda} (\bar{M}_{0, (\mu, \kappa)}^{(\lambda, 1)} - \delta_{\mu v} \bar{L}_0^{(\mu, \mu)}) U_{\iota, j, (\mu, v)}^{(\lambda, \kappa-1)} + \sum_{\kappa \in A(\mu, v)} (\bar{M}_{0, (\kappa, v)}^{(\lambda, 1)} - \delta_{\mu v} \bar{L}_0^{(\mu, \mu)}) U_{\iota, j, (\mu, v)}^{(\lambda, \kappa-1)},
\]

\[
F_{\iota, j, (\mu, v)}^{(\lambda, k, 2)} = \sum_{(i', j', (\mu, \kappa)) \in A(\mu, v)} \frac{1}{\alpha!} \partial_{x^d} \bar{M}_{i', j', (\mu, \kappa)}^{(\lambda, 1)} \partial_{x^e} U_{i', j', (\mu, \kappa)}^{(\lambda, k-1)} + \sum_{(i'', j'', (\mu, \kappa)) \in A(\mu, v)} \frac{1}{\alpha!} \partial_{x^d} \bar{M}_{i'', j'', (\mu, \kappa)}^{(\lambda, 1)} \partial_{x^e} U_{i'', j'', (\mu, \kappa)}^{(\lambda, k-1)},
\]

\[
F_{\iota, j, (\mu, v)}^{(\lambda, k, 3)} = \sum_{(i', j', (\mu, \kappa)) \in A(\mu, v)} U_{i', j', (\mu, \kappa)}^{(\lambda, k-1)} \circ \bar{M}_{i', j', (\mu, \kappa)}^{(\lambda, 1)} \circ U_{i', j', (\mu, \kappa)}^{(\lambda, k-1)}.
\]

Here we have defined

\[
i' + i'' + |\alpha'| = i, \quad i'' \neq i,
\]

\[
j' + j'' = j, \quad j' \leq i', \quad j'' \leq i'', \quad \lambda \leq \kappa \leq m,
\]

and

\[
i' + i'' + |\alpha'| = i, \quad i'' \neq i,
\]

\[
j' + j'' = j, \quad j' \leq i', \quad j'' \leq i'', \quad \kappa \in A(\mu, v)
\]

for \( i', i'', j', j'', \kappa \in \mathbb{Z}_+, \) and \( \alpha' \in \mathbb{Z}_+^{\alpha - 1} \). (We have written \( U^{(\lambda, k)} = F^{(\lambda, k)} = F^{(\lambda, k, l)} \) if \( k < 0 \)). Let

\[
\delta_{\mu v}(x_1) = \delta_{\mu v}^+(x_1)
\]

if \( \mu < v \), and

\[
\delta_{\mu v}(x_1) = \delta_{-\mu v}^-(x_1), \varphi_{\mu v}(x, \xi) = -\varphi_{v, \mu}(x, \xi)
\]

if \( \mu > v \). Note that if we have decided \( U_{(\mu, v)}^{(\lambda, k-1)} \), then we can determine \( F_{(\mu, v)}^{(\lambda, k, l)} \). Therefore we may define \( U_{(\mu, v)}^{(\lambda, k)} \) by
\[ U^{(\lambda,k)}_{(\mu,v)}(x,\xi') = \int_{\mathbb{R}^d} \exp(\varphi_{\mu,v}(x,s,\xi')) F^{(\lambda,k)}_{(\mu,v)}(s,x',\xi') ds. \]

We want to prove that \( \lim_{k \to \infty} U^{(\lambda,k)}_{(\mu,v)} \in \mathcal{B}^0 \) exists and it satisfies (26). For this purpose, we define \( \tilde{U}^{(\lambda,k)}_{(\mu,v)} = U^{(\lambda,k)}_{(\mu,v)} - U^{(\lambda,k-1)}_{(\mu,v)} \), \( \tilde{F}^{(\lambda,k,\ell)}_{(\mu,v)} = F^{(\lambda,k,\ell)}_{(\mu,v)} - F^{(\lambda,k-1,\ell)}_{(\mu,v)} \). Therefore we need to consider

\[ \partial_{x_1} \tilde{U}^{(\lambda,k)}_{i,j,(\mu,v)} + (\tilde{L}''_{0,(\mu,v)} - \tilde{L}''_{0,(v,v)}) \tilde{U}^{(\lambda,k)}_{i,j,(\mu,v)} = \tilde{F}^{(\lambda,k)}_{i,j,(\mu,v)}, \]

