An Upper Bound for the Characteristic Variety of an Induced $\mathcal{D}$-Module

By

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Abstract

We generalise the $\text{Car}^2 (\mathcal{M})$ upper bound of Laurent & Schapira [LS87] for the characteristic variety of the induced system of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ on a hypersurface $Y$ of $X$, to the case where $Y$ is a smooth submanifold of $X$ of arbitrary codimension.

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1. Introduction

Given a complex analytic manifold $X$ and a smooth submanifold $Y$ of $X$, let $T^*X \to X$ be the cotangent bundle of $X$, $T^*Y \to Y$ the cotangent bundle of $Y$, $T_Y X \to Y$ the normal bundle of $Y$ in $X$, $T^*_Y X \to Y$ the conormal bundle of $Y$ in $X$, and let $\rho$ and $\omega$ be the maps canonically associated to the immersion $Y \hookrightarrow X$.
Let $\mathcal{O}_X$ be the structural sheaf of $X$, $\mathcal{I}_Y$ the defining ideal of $Y$, $\mathcal{O}_Y = \mathcal{O}_X / \mathcal{I}_Y$ the structural sheaf of $Y$, $\mathcal{D}_X$ the sheaf of holomorphic differential operators of finite order in $X$, $\mathcal{D}_X|_Y$ the restriction of $\mathcal{D}_X$ to $Y$, and let

$$\mathcal{D}_{Y \rightarrow X} = \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{D}_X|_Y = \mathcal{D}_X / \mathcal{I}_Y \mathcal{D}_X$$

be the transfer bimodule from $Y$ to $X$. Given a coherent $\mathcal{D}_X$-Module $\mathcal{M}$, let $\mathcal{M}_Y = \mathcal{D}_{Y \rightarrow X} \otimes_{\mathcal{O}_X} \mathcal{M}$ be the induced $\mathcal{D}_Y$-Module in $Y$. Define

$$\mathcal{M}_Y^F = H^{-k}(\mathcal{M}_Y) = \text{Tor}_k^\mathcal{O}_Y(\mathcal{D}_{Y \rightarrow X}, \mathcal{M}).$$

Kashiwara [Ka83a] proved that, if $\mathcal{M}$ is non-characteristic for $Y$, then

- the cohomology of the complex $\mathcal{M}_Y$ is concentrated in degree 0;
- $\mathcal{M}_Y^F$ is a coherent $\mathcal{D}_Y$-module;
- $\text{Car}(\mathcal{M}_Y^F) = \rho \bar{\omega}^{-1} \text{Car}(\mathcal{M})$.

Consider now in $\mathcal{D}_X|_Y$ the Kashiwara [Ka83b] $V$-filtration associated to the embedding $Y \rightarrow X$ and defined in degree $k$ by

$$F_Y^k \mathcal{D}_X = \{ \mathcal{P} \in \mathcal{D}_X|_Y : \mathcal{P} \mathcal{I}_Y \subset \mathcal{I}_Y \mathcal{P}^{1-k} \quad \forall i \in \mathbb{N} \}.$$ 

and let $F_Y^k \mathcal{D}_{Y \rightarrow X} = \mathcal{D}_{Y \rightarrow X} / \mathcal{I}_Y \mathcal{D}_{Y \rightarrow X}$ be the degree $k$ of the corresponding $F_Y \mathcal{D}_{Y \rightarrow X}$ quotient filtration.

Let $\mathcal{M}$ be an arbitrary coherent $\mathcal{D}_X$-module not necessarily non-characteristic for $Y$. In [LS87] Laurent & Schapira proved that

- $\mathcal{M}_Y^F$ is a union of an increasing sequence of coherent $\mathcal{D}_Y$-modules.

So they could define the notion of characteristic variety of $\mathcal{M}_Y^F$, $\text{Car}(\mathcal{M}_Y^F)$. Moreover by [Sch85] the sheaf of graded rings $gr_Y(\mathcal{D}_X)$ is isomorphic to the subsheaf $\lambda_* \mathcal{D}_{T^*X}$ of rings of holomorphic differential operators of finite order on $T^*X$ that are algebraic in the fibers, and if $F_Y \mathcal{M}$ is a $F_Y \mathcal{D}_X$-good filtration on $\mathcal{M}$ then the graded module of $\mathcal{M}$ for this filtration, $gr_Y(\mathcal{M})$, is a $gr_Y(\mathcal{D}_X)$-coherent module. Denoting by $\mathcal{C}_{T^*X}(\mathcal{M}) \subset T^*T^*X$ the formal microcharacteristic variety of $\mathcal{M}$ along $Y$, i.e. the characteristic variety of $\mathcal{D}_{T^*X} \otimes_{\mathcal{O}_X} [\mathcal{O}_T, T^* \mathcal{D}_Y]$ on $\mathcal{M}$, it was proved in [LS87] that

- $\text{Car}(\mathcal{M}_Y^F) \subset T^*Y \cap \mathcal{C}_{T^*X}(\mathcal{M})$.

