Atomic Positive Linear Maps in Matrix Algebras

By

Kil-Chan HA*

Abstract

We show that all of the known generalizations of the Choi maps are atomic maps.

§1. Introduction

Let $M_n$ be the $C^*$-algebra of all $n \times n$ matrices over the complex field and $P_s[M_n]$ (respectively, $P[M_n]$) the convex cone of all $s$-positive (respectively, $s$-copositive) linear maps between $M_n$. One of the basic problems about the structures of the positive cone $P_1[M_n]$ is if this cone can be decomposed as the algebraic sum of some simpler subcones. It is well known [22, 25] that every positive linear map between $M_2$ is decomposable, that is, it can be written as the sum of a completely positive linear map and a completely copositive linear map. But, this is not the case for higher dimensional matrix algebras.

The first example of an indecomposable positive linear map in $M_3$ was given by Choi [5]. Choi and Lam [7] also gave an example of an indecomposable extremal positive linear map in $M_3$ (see also [6]). Another examples of indecomposable extremal positive linear maps are found in [9, 16, 21]. These maps are neither 2-positive nor 2-copositive, and so they become atomic maps in the sense in [24], that is, they can not be decomposed into the sums of 2-positive linear maps and 2-copositive linear maps. Several authors [1, 2, 10, 15, 17, 24] considered indecomposable positive linear maps as extensions of the Choi’s example. The first two examples [2, 10] are generalizations of the Choi’s map [5, 6] in $M_3$ and the other maps $\tau_{n,k}$ in [1, 15, 17, 24] (see Section 2 for the definition) are extensions of the Choi map [7] in...
higher dimensional matrix algebra $M_n$. Among them, examples in [10] and the map $\tau_{n,1}$ $(n \geq 3)$ [17] are known to be atomic maps. But the atomic properties of the other indecomposable maps are not determined. Even decomposabilities of the maps $\tau_{n,k}$ are remained open except for some special cases [17]. The usual method to show the atomic property of a positive linear map depends on a tedious matrix manipulation.

The purpose of this note is to show that all of the above mentioned examples are atomic maps, using the recent result of Eom and Kye [8]. Generalizing the Woronowicz's argument [25], they considered the duality between the space $M_n \otimes M_m (= M_{nm})$ of all $nm \times nm$ matrices over the complex field and the space $\mathcal{L}(M_m, M_n)$ of all linear maps from $M_m$ into $M_n$, which is given by

$$\langle A, \phi \rangle = \text{Tr} \left[ \sum_{i,j=1}^{m} (\phi(e_{i,j}) \otimes e_{i,j}) A^t \right] = \sum_{i,j=1}^{m} \langle \phi(e_{i,j}), a_{i,j} \rangle,$$

for $A = \sum_{i,j=1}^{m} a_{i,j} \otimes e_{i,j} \in M_n \otimes M_m$ and a linear map $\phi : M_m \to M_n$, where $\{e_{i,j}\}$ is the matrix units of $M_m$ and the bilinear form in the right-side is given by $\langle X, Y \rangle = \text{Tr}(Y X^t)$ for $X, Y \in M_n$ with the usual trace Tr. For the convenience of readers, we briefly explain the results in [8]. For a matrix $A = \sum_{i,j=1}^{m} x_{i,j} \otimes e_{i,j} \in M_n \otimes M_m$, we denote by $A^T$ the block-transpose $\sum_{i,j=1}^{m} x_{j,i} \otimes e_{i,j}$ of $A$. We say that a vector $z = \sum_{i=1}^{s} z_i \otimes e_i \in \mathbb{C}^r \otimes \mathbb{C}^m$ is an $s$-simple if the linear span of $\{z_1, \cdots, z_m\}$ has the dimension $\leq s$, where $\{e_1, \cdots, e_m\}$ is the usual orthonormal basis of $\mathbb{C}^m$. Let $V_s[M_n]$ (respectively $V^*[M_n]$) denote the convex cone in $M_n \otimes M_n$ generated by $zz^* \in M_n \otimes M_n$ (respectively $(zz^*)^t \in M_n \otimes M_n$) with all $s$-simple vectors $z \in \mathbb{C}^n \otimes \mathbb{C}^m$. It turns out that $V_s[M_n]$ (respectively $V^*[M_n]$) is the dual cone of $P_s[M_n]$ (respectively $P^*[M_n]$) with respect to the pairing (1.1). With this machinery, the maximal faces of $P_s[M_n]$ and $P^*[M_n]$ are characterized in terms of $s$-simple vectors (see also [12, 13, 14]). Another consequence is a characterization of the cone $P_s[M_n] + P^*[M_n]$: For a linear map $\phi : M_n \to M_n$, the map $\phi$ is the sum of an $s$-positive linear map and a $t$-cosiposite linear map if and only if $\langle A, \phi \rangle \geq 0$ for each $A \in V_s[M_n] \cap V^*[M_n]$. This result provides us a useful method to examine the atomic property for the generalizations of the Choi maps mentioned before.