where \( \tilde{F}^{(\lambda,k)}_{i,j,(\mu,v)} = \sum_{0 \leq \ell \leq 3} \tilde{F}^{(\lambda,k,\ell)}_{i,j,(\mu,v)} \) and

\[ \tilde{F}^{(\lambda,k,0)}_{(\mu,v)} = -\delta_{k0} \tilde{M}^{(\lambda-1)}_{(\mu,v)}, \]
\[ \tilde{F}^{(\lambda,k,1)}_{(\mu,v)} = -\sum_{\kappa \geq \lambda} (\tilde{M}^{(\lambda-1)}_{0,(\mu,\kappa)} - \delta_{\mu\kappa} \tilde{L}''_{0,(\mu,v)}) \tilde{U}^{(\lambda,k-1)}_{(\mu,v)}, \]
\[ + \sum_{\kappa \in A(\mu,v)} (\tilde{M}^{(\lambda-1)}_{0,(\kappa,v)} - \delta_{\kappa v} \tilde{L}''_{0,(v,v)}) \tilde{U}^{(\lambda,k-1)}_{(\mu,v)}, \]
\[ \tilde{F}^{(\lambda,k,2)}_{i,j,(\mu,v)} = -\sum_{(i',j') \in B(\mu,v)} \left\{ \frac{1}{\alpha^4} \partial_{x'} \tilde{M}^{(\lambda-1)}_{i',j',(\mu,\kappa)} \partial_{x'} \tilde{U}^{(\lambda,k-1)}_{i',j',(\mu,\kappa)} \right\}, \]
\[ + \sum_{(i',j') \in B(\mu,v)} \left\{ \frac{1}{\alpha^4} \partial_{x'} \tilde{M}^{(\lambda-1)}_{i',j',(\mu,v)} \partial_{x'} \tilde{U}^{(\lambda,k-1)}_{i',j',(\mu,v)} \right\}, \]
\[ \tilde{F}^{(\lambda,k,3)}_{(\mu,v)} = \sum_{(i',j') \in B(\mu,v)} \left\{ U^{(\lambda,k-1)}_{(\mu,i)} \circ \tilde{M}^{(\lambda-1)}_{(i',j')} \circ \tilde{U}^{(\lambda,k-1)}_{(\mu,j)} \right\}, \]
\[ + \tilde{U}^{(\lambda,k-1)}_{(\mu,i)} \circ \tilde{M}^{(\lambda-1)}_{(i',j')} \circ \tilde{U}^{(\lambda,k-2)}_{(\mu,j)} \}

Let us define

\[ \tilde{U}^{(\lambda,k)}_{(\mu,v)}(x,\xi') = \int_{\mathbb{R}^d} \exp(\varphi_{\mu,v}(x,s,\xi')) \tilde{F}^{(\lambda,k)}_{(\mu,v)}(s,x',\xi') ds. \]

Then we have the following

**Lemma 12.** If \( \min(\mu,v) = \lambda, \mu \neq v, 0 < t_1 < C^{-1}_\lambda, 0 < t_2 < a^{7(m+1-\lambda)} \), then we have

\[ \mathcal{N}(\tilde{U}^{(\lambda,k)}_{(\mu,v)}, \Omega'_\lambda, t_1, t_2) \leq 2^{-k-1}a^{-1}. \]

**Proof.** Let \( k_0 \in \mathbb{Z}_+ \) and assume that Lemma 12 is true if \( 0 \leq k \leq k_0 - 1 \) (We have also assumed that Proposition 2 is true if we replace the number \( \lambda \) by \( \lambda - 1 \)). Then we may assume that
and we can prove

\[ \mathcal{N}(F(\lambda,k,\ell), \Omega'_\lambda) \leq \{ \begin{array}{ll}
4ma^{-7}2^{-k}|x_1|^{q/2} \text{Im} \xi_n, & \ell = 2, \\
4ma^{-7}2^{-k}|x_1|^{\frac{q(\ell-1)}{2}} & \ell = 1,3,
\end{array} \]