Moreover, when $Y$ is smooth embedded hypersurface of $X$, in [LS87] was defined a new subset of $T^*Y$, denoted $\mathcal{C}_{T^*X}(\mathcal{M})$, and it was proved that

- $\text{Car}(\mathcal{M}_Y^F) \subset \mathcal{C}_{T^*X}(\mathcal{M}) \subset T^*Y \cap \mathcal{C}_{T^*X}(\mathcal{M})$.

providing a better upper bound for $\text{Car}(\mathcal{M}_Y^F)$.

The aim of this work is to generalize the construction of the $\text{Car}_r(\mathcal{M})$ of [LS87] to the case where $Y$ is a smooth embedded submanifold of $X$ of arbitrary codimension.

To finish this introductory section some of the above globally defined objects are computed in a special coordinate system.
The above objects in local coordinates. Let \((y, t) = (y_1, ..., y_{m-q}, t_1, ..., t_q)\) be a local coordinate system in \(X\) such that \(Y = \{(y, t) : t = 0\}\). Then:

\[ T_Y X = \{(y, \tau) : y \in \mathbb{C}^{m-q}, \tau \in \mathbb{C}^q \}, \]

and

\[ \mathcal{D}_{Y-X} \cong \frac{\mathcal{D}_X}{t_1\mathcal{D}_X + \cdots + t_q\mathcal{D}_X}. \]

Let \(\delta^a = \delta^{(a_1, \ldots, a_r)}\) be the image of \(\partial_1^{a_1}, \ldots, \partial_r^{a_r} \in \mathcal{D}_X\) by the canonical projection \(\mathcal{D}_X \to \mathcal{D}_{Y-X} = \frac{\mathcal{D}_X}{t_1\mathcal{D}_X + \cdots + t_q\mathcal{D}_X}\). Then

\[ F_0^{k} \mathcal{D}_{Y-X} \cong \bigoplus_{|\alpha| \leq k} \mathcal{D}_Y \delta^\alpha, \]

and

\[ \mathcal{D}_{Y-X} \cong \bigoplus_{k \geq 0} \mathcal{D}_Y \delta^\alpha. \]

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2. Definition of \(\text{Car}(\mathcal{M}_{f})\)

Proposition 2.1. [LS87]. Let \((X, \mathcal{O}_X)\) be a complex analytic manifold and let \(Y\) be a smooth submanifold of \(X\). Let \(\mathcal{M}\) be a coherent \(\mathcal{O}_X\)-module. Then the \(\mathcal{D}_Y\)-modules \(\mathcal{M}_{f}\) may be locally written as a union of an increasing sequence of coherent \(\mathcal{D}_Y\)-modules.

Proof. Consider a local finite type free resolution of \(\mathcal{M}\):

\[ 0 \to \mathcal{D}_X^{m^p} \xrightarrow{A_{p-1}} \cdots \xrightarrow{A_0} \mathcal{D}_X^{m^0} \to \mathcal{M} \to 0, \]

where \(A_i(i = 0, ..., p - 1)\) is a \((m_{i+1} \times m_i)\) matrix of differential operators that acts on the right of \(\mathcal{D}_X^{m^{i+1}}\). Tensoring (1) on the left by \(\mathcal{D}_{Y-X} \otimes \mathcal{D}_X\) we get the complex

\[ (\mathcal{D}_{Y-X})^{m^p} \to \cdots \to (\mathcal{D}_{Y-X})^{m^0} \]

which is quasi-isomorphic to \(\mathcal{M}_{f}\). Then

\[ \text{Ker}(A_{i-1}) = \bigcup_{k \in \mathbb{N}} \text{Ker}(F_0^{k} \mathcal{D}_{Y-X}^{m^i} \to \mathcal{D}_{Y-X}^{m^{i-1}}) \]

for a big enough \(l \geq 0\). Setting
we have that $K_i(k) \subset K_i(k+1)$ and that $K_i(k)$ is a coherent $\mathcal{D}_Y$-module. This proves that $\text{Ker}(A_{i-1})$ is a union of an increasing sequence of coherent $\mathcal{D}_Y$-modules. On the other hand we have:

$$\text{Im}(A_i) = \bigcup_{k \in \mathbb{N}} \text{Im}(\mathcal{D}_Y^{-x^m_i} \rightarrow \mathcal{D}_Y^{-x^m_i}) \cap F^k \mathcal{D}_Y^{-x^m_i}$$

$$= \bigcup_{k \in \mathbb{N}} \bigcup_{i \in \mathbb{N}} (F^k \mathcal{D}_Y^{-x^m_i} \rightarrow \mathcal{D}_Y^{-x^m_i}) \cap F^k \mathcal{D}_Y^{-x^m_i}.$$  

Setting

$$I_i(k) = \bigcup_{j \in \mathbb{N}} \text{Im}(F^j \mathcal{D}_Y^{-x^m_i} \rightarrow \mathcal{D}_Y^{-x^m_i}) \cap F^k \mathcal{D}_Y^{-x^m_i},$$

we see that $I_i(k)$ is a union of an increasing sequence of coherent sub-$\mathcal{D}_Y$-modules of the coherent $\mathcal{D}_Y$-module $F^k \mathcal{D}_Y^{-x^m_i}$. Being $\mathcal{D}_Y$ a noetherian sheaf of rings, $I_i(k)$ is a coherent $\mathcal{D}_Y$-module. Finally we have $\text{Im}(A_i) = \bigcup_{k \in \mathbb{N}} I_i(k)$ and $I_i(k) \subset I_i(k+1)$. Hence it follows that $\text{Im}(A_i)$ is also a union of an increasing sequence of coherent $\mathcal{D}_Y$-modules. \[ \square \]