Throughout this note, every vector in the space $\mathbb{C}^r$ will be considered as an $r \times 1$ matrix. The usual orthonormal basis of $\mathbb{C}^r$ and matrix units of $M_r$ will be denoted by $\{e_{i}: i=1, \cdots, r\}$ and $\{e_{i,j}: i, j=1, \cdots, r\}$ respectively, regardless of the dimension $r$. 


§2. The Maps $\tau_{n,k}$

Let $\varepsilon$ be the canonical projection of $M_n$ to the diagonal part and $S$ be the rotation matrix in $M_n$ which sends $e_i$ to $e_{i+1} \ (\text{mod} \ n)$ for $i = 1, \ldots, n$. The map $\tau_{n,k} : M_n \to M_n$ is defined by

$$
\tau_{n,k}(X) = (n-k)e(X) + \sum_{i=1}^{k} e(S^iXS^*) - X, \quad X \in M_n,
$$

for $k = 1, 2, \ldots, n-1$. The map $\tau_{n,0} : M_n \to M_n$ is also defined by

$$
\tau_{n,0}(X) = ne(X) - X, \quad X \in M_n.
$$

It is easy to see that $\tau_{n,0}$ is completely positive and $\tau_{n,n-1}$ is completely copositive. The positivity of $\tau_{n,k}$ is equivalent [1] to a certain cyclic inequality, which was shown by Yamagami [26]. The map $\tau_{3,1}$ is the Choi and Lam's example mentioned in the introduction. For $n \geq 4$, Osaka showed that $\tau_{n,n-2}$ is not the sum of a 3-positive linear map and a 3-copositive linear map [15], and $\tau_{n,1}$ is atomic [17]. In this section, we show that the map $\tau_{n,k}$ is an atomic map for each $n \geq 3$ and $k = 1, 2, \ldots, n-2$.

For each fixed natural number $n = 1, 2, \ldots$, let \{\(\omega_i : 1 \leq i \leq 3^n\)\} be the $3^n$-th roots of unity. Then we have

$$
(2.1) \quad \sum_{i=1}^{3^n} \omega_i^k = 0, \quad 1 \leq k \leq 3^n - 1.
$$

For each $k = 1, 2, \ldots, n$, we define $m_k \in \mathbb{Z}$ by $m_k = 3(2^{k-1} - 1)$. Then it is easy to see the following:

$$
(2.2) \quad m_k - m_l = m_i - m_j \text{ if and only if } (k, i) = (l, j) \text{ or } (k, l) = (i, j).
$$

For any $\gamma > 0$, we define $a_{ik}, c_r \in \mathbb{C}^n$ by

$$
a_{i1} = \sum_{j=1}^{n} \omega_i^{m_j} e_j, \quad 1 \leq i \leq 3^n,
$$

$$
a_{ik} = \omega_i^{-m_k} a_{i1}, \quad 1 \leq i \leq 3^n, \quad 2 \leq k \leq n,
$$

$$
c_1 = e_1 + \gamma e_2 + \sum_{k=3}^{n-1} e_k + \frac{1}{\gamma} e_n,
$$

$$
c_r = S^{r-1} c_1, \quad 2 \leq r \leq n.
$$
For each \(r=1,2,\cdots,n\) and \(i=1,2,\cdots,3^n\) and \(j=1,2,\cdots,n\), we define \(b_{rij} \in \mathbb{C}^n\) by

\[
b_{rij} = \begin{cases} a_{ij}, & j \neq r \\ c_j \circ a_{ij}, & j = r \\ \end{cases}
\]

where \(\circ\) is the Schur product of \(n \times 1\) matrices \(c_j\) and \(a_{ij}\). We also define \(z_{ri} \in \mathbb{C}^n \otimes \mathbb{C}^n\) and \(A_r \in M_n \otimes M_n\) by

\[
z_{ri} = \sum_{j=1}^{n} b_{rij} \otimes e_j, \quad 1 \leq r \leq n, \quad 1 \leq i \leq 3^n,
\]

\[
A_r = \frac{1}{3^n} \sum_{i=1}^{3^n} z_{ri} z_{ri}^*, \quad 1 \leq r \leq n.
\]

Then we see that each \(z_{ri}\) is a 2-simple vector and so \(A_r \in V_2[M_n]\). If we write \(A_r\) by \(A_r = \sum_{p,q=1}^{n}(A_{r})_{p,q} \otimes e_{p,q} \in M_n \otimes M_n\), then it is easy to see that

\[
(A_r)_{p,q} = \begin{cases} e_{p,q}, & p \neq q, \\ \Sigma_{i=1}^{n} e_{i,i}, & p = q, p \neq r, \\ S^{r-1}(e_{1,1} + \frac{\gamma^2}{n} e_{2,2} + \Sigma_{i=3}^{n} e_{i,i} + \frac{1}{\gamma^2} e_{n,n}) S^{r-1}, & p = q = r, \end{cases}
\]

by (2.1) and (2.2).