Let us prove (30) for the case \( \ell = 2 \) (The other cases are easier). We have

\[ \mathcal{N}(F(\lambda,k,\ell), \Omega'_\lambda) \]

\[ \leq \sum_{(32)} \frac{2(2n)^{-i'-i''-|\gamma'|}(i'+i''+|\gamma'|)!|x^{(1)}(1) + x^{(2)}(2)!|!(\beta^{(1)} + \beta^{(2)}))!}{(i'+i''+|x^{(1)}(1) + x^{(2)}(2) + \gamma'|)!|x^{(1)}(1) + x^{(2)}(2)!|!(\beta^{(1)} + \beta^{(2)} + \gamma')!}|x^{(1)}(1) + x^{(2)}(2)!|!(\beta^{(1)} + \beta^{(2)})!|\gamma'!}
\]

\[ \times \rho_{x^{(1)}}(\lambda, j', j', \mu, \kappa) \rho_{x^{(2)}}(\lambda, j', j', \mu, \kappa) \rho_{x^{(1)}}(\lambda, j', j', \mu, \kappa) \]

\[ \times |\xi|^{i'+i''+|\beta^{(1)} + \beta^{(2)} + \gamma'| + |z^{(1)} + x^{(2)}(2) + \beta^{(1)} + \beta^{(2)} + \gamma'|}!|x^{(1)}(1) + x^{(2)}(2)!|!(\beta^{(1)} + \beta^{(2)})|\gamma'!}
\]

where

\[ (31) \quad i' + j' + |\gamma'| \neq 0, \quad \lambda \leq \kappa \leq m \]

and

\[ (32) \quad i' + j' + |\gamma'| \neq 0, \quad \kappa \in A(\mu, \nu) \]

for \( i', i'', j', j'', \kappa \in \mathbb{Z}_+ \), and \( x^{(1)}, x^{(2)}, \beta^{(1)}, \beta^{(2)}, \gamma' \in \mathbb{Z}_+^{-1} \). Here we have

\[ \frac{2(2n)^{-i'-i''-|\gamma'|}(i'+i''+|\gamma'|)!|x^{(1)}(1) + x^{(2)}(2)!|!(\beta^{(1)} + \beta^{(2)}))!}{(i'+i''+|x^{(1)}(1) + x^{(2)}(2) + \gamma'|)!|x^{(1)}(1) + x^{(2)}(2)!|!(\beta^{(1)} + \beta^{(2)} + \gamma')!}|x^{(1)}(1) + x^{(2)}(2)!|!(\beta^{(1)} + \beta^{(2)})!|\gamma'!}
\]

\[ \leq \frac{2(2n)^{-i'-i''-|\gamma'|}(i'+i''+|\gamma'|)!|x^{(1)}(1)!|!(\beta^{(1)} + \beta^{(2)})!|\gamma'!}{(i'+|x^{(1)}(1))!|!(i' + |\beta^{(1)} + \beta^{(2)} + \gamma'|)!|x^{(1)}(1) + x^{(2)}(2) + \gamma'|!|(i' + |\beta^{(1)} + \beta^{(2)} + \gamma'|)!|x^{(1)}(1)!|!(\beta^{(1)} + \beta^{(2)})!|\gamma'!}
\]