Now let $\mathfrak{M}$ be a left $\mathcal{D}_Y$-module, locally a union of an increasing sequence of coherent $\mathcal{D}_Y$-modules $(\mathfrak{M}_k)_{k \in \mathbb{N}}$. Then the subset

$$\text{Car}(\mathfrak{M}) := \bigcup_{k \in \mathbb{N}} \text{Car}(\mathfrak{M}_k)$$

does not depend on the sequence $(\mathfrak{M}_k)_{k \in \mathbb{N}}$ and is called the Characteristic Variety of $\mathfrak{M}$.

If $0 \longrightarrow \mathfrak{N} \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{N}'' \longrightarrow 0$ is an exact sequence of $\mathcal{D}_Y$-modules of the preceding type then

$$\text{Car}(\mathfrak{N}) = \text{Car}(\mathfrak{N}') \cup \text{Car}(\mathfrak{N}'').$$

**Definition 2.2.** If $\mathfrak{M}^\circ$ is a bounded complex of $\mathcal{D}_Y$-modules such that the cohomology groups are $\mathcal{D}_Y$-modules of the preceding type the characteristic variety of the complex $\mathfrak{M}^\circ$ is defined to be the following subset of $T^*Y$:

$$\text{Car}(\mathfrak{M}^\circ) = \bigcup_{j \in \mathbb{Z}} \text{Car}(\mathfrak{H}^j(\mathfrak{M}^\circ)).$$

In particular if $\mathfrak{M}$ is a coherent $\mathcal{D}_X$-module, then the characteristic variety of $\mathfrak{M}_Y$ is the following subset of $T^*Y$

$$\text{Car}(\mathfrak{M}_Y) := \bigcup_{j \in \mathbb{N}} \text{Car}(\mathfrak{H}^{-j}(\mathfrak{M}_Y)).$$

3. **Differential Operators on a Holomorphic Vector Bundle**

Given a holomorphic vector bundle of rank $q$ over the complex analytic manifold $Y$,
let $\theta = e_1$ be the Euler–vector field of $\Lambda$. Given an integer $k$ let

$$\mathcal{D}_{\Lambda} [k] = \{ f \in \mathcal{D}_\Lambda : \theta f = k f \}$$

be the sheaf of holomorphic functions on $\Lambda$ that are homogeneous of degree $k$ in the fibers, and let

$$\mathcal{D}_{\Lambda} [k] = \{ P \in \mathcal{D}_\Lambda : [\theta, P] = -k P \}$$

be the sheaf of holomorphic differential operators on $\Lambda$ that are homogeneous of degree $k$ in the fibers. The following proposition is clear:

**Proposition 3.1.** [LS87] The map $\lambda \mathcal{O}_{\Lambda} [0] \longrightarrow \mathcal{O}_Y$, $f \mapsto f|_Y$ is an isomorphism of $\mathcal{O}_Y$-modules.

The sheaf $\lambda \mathcal{O}_{\Lambda} [0]$ acts on the left of $\lambda \mathcal{O}_{\Lambda} [0]$ and so also on $\mathcal{O}_Y$. This defines a morphism of sheaf of rings $\lambda \mathcal{D}_{\Lambda} [0] \longrightarrow \mathcal{D}_Y$.

If $(y, t)$ is a local trivialization of $\lambda$ such that $\lambda (y, t) = y$, then the differential operators $P \in \mathcal{D}_{\Lambda} [k]$ are those that may be written in that coordinate system in the form:

$$P = \sum_{|\alpha| - |\beta| = k} P_{\alpha, \beta} (y, \partial_y) t^\alpha \partial^\beta.$$ 

In particular the differential operators $P \in \mathcal{D}_\Lambda [0]$ are those that may be written in the form:

$$P = \sum_{|\alpha| = |\beta|} P_{\alpha, \beta} (y, \partial_y) t^\alpha \partial^\beta.$$ 

and we have

$$\rho (P) = P_{0,0} (y, \partial_y).$$

Thus, locally, $\lambda \mathcal{D}_{\Lambda} [0]$ is identified to

$$\mathcal{D}_Y \langle \theta \rangle := \mathcal{D}_Y [\theta_{t_1}, \theta_{t_2}, ..., \theta_{t_4}, ..., \theta_{t_4}] / \{ \text{commutation relations} \}$$

where, by definition, $\theta_{t_i} = t_i \partial_u$, and the commutation relations between the variables $\theta_{t_i}$ are the following ones:

$$[\theta_{t_i}, \theta_{t_j}] = [t_i \partial_u, t_j \partial_u] = \begin{cases} 0 & \text{if } j \neq k \text{ and } i \neq l \\ \theta_{t_j} & \text{if } j = k \text{ and } i \neq l \\ \theta_{t_i} - \theta_{t_j} & \text{if } j = k \text{ and } i = l \\ -\theta_{t_j} & \text{if } j \neq k \text{ and } i = l \\ \end{cases}$$