Now we define \(A \in V_2[M_n]\) by \(A = \frac{1}{n} \Sigma_{r=1}^{n} A_r = \Sigma_{p,q=1}^{n} A_{p,q} \otimes e_{p,q} \in M_n \otimes M_n\). Then, by (2.3), we have

\[
A_{p,q} = \begin{cases} e_{p,q}, & p \neq q, \\ e_{1,1} + \frac{1}{n} (\gamma^2 + (n-1)) e_{2,2} + \Sigma_{i=3}^{n} e_{i,i} + \frac{1}{\gamma^2} e_{n,n}, & p = q = 1, \\ S^{p-1} A_{1,1} S^{q-1}, & p = q, p \neq 1. \end{cases}
\]

In order to show that \(A \in V_2[M_n]\), define \(u_i, v_i, \alpha_i\) and \(\beta_{ij} \in \mathbb{C}^n\) by

\[
u_i = \frac{\gamma}{\sqrt{n}} e_{i+1} \otimes e_i + \frac{1}{\sqrt{n\gamma}} e_i \otimes e_{i+1}, \quad 1 \leq i \leq n,
\]

\[
u_i = \sqrt{\frac{n-1}{n}} e_{i+1} \otimes e_i + \sqrt{\frac{n-1}{n}} e_i \otimes e_{i+1}, \quad 1 \leq i \leq n,
\]
where suffixes are understood in mod n. A direct calculation shows

$$A^T = \sum_{i=1}^{n} (u_i u_i^* + v_i v_i^* + \alpha_i \beta_i^*) + \sum_{j=3}^{n-1} \beta_{1j} \beta_{1j}^* + \sum_{i=2}^{n-2} \sum_{j=i+2}^{n} \beta_{ij} \beta_{ij}^*,$$

and so, we have $A \in V^2[M_n]$. Furthermore, we also have $\langle A, \tau_{n,k} \rangle = \gamma^2 - 1$, for each $n = 3, 4, \ldots$ and $k = 1, 2, \ldots, n-2$, and so we see that $\langle A, \tau_{n,k} \rangle < 0$ for $0 < \gamma < 1$. By the result in [8] mentioned in the introduction, we conclude the following:

**Theorem 2.1.** For $n \geq 3$ and $1 \leq k \leq n-2$, the map $\tau_{n,k} : M_n \rightarrow M_n$ is an atomic positive linear map.

### §3. The Generalized Choi Map

The other generalization of the Choi map is given in [2]. For nonnegative real numbers $a$, $b$, and $c$, the map $\Phi[a, b, c] : M_3 \rightarrow M_3$ is defined by

$$\Phi[a, b, c](X) = 
\begin{pmatrix}
ax_{1,1} + bx_{2,2} + cx_{3,3} & 0 & 0 \\
0 & ax_{2,2} + bx_{3,3} + cx_{1,1} & 0 \\
0 & 0 & ax_{3,3} + bx_{1,1} + cx_{2,2}
\end{pmatrix} - X$$

for each $X = (x_{ij}) \in M_3$. The map $\Phi[2, 0, \mu]$ with $\mu \geq 1$ is the example of an indecomposable positive linear map given by Choi [6], and $\Phi[2, 0, 2]$ and $\Phi[2, 0, 1]$ are the examples given in [5] and [7] respectively. Choi and Lam [7] showed that $\Phi[2, 0, 1]$ is an extremal positive linear map using the theory of biquadratic forms. Later, Tanahashi and Tomiyama [24] showed the atomic property of the map $\Phi[2, 0, 1]$ which is same as the map $\tau_{3,1}$. It was shown [2] that the map $\Phi[a, b, c]$ is an indecomposable positive linear map if and only if the following conditions are satisfied:
In this section we show that these conditions imply that \( \Phi[a, b, c] \) is an atom.