Let \( \bar{a}^{(1)} = a^{(1)}(1), \bar{a}^{(2)} = a^{(2)} + \gamma', \bar{b}^{(1)} = b^{(1)} + \gamma', \bar{b}^{(2)} = b^{(2)}. \) Then it follows that
\[
\sum_{(31)} \leq \sum_{(31)} \left( \frac{2(2n)^{-i'-l''-|r'| i' i'' |r'|}}{(i' + |\tilde{a}^{(1)}|)(i' + |\tilde{\beta}^{(1)}|)(i'' + |\tilde{a}^{(2)}|)(i'' + |\tilde{\beta}^{(2)}|) t_1^l t_2^m t_1^l t_2^m} \times \left| \frac{\partial \tilde{a}^{(2)}}{\partial x^{(2)}} \frac{\partial \tilde{\beta}^{(2)}}{\partial x^{(2)}} M^{(\lambda - 1)}_{i'' + j''} \right| \right.
\]
where
\[
(31)' \quad i' + j' + |\tilde{a}^{(1)}| + |\tilde{\beta}^{(1)}| \neq 0, \quad \lambda \leq \kappa \leq m
\]
for \( i', i'', j', j'', \kappa, \tilde{a}^{(1)}, \tilde{a}^{(2)}, \tilde{\beta}^{(1)}, \tilde{\beta}^{(2)}, \gamma' \). Here we have
\[
\left| \frac{\partial \tilde{a}^{(2)}}{\partial x^{(2)}} \frac{\partial \tilde{\beta}^{(2)}}{\partial x^{(2)}} M^{(\lambda - 1)}_{i'' + j''} \right| \leq \frac{2(2n)^{-i'}}{|(i' + |\tilde{a}^{(1)}|)(i' + |\tilde{\beta}^{(1)}|)|} \times |\xi|^{i''-i'-|\tilde{a}^{(1)}|} \times |\xi|^{i''-|\tilde{\beta}^{(1)}|} \times \frac{2a^{-7(m+2-\lambda)}}{a^{-7}} \left| a x_1 \right|^{\infty} \text{Im} \xi_n.
\]
by the assumption of induction (on \( \lambda \)). It follows that
\[
\sum_{(31)} \leq \sum_{(31)} \left( \frac{2(2n)^{-i'-l''-|r'| i' i'' |r'|}}{(i' + |\tilde{a}^{(2)}|)(i' + |\tilde{\beta}^{(2)}|) t_1^l t_2^m} \times \left| \frac{\partial \tilde{a}^{(2)}}{\partial x^{(2)}} \frac{\partial \tilde{\beta}^{(2)}}{\partial x^{(2)}} M^{(\lambda - 1)}_{i'' + j''} \right| \right.
\]
Here we note \( 0 < t_1 C_{\lambda - 1} < C_{\lambda - 1} / C_\lambda \ll 1 \) and \( 0 < t_2 a^{-7(m+2-\lambda)} < a^{-7} \). By the assumption of induction on \( k \), it follows that \( \sum_{(31)} \leq 2^{1-k} ma^{-7}|x_1|^q \text{Im} \xi_n \). We can similarly prove the same inequality for \( \sum_{(32)} \).

From (29) and (30) we obtain
\[
\mathcal{N}(F^{(\lambda, k)}_{(\mu, v)}, \Omega'_k) \leq 2^{1-k} C_{\lambda - 1} |x_1|^{-q(m-1)-1} + 4ma^{-7} 2^{-k} |x_1|^q \text{Im} \xi_n
\]
\[
\leq 5ma^{-7} 2^{-k} |x_1|^q \text{Im} \xi_n.
\]
From Corollary of Lemma 11 it follows that
\[
\mathcal{N}(\exp(\pm \text{Re} \varphi_{\lambda, \mu}(x, t, \xi')), \Omega'_k) \leq 4 \exp(-a^{-5}|t q'_1 + 1 - x'_q + 1|) \text{Im} \xi_n
\]
if \( t \in \delta_{\lambda, \mu}' \). We have
\[
\mathcal{N}(\tilde{U}^{(\lambda, k)}_{(\mu, v)}(x, \xi'), \Omega'_{\lambda / 2}) \leq \int \mathcal{N}(\tilde{U}^{(\lambda, k)}_{(\mu, v)}(s, x', \xi'), \Omega'_{\lambda / 2}) \mathcal{N}(\exp(\text{Re} \varphi_{\mu, v}(x, s, \xi')) \Omega'_k) |ds|
\]
Now we can prove Proposition 2. From Lemma 12 we obtain \( \mathcal{N}(U_{(\nu, \lambda)}^{(\mu, \nu)}), \Omega_{1}^{(\lambda)}, t_{1}, t_{2}) \leq a^{-1}. \) If we define \( \tilde{M}^{(\lambda)} \) by (24), we have (25). We can prove (22) directly from this.

We define \( U^{(\lambda)} = U^{(\lambda)} \), and calculate its inverse series \( U^{(\lambda)} \) as follows.

**Lemma 13.** There exists some \( U^{(\lambda)}(\mu, \nu, \lambda) \) such that

\[
U^{(\lambda)} = U^{(\lambda)}(\mu, \nu) \in (C(\Omega_{1}^{(\lambda)}))^{m \times m},
\]

\[
U^{(\lambda)}(\mu, \nu) \circ U^{(\lambda)}(\mu, \nu) = I_{m},
\]

\[
\mathcal{N}(U^{(\lambda)}(\mu, \nu), t_{1}, t_{2}) \leq 2ma^{-1}
\]

if \( 0 < t_{1} < C_{\lambda}^{-1}, 0 < t_{2} < a^{-1} \).