In particular, locally, $\rho$ is identified to $\rho (P(y, \partial_y, \theta_{t_i})) = P(y, \partial_y, 0)$

If $\mathcal{R}$ is a coherent $\lambda \mathcal{O}_{\Lambda} [0]$-module the coherent $\mathcal{D}_Y$-module $\rho (\mathcal{R})$ is
defined by "extension" of scalars:
\[ \rho(\mathcal{R}) = \mathcal{D}_Y \otimes_{\mathcal{D}_{\mathcal{M}[0]}} \mathcal{R}, \]
thus having a characteristic variety \( \text{Car}(\rho(\mathcal{R})) \) in its own right, which is an involutive analytic subset of \( T^*Y \).

**Proposition 3.2.**

(a) If \( 0 \to \mathcal{R} \to \mathcal{R} \to \mathcal{R}'' \to 0 \) is an exact sequence of coherent \( \lambda_*\mathcal{D}_{\mathcal{M}[0]} \)-modules then
\[ \text{Car}(\rho(\mathcal{R})) = \text{Car}(\rho(\mathcal{R}')) \cup \text{Car}(\rho(\mathcal{R}'')). \]

(b) If \( \mathcal{J} \) is coherent ideal of \( \lambda_*\mathcal{D}_{\mathcal{M}[0]} \) and if \( \mathcal{R} = \lambda_*\mathcal{D}_{\mathcal{M}[0]} / \mathcal{J} \) then
\[ \text{Car}(\rho(\mathcal{R})) = \{ y^* \in T^*Y : \forall P \in \mathcal{J} \sigma(\rho(P))(y^*) = 0 \} \]

**Proof.** The problem being of local character we can set
\[ \lambda_*\mathcal{D}_{\mathcal{M}[0]} = \mathcal{D}_Y(\theta). \]

Let \( I \) be the left ideal of \( \mathcal{D}_Y(\theta) \) generated by \( \theta_{11}, \theta_{12}, ..., \theta_{1q}, ..., \theta_{q1}, ..., \theta_{qq} \). Then:
\[ \mathcal{D}_Y = \frac{\mathcal{D}_Y(\theta)}{I}, \]
and the commutation relations (2) give
\[ \theta_{ij} \in I \quad \forall i, j \]
and the commutation relations (2) give
\[ \theta_{ij} \in I^k \quad \text{if} \ i \neq j \]
\[ \theta_{ij} - \theta_{ji} \in I^k \quad \forall i, j \]
\[ \theta_{ik} \theta_{jk} \in I^k \quad \forall i, j, k \in \mathbb{N} \]

Let \( F_k \mathcal{D}_Y(\theta) \) be the non-separated filtration on \( \mathcal{D}_Y(\theta) \) defined by
\[ F_k \mathcal{D}_Y(\theta) = \begin{cases} \mathcal{D}_Y(\theta) & \text{if } k \geq 0 \\ I^{-k} & \text{if } k < 0 \end{cases} \]

The properties of \( \mathcal{D}_Y(\theta) \) listed above imply that the graded ring of \( \mathcal{D}_Y(\theta) \) for this filtration is isomorphic to the ring of polynomials \( \mathcal{D}_Y(\bar{\theta}) \) in one variable \( \bar{\theta} \) and with coefficients in \( \mathcal{D}_Y \), where \( \bar{\theta} \) is the image of all the \( \theta_{ii} \in I^k \) \((i=1, ..., q)\) in the quotient \( I^k/I^2 \).

As \( \text{gr}\mathcal{D}_Y(\theta) = \mathcal{D}_Y[\bar{\theta}] \) is a noetherian graded ring and
\[ F_0 \mathcal{D}_Y(\theta) = \mathcal{D}_Y(\theta) \]
is a noetherian filtered ring, proposition 1.1.8 of Chap. II of [Sch85] implies that the filtration \( F_k \mathcal{D}_Y(\theta) \) is a noetherian one.

Now let \( \text{gr}\mathcal{D}_Y(\theta) \) be filtered by the order of holomorphic differential operators in \( Y \).

If \( \mathcal{R} \) is a coherent \( \mathcal{D}_Y(\theta) \)-module equipped with a good \( F_k \mathcal{D}_Y(\theta) \)-filtration the graded module of \( \mathcal{R} \) for this filtration, \( \text{gr}(\mathcal{R}) \), is a graded coherent
$D_Y[\mathfrak{H}]$-module whose characteristic variety $\text{Car}(gr\mathcal{R})$ is an analytic subset of $T^*(Y) \times \mathbb{C}$.

By Proposition 1.3.1 of Chap. II of [Sch85], the characteristic variety $\text{Car}(gr\mathcal{R})$ is independent of the choice of the good filtration on $\mathcal{R}$ and the map that sends $\mathcal{R}$ to $\text{Car}(gr\mathcal{R})$ is an additive map, that is, if $0 \to \mathcal{R} \to \mathcal{R} \to \mathcal{R} \to 0$ is an exact sequence of coherent $D_Y[\mathfrak{H}]$-modules then $\text{Car}(gr\mathcal{R}) = \text{Car}(gr\mathcal{R}') \cup \text{Car}(gr\mathcal{R}^*)$.