Let \( \{\omega_i : i = 1, 2, 3\} \) be the cube roots of unity and \( s \) be any positive real number. Define \( a_{ik} \in C^3 \), \( z_i, u_i \in C^3 \otimes C^3 \) and \( B \in V_2[M_3] \) by

\[
a_{i1} = (\omega_i, 0, 0), \quad a_{i2} = (0, \omega_i^2, s), \quad a_{i3} = (\frac{s}{\omega_i})a_{i2}, \quad i = 1, 2, 3,
\]

\[
z_i = \sum_{k=1}^{3} a_{ik} \otimes e_k, \quad i = 1, 2, 3,
\]

\[
u_1 = e_2 \otimes e_1, \quad u_2 = e_1 \otimes e_3, \quad u_3 = e_3 \otimes e_1, \quad u_4 = e_1 \otimes e_2,
\]

\[
B = \frac{1}{3} \left( \sum_{i=1}^{3} z_i z_i^* \right) + \frac{1}{s^2} \left( \sum_{i=1}^{2} u_i u_i^* \right) + s^2 \left( \sum_{i=3}^{4} u_i u_i^* \right).
\]

It is clear that \( B \in V_2[M_3] \). To show that \( B \in V^2[M_3] \), we define \( z_i \) and \( u_i \in C^3 \otimes C^3 \) by

\[
z_i = \frac{1}{s} (e_{i+1} \otimes e_i) + s(e_i \otimes e_{i+1}), \quad i = 1, 2, 3,
\]

\[
u_i = e_i \otimes e_i, \quad i = 1, 2, 3,
\]

where suffixes are understood in mod 3. A direct calculation show that

\[
B^T = \sum_{i=1}^{3} (z_i z_i^* + u_i u_i^*) \in V_2[M_3].
\]

It is also easy to calculate

\[
\langle B, \Phi[a, b, c] \rangle = 3((a - 3) + \frac{c}{s^2} + s^2 b).
\]
We proceed to show the conditions in (3.1) imply that the pairing in (3.2) is negative. We first consider the case $bc=0$. If $b=0$ then the pairing (3.2) becomes negative for $s > \sqrt{\frac{c}{3-a}}$. If $c=0$ then (3.2) is negative for $0 < s < \sqrt{\frac{3-a}{b}}$. When $bc \neq 0$, we take $s = \left(\frac{c}{b}\right)^{1/4}$, then the pairing (3.2) is reduced to

$$\langle B, \Phi[a, b, c] \rangle = 3((a-3) + 2\sqrt{bc}),$$

which is also negative since $\sqrt{bc} < \frac{3-a}{2}$ in (3.1). Therefore we have Theorem 3.1.

**Theorem 3.1.** The map $\Phi[a, b, c]$ is an indecomposable positive linear map if and only if it is an atomic positive linear map.

For the Choi's map $\Phi[2, 0, \mu]$, the condition (3.1) is reduced to the condition $\mu \geq 1$. Therefore, we see that the Choi's map $\Phi[2, 0, \mu]$ is atomic whenever $\mu \geq 1$.

§4. The Robertson's Map

An example of an indecomposable positive linear map on $M_4$ was given by Robertson [18] by considering an extension of an automorphism on a certain spin factors. To describe this map, let $\sigma : M_2 \rightarrow M_2$ be the symplectic involution defined by

$$\sigma \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) = \left( \begin{array}{cc} \delta & -\beta \\ -\gamma & \alpha \end{array} \right).$$

The Robertson's map $\Psi : M_4 \rightarrow M_4$ is defined by

$$\Psi \left( \begin{array}{cc} X & Y \\ Z & W \end{array} \right) = \left( \begin{array}{cc} \text{tr}(W)I_2 & \frac{1}{2}(Y+\sigma(Z)) \\ \frac{1}{2}(Z+\sigma(Y)) & \text{tr}(X)I_2 \end{array} \right)$$

for $X, Y, Z, W \in M_2$, where $\text{tr}$ is the normalized trace on $M_2$. The indecomposability of this map was shown in [20] by using the Størmer's characterization [23] of decomposability.

It turns out that this map is an extremal positive linear map which is neither 2-positive nor 2-copositive [19,21]. So this map is an atomic map. We
provide a simple proof.

Define \( z_1 \in \mathbb{C}^4 \otimes \mathbb{C}^4 \) and \( D \in V_2[[M_4]] \) by

\[
z_1 = e_1 \otimes e_1, \quad z_2 = e_1 \otimes e_3, \quad z_3 = e_2 \otimes e_1, \quad z_4 = e_2 \otimes e_4, \\
z_5 = e_3 \otimes e_1, \quad z_6 = e_3 \otimes e_3, \quad z_7 = e_4 \otimes e_4,
\]

\[
D = (z_1 - z_6)(z_1 - z_6)^* + (z_5 + z_4)(z_5 + z_4)^* + z_2z_2^* + z_3z_3^* + z_7z_7^*.
\]

Then we see that

\[
D^T = (z_5 - z_2)(z_5 - z_2)^* + (z_3 + z_7)(z_3 + z_7)^* + z_1z_1^* + z_6z_6^* + z_4z_4^* \in V_2[[M_4]],
\]

and so \( D \in V_2[[M_4]] \cap V^2[[M_4]] \). Furthermore, we can show that the pairing \( \langle D, \Psi \rangle = -\frac{1}{2} \) by an easy calculation. Consequently, we conclude that \( \Psi \) is an atomic map.

References