**Proof.** Let \( U^{(\lambda)} = I_{m} + \sum_{k \geq 1} \overbrace{U^{(\lambda)} \circ \cdots \circ U^{(\lambda)}}^{k \text{-times}} \). Then it is easy to see that

\[
\mathcal{N}(U^{(\lambda)}(\mu, \nu), \Omega_{1}^{(\lambda)}, t_{1}, t_{2}) \leq \sum_{k \geq 1} (m \times \max_{i, k} \mathcal{N}(U^{(\lambda)}(\mu, \nu), \Omega_{1}^{(\lambda)}))^{k} \leq \sum_{k \geq 1} (ma)^{k} \leq 2ma^{-1},
\]

and we have \( U^{(\lambda)}(\mu, \nu) \circ U^{(\lambda)}(\mu, \nu) = I_{m}. \) Q.E.D.

If we let \( U^{(\pm)} = U^{(\pm)} \circ \cdots \circ U^{(\pm)} \) and \( U^{(\mp)} = U^{(\mp)} \circ \cdots \circ U^{(\mp)} \), we obtain Proposition 1. Our next step is the following

**Proposition 3.** Let \( C' \gg C_{m} \), and let \( x_{j}^{(\pm)} = C' \exp(\sqrt{-1} \theta_{m}^{(\pm)}(q_{m} + 1)). \) There exist \( V_{i, j, (\mu, \nu)}^{(\pm)} \in C(\Omega_{m}^{(\lambda)}(C')) \), \( 1 \leq j \leq i \), such that \( V^{(\pm)} = \sum_{0 \leq j \leq i} V_{i, j}^{(\pm)} \) satisfy

\[
V^{(\pm)} = I_{m},
\]

\[
V_{i, j}^{(\pm)} = 0, \quad \mu \neq \nu,
\]

\[
\partial_{x_{1}} V^{(\pm)} + \tilde{M}^{(\mu)}(\lambda) = O, \quad V^{(\pm)}(x_{1}, x_{j}, \xi_{j}) = I_{m},
\]

\[
(33) \quad \partial_{x_{1}} V^{(\pm)} = \tilde{M}^{(\mu)}(\lambda) O = O, \quad V^{(\pm)}(x_{1}, x_{j}, \xi_{j}) = I_{m},
\]

\[
(34) \quad \partial_{x_{1}} V^{(\pm)} - \tilde{M}^{(\mu)}(\lambda) = O, \quad V^{(\pm)}(x_{1}, x_{j}, \xi_{j}) = I_{m},
\]
Proof. Let $M_{i,j'} = \delta_{i'0}\delta_{j'0}$ and $M_{0,0,(\mu,v)}^{(m)} = M_{0,0,(\mu,v)}^{(m)}$. Since all the matrices in Proposition 3 are diagonal matrices, we can easily calculate $V_{i,j,(\mu,\nu)}^{(0)}$ by solving the following equation:

$$
\partial_{x_1} V_{i,j,(\mu,\nu)}^{(0)} = -\sum_{i' + i'' + [\alpha] = i, j' = j} \frac{1}{\alpha^2} (\partial_{x_1} M_{i',j',(\mu,\nu)}^{(m)} \partial_{x_1} V_{i'',j'',(\mu,\nu)}^{(0)} - \partial_{x_1} M_{i',j',(\mu,\nu)}^{(m)} \partial_{x_1} V_{i'',j'',(\mu,\nu)}^{(0)}).
$$

We solve this equation by successive approximation. Let

$$
\begin{align*}
G_{i,j,(\mu,\nu)}^{(+,k)} &= -\sum_{i' + i'' + [\alpha] = i, j' = j} \frac{1}{\alpha^2} (\partial_{x_1} M_{i',j',(\mu,\nu)}^{(m)} \partial_{x_1} V_{i'',j'',(\mu,\nu)}^{(0,k-1)} - \partial_{x_1} M_{i',j',(\mu,\nu)}^{(m)} \partial_{x_1} V_{i'',j'',(\mu,\nu)}^{(0,k-1)}),
\end{align*}
$$

for $k \geq 0$ (Here $V_{i,j,(\mu,\nu)}^{(+,-)} = 0$). Then by a direct calculation we can prove that

$$
|\partial_{x_1} \partial_{x_1'} V_{i,j,(\mu,\nu)}^{(+,k)}| \leq 2^{-k-1} \sum_{\ell \leq i} \frac{(i + [\alpha' + \beta'])!}{\ell!} C^{\alpha - \ell |\alpha'/\beta'| + 1} a^{-\ell} \times |x_1 - x_1'| \exp(C' |x_1 - x_1'| (\text{Im } \xi_n)^{(m+1)}/(m+1))
$$