Hence, to prove the first part of the proposition it is enough to prove that $\text{Car}(gr\mathcal{R}) = \text{Car}(\mathcal{R}) \times \mathbb{C}$.

Suppose that $\mathcal{R}$ is a coherent $D_Y[\mathfrak{H}]$-module. Then the filtration on $\mathcal{R}$ defined by

$$\mathcal{R}_k = \begin{cases} \mathcal{R} & \text{if } k \geq 0 \\ I^{-k}\mathcal{R} & \text{if } k < 0 \end{cases}$$

is a good filtration, and the graded module of $\mathcal{R}$ for this filtration is

$$gr(\mathcal{R}) = \bigoplus_{k \geq 0} \frac{I^k\mathcal{R}}{I^{k+1}\mathcal{R}}.$$ 

So, for all $k \in \mathbb{Z}$, $\frac{\mathcal{R}}{I^k\mathcal{R}}$ is a coherent $D_Y$-module and we have a surjective morphism of coherent $D_Y$-modules

$$\mathcal{R} \xrightarrow{\rho} \bigoplus_{k \geq 0} \frac{I^k\mathcal{R}}{I^{k+1}\mathcal{R}}.$$ 

Thus

$$\text{Car}\left(\frac{I^k\mathcal{R}}{I^{k+1}\mathcal{R}}\right) \subseteq \text{Car}\left(\frac{\mathcal{R}}{I^k\mathcal{R}}\right) \subseteq T^*Y,$$

and

$$\text{Car}(gr\mathcal{R}) = \left(\bigoplus_{k \geq 0} \text{Car}\left(\frac{I^k\mathcal{R}}{I^{k+1}\mathcal{R}}\right)\right) \times \mathbb{C} = \text{Car}\left(\frac{\mathcal{R}}{I^k\mathcal{R}}\right) \times \mathbb{C}.$$ 

Part b) of the proposition follows from $\rho(\mathcal{R}) = \frac{\rho}{\rho(\mathfrak{H})}$, where

$$\rho(\mathfrak{H}) = (p(y, \partial_y, \theta_{11}, \theta_{12}, \theta_{14}, \ldots, \theta_{41}, \ldots, \theta_{4q})_{(q_{ij} = 0)} : p \in \mathfrak{H}).$$

\[\square\]

**Notation.** For $k \in \mathbb{Z}$ the module $D_{(\Lambda[1])}[k]$ is a coherent $D_{(\Lambda[1])}[0]$-bimodule (in fact it is locally free). Therefore, given a coherent $\lambda_\mathfrak{H}D_{(\Lambda[1])}$-module $\mathcal{R}$, we may consider the coherent $D_Y$-module

$$\mathcal{R}_Y, k = D_Y \otimes_{\lambda_\mathfrak{H}D_{(\Lambda[1])}[0]} (\lambda_\mathfrak{H}D_{(\Lambda[1])}[k] \otimes_{\lambda_\mathfrak{H}D_{(\Lambda[1])}[0]} \mathcal{R}).$$

$$= \rho\left(\lambda_\mathfrak{H}D_{(\Lambda[1])}[k] \otimes_{\lambda_\mathfrak{H}D_{(\Lambda[1])}[0]} \mathcal{R}\right).$$
Example 3.3. Let $P \in F^Y_1 \mathcal{D}_X$ and let $\mathcal{R} = \text{gr}^b \left( \frac{\mathcal{D}_X}{\mathcal{D}_X^P} \right)$, where $\mathcal{D}_X^P$ is equipped with the induced filtration $F^Y_1 \mathcal{D}_{Y-X}$. Then

$$\mathcal{R}_{Y,k} = \mathcal{D}_Y \otimes_{\lambda^*_\mathcal{D}_{[A]}} \left( \lambda^* \mathcal{D}_{[A]} \left( k \right) \otimes_{\lambda^*_\mathcal{D}_{[A]}} \text{gr}^0 \left( \frac{\mathcal{D}_X}{\mathcal{D}_X^P} \right) \right)$$

$$= \mathcal{D}_Y \otimes_{\lambda^*_\mathcal{D}_{[A]}} \frac{\text{gr}^k \left( \mathcal{D}_X \right)}{\text{gr}^k \left( \mathcal{D}_X^P \right)} \sigma_0 \left( P \right)$$

$$= \frac{\text{gr}^k \left( \mathcal{D}_Y \right)}{\text{gr}^k \left( \mathcal{D}_{Y-X} \right)} \sigma_0 \left( P \right).$$

Example 3.4. Given $k \geq 1$ let $P \in F^Y_{k-1} \mathcal{D}_X \setminus F^Y_k \mathcal{D}_X$. Then, in the special local coordinate system chosen in the introductory section,

$$P = Q + \sum_{|\beta|=k} \partial^\beta Q_\beta,$$

where $Q \in F^Y_{k-1} \mathcal{D}_X$ and $Q_\beta \in F^Y_{0} \mathcal{D}_X$. Thus, locally,

$$\text{gr}^0 \left( \frac{\mathcal{D}_X}{\mathcal{D}_X^P} \right) = \bigoplus_{|\alpha|=k} \text{gr}^0 \left( \mathcal{D}_X \right) \tau^a \sum_{|\beta|=k} \partial^\beta Q_\beta.$$

Since

$$\mathcal{D}_Y = \frac{\text{gr}^0 \mathcal{D}_X}{\text{gr}^0 \mathcal{D}_X \left( \tau_1 \partial_{\tau_1} \ldots, \tau_q \partial_{\tau_q} \right)}.$$  

it follows that

$$\mathcal{R}_{Y,k} = \mathcal{D}_Y \otimes_{\text{gr}^0 \mathcal{D}_X} \text{gr}^0 \left( \frac{\mathcal{D}_X}{\mathcal{D}_X^P} \right)$$

$$= \frac{\text{gr}^0 \mathcal{D}_X}{\text{gr}^0 \mathcal{D}_X \left( \tau_1 \partial_{\tau_1} \ldots, \tau_q \partial_{\tau_q} \right) \otimes \text{gr}^0 \mathcal{D}_X \left( \frac{\mathcal{D}_X}{\mathcal{D}_X^P} \right)}$$

$$= 0.$$

Proposition 3.5. [LS87]

(i) Let $\mathcal{M}$ be a coherent $\lambda^*_\mathcal{D}_{[A]}$-module and let $\mathcal{R}$ be a coherent sub-$\lambda^*_\mathcal{D}_{[A]}[0]$-module of $\mathcal{M}$ that generates $\mathcal{M}$ over $\lambda^*_\mathcal{D}_{[A]}$. Then

$$\mathcal{S}(\mathcal{M}) := \bigcup_{k \in \mathbb{Z}} \text{Car}(\mathcal{R}_{Y,k})$$

is a subset of $T^*Y$ which does not depend on the choice of $\mathcal{R}$.

(ii) If $0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$ is an exact sequence of coherent $\lambda^*_\mathcal{D}_{[A]}$-modules then

$$\mathcal{S}(\mathcal{M}) = \mathcal{S}(\mathcal{M}') \cup \mathcal{S}(\mathcal{M}'').$$

Proof. (i) Let $\mathcal{R}$ and $\mathcal{R}'$ two coherent $\lambda^*_\mathcal{D}_{[A]}[0]$-modules that generate $\mathcal{M}$. As $\mathcal{R}$ is a generator of $\mathcal{M}$ we have
\[ \mathcal{V}' = \sum_{k=\mathbb{Z}} (\lambda_* \mathcal{D}_{[\lambda]}[k] \mathcal{V}) \cap \mathcal{V}' \]

and so \( \mathcal{V}' = \bigcup_{k \in \mathbb{N}} \mathcal{V}'^{(k)} \) where

\[ \mathcal{V}'^{(k)} = \sum_{-k \leq j \leq k} (\lambda_* \mathcal{D}_{[\lambda]}[j] \mathcal{V}) \cap \mathcal{V}' . \]

The sequence \( (\mathcal{V}'^{(k)})_{k \in \mathbb{Z}} \) is a sequence of coherent \( \lambda_* \mathcal{D}_{[\lambda]}[0] \)-modules of \( \mathcal{V}' \), and being \( \mathcal{V}' \) of finite type this sequence must stabilize. Let \( k_0 \) be an integer such that \( \mathcal{V}' = \mathcal{V}'^{(k_0)} \) and let \( \mathcal{V}'' = \sum_{-k_0 \leq j \leq k_0} (\lambda_* \mathcal{D}_{[\lambda]}[j] \mathcal{V}) \).

Then

\[ \bigcup_{k \in \mathbb{Z}} \text{Car}(\mathcal{V}_{Y,k}) \subset \text{Car}(\mathcal{V}'_{Y,k}) = \bigcup_{k \in \mathbb{Z}} \text{Car}(\mathcal{V}_{Y,k}). \]

Reversing the roles of \( \mathcal{V} \) and \( \mathcal{V}' \) we get the first part of the proposition.

(ii) It is enough to prove that if \( 0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}'' \longrightarrow 0 \) is an exact sequence of coherent \( \lambda_* \mathcal{D}_{[\lambda]}[0] \)-modules then

\[ \text{Car}(\mathcal{V}_{Y,k}) = \text{Car}(\mathcal{V}'_{Y,k}) \cup \text{Car}(\mathcal{V}_{Y,k'}). \]

But this is an immediate consequence of Proposition 3.2 and of the flatness of \( \lambda_* \mathcal{D}_{[\lambda]}[k] \) over \( \lambda_* \mathcal{D}_{[\lambda]}[0] \).