by induction on $k$. This means that $V_{i,j,(\mu,\nu)}^{(0)} = V_{i,j,(\mu,\nu)}^{(+,k)}$ satisfies (33) and (35). We can similarly construct $V_{i,j,(\mu,\nu)}^{(-)}$ satisfying (34) and (35). It follows that

$$
\partial_{x_1} (V^{(+)} \circ V^{(-)}) + M^{(m)} \circ (V^{(+)} \circ V^{(-)}) - (V^{(+)} \circ V^{(-)} \circ M^{(m)}) = 0,
$$

whose unique solution is $V^{(+)} \circ V^{(-)} = I_m$. We can similarly prove $V^{(-)} \circ V^{(+)} = I_m$. Q.E.D.
§7. The Last Part of the Proof of Theorem 2

Here we rewrite the results we have already proved. Let $C \gg C' (\gg C_m)$, $\theta \in [0, 2\pi), k \in \mathbb{Z}_+$. We define

$$\Xi(C) = \{(x, \xi') \in \mathbb{C}^n \times \mathbb{C}^{n-1}; C|x| < 1, C|x''| < \operatorname{Im} \xi_n,$$

$$C|\operatorname{Re} \xi_n| < \operatorname{Im} \xi_n, C^{4m(q+1)} < \operatorname{Im} \xi_n\},$$

$$\Xi'(C, \theta) = \{(x, \xi') \in \Xi(C); C(\operatorname{Im} \xi_n)^{-1/(mq+m)} < |x_1|, C|\arg x_1 - \theta| < 1\},$$

$$\Xi''(C) = \{(x, \xi') \in \Xi(C); |x_1| < C^{-1/2}(\operatorname{Im} \xi_n)^{-1/(2mq+2m)}\},$$

$$\Xi''(C) = \{(x, \xi') \in \Xi''(C); x_1 \neq 0, |\arg x_1| < 2\pi\},$$

$$\Xi(C, \theta) = \Xi'(C, \theta) \cup \Xi''(C),$$

$$\Xi(C, \theta) = \Xi'(C, \theta) \cup \Xi''(C),$$

$$\Xi_k(C) = \{(x, \xi') \in \Xi(C); C^{4m(q+1)}(k + 1) < \operatorname{Im} \xi_n\},$$

$$\Xi'_k(C, \theta) = \Xi'(C, \theta) \cap \Xi_k(C),$$

$$\Xi''_k(C) = \Xi''(C) \cap \Xi_k(C).$$

Let $W^{(+, \theta)} = \mathcal{A}^+ \circ U^{(+)} \circ V^{(+)}$, and $W^{(-, \theta)} = V^{(-)} \circ U^{(-)} \circ A^-$ on $\Xi'(C, \theta) (\subset \Omega'_m)$. Precisely speaking, $U^{(\pm)}$, $V^{(\pm)}$, and $\Omega'_m$ are dependent on $\theta$, and we should have written as $U^{(\pm, \theta)}$, $V^{(\pm, \theta)}$, and $\Omega^{(\theta)}$, respectively. To the contrary, $A^{(\pm)}$ are independent of $\theta$. Let $\sigma_+(\mu, \nu) = \mu - 1$, and $\sigma_-(\mu, \nu) = 1 - \nu$. Then we have

$$\left\{ \begin{array}{l}
\partial_{x_1} W^{(+, \theta)} + \overline{L} \circ W^{(+, \theta)} - W^{(+, \theta)} \circ \overline{M} = O,
\partial_{x_1} W^{(-, \theta)} + \overline{M} \circ W^{(-, \theta)} - W^{(-, \theta)} \circ \overline{L} = O,
\end{array} \right.$$  

and

$$\left| \partial_{x_1}^{\alpha'} \partial_{\xi'}^{\beta'} W^{(\pm, \theta)}_{l, (\mu, \nu)} \right| \leq \sum_{j \leq l} C^{i-j+|\alpha'|+|\beta'|+1} a^{-\delta j(i - j)|\alpha'|\beta'|} \left| x_1 \right|^{-q(m-1)}$$

$$\times \left( \operatorname{Im} \xi_n \right)^{-i-j-|\beta'|+1} \exp(C(\operatorname{Im} \xi_n)^{mq+1}/(mq+m)^p)$$

on $\Xi'(C, \theta)$. Finally we need to calculate $W^{(\pm, \theta)}$ on $\Xi''(C)$. This means that we have already calculated $W^{(\pm, \theta)}$ on $\{ |\xi_n| > 1/|x_1| \}$, and we need to extend them to the remaining domain. This is very easy. Let $x_1^{(k)} = C^{-11/4}(k + 2)^{-1/(2mq+2m)} \exp(-i \theta)$. We define the distance $\rho(k)(x_1)$ from $x_1^{(k)}$ to $x_1$ by

$$\rho(k)(x_1) = |x_1|^{1/m} - |x_1^{(k)}|^{1/m} + \frac{1}{m} |x_1|^{1/m} |\arg x_1 - \theta|.$$ 