4. Definition of \( \text{Car}^x(M) \)

Now let \( \Lambda = T_YX \overset{\lambda}{\longrightarrow} Y \) be the normal bundle of \( Y \) in \( X \). Let \( \mathcal{W} \) be a coherent \( \mathcal{D}_X \)-module and let \( F_Y\mathcal{W} \) be a good filtration on \( \mathcal{W} \). Then the graded module for this filtration, \( \text{gr}_Y\mathcal{W} \), is a coherent \( \mathcal{D}_{[\lambda]} \)-module and \( \mathcal{W} = \text{gr}_Y^x(\mathcal{W}) \) generates \( \mathcal{W} \) over \( \mathcal{D}_{[\lambda]} \). Thus we can associate to \( \text{gr}_Y(\mathcal{W}) \) the subset \( \mathcal{S}(\text{gr}_Y\mathcal{W}) \) of \( T^*Y \). By Proposition 3.5, the functor \( \mathcal{W} \mapsto \mathcal{S}(\text{gr}_Y\mathcal{W}) \) is an additive one. By Proposition 1.3.1. of Chap. II of [Sch85], \( \mathcal{S}(\text{gr}_Y\mathcal{W}) \) is independent of the choice of the good \( F_Y\mathcal{D}_X \)-filtration and the functor \( \mathcal{W} \mapsto \mathcal{S}(\text{gr}_Y\mathcal{W}) \) is an additive one. Therefore we have the following proposition

**Proposition 4.1.** Let \( \mathcal{W} \) be a coherent \( \mathcal{D}_X \)-module and let \( F_Y\mathcal{W} \) be a good \( F_Y\mathcal{D}_X \)-filtration on \( \mathcal{W} \). Then

(i) \( \mathcal{S}(\text{gr}_Y\mathcal{W}) \) is a subset of \( T^*Y \) and does not depend on the choice of the good \( F_Y\mathcal{D}_X \)-filtration on \( \mathcal{W} \).

(ii) if \( 0 \longrightarrow \mathcal{W} \longrightarrow \mathcal{W} \longrightarrow \mathcal{W}'' \longrightarrow 0 \) is an exact sequence coherent \( \mathcal{D}_X \)-modules then

\[ \mathcal{S}(\text{gr}_Y\mathcal{W}) = \mathcal{S}(\text{gr}_Y\mathcal{W'}) \cup \mathcal{S}(\text{gr}_Y\mathcal{W''}). \]

This proposition enables us to make the following definition, as in [LS87]:

\[ \text{Characteristic Variety} \]
Definition 4.2. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and let $F_Y\mathcal{M}$ be a good $F_Y\mathcal{D}_X$-filtration on $\mathcal{M}$. We define

$$\text{Car}_Y^Z(\mathcal{M}) := \mathfrak{S}(\text{gr}_Y\mathcal{M}) = \bigcup_{k \in \mathbb{Z}} \text{Car}(\mathcal{M}_{Y,k})$$

where $\mathcal{M}_{Y,k} = \mathcal{D}_Y \otimes_{\mathcal{D}_X(\mathfrak{m})} \mathcal{D}_X[k] \otimes_{\mathcal{D}_X(\mathfrak{m})} \text{gr}_Y^Z(\mathcal{M})$.

The goal of the remaining sections is to prove that $\text{Car}_Y^Z(\mathcal{M})$ is an upper bound for $\text{Car}(\mathcal{M}_{Y^*})$.

5. The Case Where $\mathcal{M}$ is the Module Defined by One Operator

Let $P \in \mathcal{D}_X$ and $\mathcal{M} = \mathcal{D}_X / \mathcal{D}_XP$. Then

$$0 \rightarrow \mathcal{D}_X \xrightarrow{\rho} \mathcal{D}_X \rightarrow \mathcal{M} \rightarrow 0$$

is a free resolution of $\mathcal{M}$. So, in the derived category,

$$\mathcal{M}_Y^* \cong \mathcal{D}_{Y,X} \xrightarrow{\rho} \mathcal{D}_{Y,X} \cong \mathcal{D}_X \xrightarrow{\rho} \mathcal{D}_X \xrightarrow{\rho} \mathcal{D}_X$$

and

$$\mathcal{M}_Y^* = \ker P, \quad \mathcal{M}_Y = \text{coker } P.$$ 

Let $P \in F^*_Y \mathcal{D}_X$. Then $(F^*_Y \mathcal{D}_{Y,X}).P \subset F^*_Y \mathcal{D}_{Y,X}$. Hence we can define

$$\bar{\rho}(P) : F^*_Y \mathcal{D}_{Y,X} \rightarrow F^*_Y \mathcal{D}_{Y,X}$$

where $\bar{\rho}(P)$ is the restriction of $P$ to $F^*_Y \mathcal{D}_{Y,X}$.

Consider now the action of $P$ on the vectors $\delta^\alpha$ of the base $(\delta^\gamma)_{|\gamma| \leq k}$ of $F^*_Y \mathcal{D}_{Y,X} = \bigoplus_{|\gamma| \leq k} \mathcal{D}_Y \delta^\gamma$. If $P$ is locally formally written as

$$P = \sum_{|\alpha| \leq |\beta|} P_{\alpha,\beta} (y, \partial_y) \delta^\alpha \delta^\beta$$

then

$$\delta^\gamma, P = \sum_{|\alpha| \geq |\beta|} P_{\alpha,\beta} (y, \partial_y) \frac{\gamma!}{(\gamma-\alpha)!} \delta^{\gamma-a+\beta}.$$ 