Then we have the following
Lemma 14. If \((x, \xi') \in \mathcal{S}'_k(C, \theta) \setminus \mathcal{S}'_{k+2}(C, \theta)\), we have \((x_1^{(k)}, x', \xi') \in \mathcal{S}'(C, \theta) \cap (\mathcal{S}''_k(C) \setminus \mathcal{S}''_{k+2}(C))\).

Proof. It is easy to see that \(C(\text{Im} \xi_n)^{-1/(mq+m)} < |x_1^{(k)}| < C^{-1/2} (\text{Im} \xi_n)^{-1/(2mq+2m)}\), and the lemma follows immediately. Q.E.D.

Lemma 15. We have
\[
\left| \partial_{x'}^j \partial_{\xi'}^j \mathcal{W}^{(\pm, \theta)}_{t, (\mu, \nu)} \right| 
\leq \sum_{0 \leq j \leq l} \frac{(i - j + |x'| + |\beta'|)!}{\ell!} C^{2l - 2j + |x'| + |\beta'| + \ell + 1} a^{-5j} \times (\rho^{(k)}(x_1))^{(\text{Im} \xi_n)}^{-i + j + \ell - |\beta'| + \sigma_+(\mu, \nu) + 1} \exp(C(\text{Im} \xi_n)^{(mq+1)/(mq+m)})
\]
on \mathcal{S}''_k(C) \setminus \mathcal{S}''_{k+2}(C).

Proof. As we have already seen, \(\mathcal{W}^{(+, \theta)}\) satisfies (36) on \(\mathcal{S}'(C, \theta)\). Of course we can uniquely extend this solution to \(\mathcal{S}''_k(C) \setminus \mathcal{S}''_{k+2}(C)\). To be precise, let us consider the following successive approximation:
\[
\mathcal{W}^{(+, \theta, \ell)}(x, \xi') = \delta_{\theta 0} \mathcal{W}^{(+, \theta)}(x_1^{(k)}, x', \xi') - \int_{x_1^{(k)}}^{x_1} H^{(\ell)}(s, x', \xi') ds,
\]
where \(H^{(\ell)} = \mathcal{L} \circ \mathcal{W}^{(+, \theta, \ell - 1)} - \mathcal{W}^{(+, \theta, \ell - 1)} \circ \mathcal{M}\) for \(\ell \in \mathbb{Z}^+ (\mathcal{W}^{(+, \theta, -1)} = 0)\). By induction on \(\ell\), let us prove that
\[
\left| \partial_{x'}^j \partial_{\xi'}^j \mathcal{W}^{(+, \theta, \ell)}_{t, (\mu, \nu)} \right| 
\leq \sum_{0 \leq j \leq l} \frac{(i - j + |x'| + |\beta'|)!}{\ell!} C^{2l - 2j + |x'| + |\beta'| + \ell + 1} a^{-5j} \times (\rho^{(k)}(x_1))^{(\text{Im} \xi_n)}^{-i + j + \ell - |\beta'| + \sigma_+(\mu, \nu) + 1} \exp(C(\text{Im} \xi_n)^{(mq+1)/(mq+m)})
\]
on \mathcal{S}''_k(C) \setminus \mathcal{S}''_{k+2}(C).\] Let \(\ell = 0\). Since \(\mathcal{W}^{(+, \theta, 0)}(x, \xi') = \mathcal{W}^{(+, \theta)}(x_1^{(k)}, x', \xi')\) and \((x_1^{(k)}, x', \xi') \in \mathcal{S}'(C, \theta)\), (39) follows from (37). Assume that \(\ell_0 \geq 1\) and that (39) is true if \(0 \leq \ell < \ell_0 - 1\). Let us consider the case \(\ell = \ell_0\). Then we obtain
\[
\left| \partial_{x'}^j \partial_{\xi'}^j H^{(\ell)}_{t, (\mu, \nu)} \right| 
\leq \sum_{0 \leq j \leq l} \frac{(i - j + |x'| + |\beta'|)!}{(\ell - 1)!} C^{2l - 2j + |x'| + |\beta'| + \ell} a^{-5j} \times (\rho^{(k)}(x_1))^{(\text{Im} \xi_n)}^{-i + j + \ell - |\beta'| + \sigma_+(\mu, \nu) + 1} \exp(C(\text{Im} \xi_n)^{(mq+1)/(mq+m)})
\]
on \mathcal{S}''_k(C) \setminus \mathcal{S}''_{k+2}(C),\] and we obtain (39). (38) follows from (39), and we can similarly estimate \(\mathcal{W}^{(-, \theta)}\). Q.E.D.
Now we can conclude the following

**Proposition 4.** We have $W^{(\pm,\theta)}_{i,(\mu,v)} \in \Theta(\Xi(C,\theta))$ and

\[|\partial_{x'}^{\alpha'}\partial_{\xi'}^{\beta'} W^{(\pm,\theta)}_{i,(\mu,v)}| \leq \sum_{0 \leq j \leq i} (i-j+|\alpha'|+|\beta'|+1) a^{-5j} \times (\text{Im } \xi)^{-i-j-|\beta'|+m} \exp(C(\text{Im } \xi)^{-1-(2mq+2m)})\]

on $\Xi(C,\theta)$.

**Proof.** Let $(x, \xi') \in \Xi(C,\theta)$. If $(x, \xi') \in \Xi'(C,\theta)$, then (40) is already known. Therefore we may assume that $(x, \xi') \in \Xi''(C)$, but it is sufficient to prove (40) for the case $x_1 \neq 0$. Then we can choose some $k \in \mathbb{Z}_+$ for which we have $(x, \xi') \in \Xi''_k(C) \setminus \Xi''_{k+2}(C)$. From Lemma 15 it follows that

\[|\partial_{x'}^{\alpha'}\partial_{\xi'}^{\beta'} W^{(\pm,\theta)}_{i,(\mu,v)}| \leq \sum_{0 \leq j \leq i} (i-j+|\alpha'|+|\beta'|+1) a^{-5j} \times (\text{Im } \xi)^{-i-j-|\beta'|+m} \exp(C(\text{Im } \xi)^{-1-(2mq+2m)})\]

Since we have $\rho^{(k)}(x_1) \leq 10|x_1|^{1/m} \leq 10C^{-1/2m}(\text{Im } \xi)^{-1/(2mq+2m)}$ on $\Xi''(C)$ and $\sigma_{\pm}(\mu,v) + 1 \leq m$, we obtain Proposition 4. Q.E.D.

**Corollary.** $W^{(\pm,\theta)} \in (\mathcal{B}^\theta)^{m \times m}$.

Now we can prove Theorem 2. Let $\tilde{E}^{(+,\theta)}(x,\xi') = W^{(+,\theta)}(x,\xi') \circ X^{(+)}(x,\xi') \circ W^{(-,\theta)}(0,x',\xi')$ and $\tilde{E}^{(-,\theta)}(x,\xi') = W^{(+,\theta)}(0,x',\xi') \circ X^{(-)}(x,\xi') \circ W^{(-,\theta)}(x,\xi')$. Then $\tilde{E}^{(\pm,\theta)}$ satisfy (15) on $\Xi(C,\theta)$. Since such a solution is unique, it follows that $E^{(\pm)} = \tilde{E}^{(\pm,\theta)}$ on $\Xi(C,\theta)$.

The above discussion is valid for each fixed $\theta \in [0,2\pi]$. To consider various values of $\theta$ simultaneously, we must denote the large number by $C_0$ instead of $C$, because it may depend on $\theta$. Let us consider on the whole $\Xi(C'), C' \gg 1$. There is a finite set $\{\theta_1, \ldots, \theta_s\} \in [0,2\pi]$, such that $[0,2\pi] \subset \bigcup_{1 \leq j \leq s} (\theta_j - 1/C_0, \theta_j + 1/C_0)$. If we choose $C' = \max(C_0, \ldots, C_0)$, then we have $\Xi(C') \subset \bigcup_{1 \leq j \leq s} \Xi(C_0, \theta_j)$. It follows that $E^{(\pm)} \in \mathcal{F}^{m \times m}$. We have trivially $X^{(\pm)} \in \mathcal{F}^{m \times m}$, and it follows that $F^{(\pm,\theta)} \in \mathcal{F}^{m \times m}$.

**References**