Let $A(\gamma, \theta)$ the coefficient of $\delta^\theta$ in the expression of $\delta^\gamma P$. Then

$$A(\gamma, \theta) = \sum_{0 \leq \alpha \leq \gamma} P_{\alpha,\gamma-\alpha} (y, \partial_y) \frac{\gamma!}{(\gamma-\alpha)!}.$$ 

Ordering the base $(\delta^\gamma)_{|\gamma| \leq k}$ in such a way that all the $\delta^\gamma$ with $|\gamma| = i$ have orders lower than the $\delta^\gamma$ with $|\gamma| = i + 1$, it follows that the matrix $A(k)$ of $\bar{\rho}(k)$ in such a base is block-lower-triangular:
A(k) = \begin{pmatrix}
A_{00} & 0 & \cdots & 0 \\
A_{10} & A_{11} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_{k0} & A_{k1} & \cdots & A_{kk}
\end{pmatrix},

where $A_{ij}$ is the matrix $A(k) |_{r=r', \theta=\theta'}$. Clearly that $A(k-1)$ is the matrix $A(k)$ with the last row and last column of blocks omitted. Thus

\[
\text{Car}(\frac{F^\dag Y_\rightarrow X}{F^\dag Y_\rightarrow X A_k(P)}) = \bigcup_{j=1}^{k} \{ y^* \in T^*Y : \det(A_{k,j})(y^*) = 0 \}
\]

where $\det$ is the determinant of Sato-Kashiwara [SK75].

But $\{ y^* \in T^*Y : \det(A_{kk})(y^*) = 0 \}$ is precisely $\text{Car}(\frac{\rho_k(P)}{\rho_k(P) - 0})$ where $\rho_k(P) : \frac{D_{Y,X}}{Y} \rightarrow \frac{D_{Y,X}}{Y}$ is the linear morphism whose matrix in base $(\mathcal{O})$ is $A_{kk}$. Observe now that $\rho_k(P)$ is the morphism $\frac{A_{kk}}{A_{kk}} \rightarrow \frac{A_{kk}}{A_{kk}}$. Then, from example (3.4) it follows that

$$
\text{Proposition 5.1. Let } P \in F^\dag Y_\rightarrow X \text{ and } \mathcal{M} = \frac{\rho_k(P)}{\rho_k(P) - 0}.
\$$

Hence we have proved the following proposition

Proposition 5.1. Let $P \in F^\dag Y_\rightarrow X$ and $\mathcal{M} = \frac{\rho_k(P)}{\rho_k(P) - 0}$. Then

(a) $\text{Car}(\frac{\mathcal{M}}{\mathcal{M}}) = \text{Car}(\frac{\mathcal{M}}{\mathcal{M}})$.

(b) Moreover if $P \in F^\dag Y_\rightarrow X$ then

$$
\text{Car}(\frac{\mathcal{M}}{\mathcal{M}}) = \bigcup_{k \in \mathbb{Z}} \{ y^* \in T^*Y : \det(\rho_k(P))(y^*) = 0 \}.
$$

6. The Case Where $\mathcal{M}$ is the Module Defined by a Coherent Ideal

Let $\mathcal{J}$ be a coherent ideal of $\mathcal{D}_X$; then $gr_{\mathcal{J}}(\mathcal{F}_{\mathcal{J}})$ is generated by $gr_{\mathcal{J}}(\mathcal{F}_{\mathcal{J}})$ where $\mathcal{J}_0 = \mathcal{J} \cap F^\dag Y_\rightarrow X$. Therefore

$$
\text{Car}(\frac{\mathcal{D}_X}{\mathcal{J}}) = \bigcup_{k \in \mathbb{Z}} \text{Car}(\frac{\mathcal{D}_X \otimes \mathcal{O}_{\mathcal{D}_X}[k]}{\mathcal{J} \otimes \mathcal{O}_{\mathcal{D}_X}[k]}).
$$
Let $P$ be an element of $\mathcal{J}$ and let $\sigma_0(P)$ be the image of $P$ in $\text{gr}^0(\mathcal{J})$. Then

$$
\text{gr}^k \mathcal{D}_X \to \text{gr}^k \mathcal{D}_X \sigma_0(P) \to \text{gr}^k \mathcal{D}_X \sigma_0 \to 0,
$$

is an exact sequence of $\text{gr}^k \mathcal{D}_X$-modules. Thus, taking into account equation (3),

$$
\text{Car}_Y \left( \frac{\mathcal{D}_X}{\mathcal{J}} \right) \subseteq \bigcap_{P \in \mathcal{J}} \text{Car}_Y \left( \frac{\mathcal{D}_X}{\mathcal{D}_X P} \right),
$$

i.e.

$$
\text{Car}_Y \left( \frac{\mathcal{D}_X}{\mathcal{J}} \right) \subseteq \bigcup_{k \in \mathbb{Z}} \{ y^* \in T^* Y : \text{det} (\rho_k(P)) (y^*) = 0 \quad \forall P \in \mathcal{J} \}.
$$

The following proposition shows that the above inclusion is in fact an equality.

**Proposition 6.1.** Let $\mathcal{J}$ be a coherent ideal of $\mathcal{D}_X$. Then

$$
\text{Car}_Y \left( \frac{\mathcal{D}_X}{\mathcal{J}} \right) = \bigcup_{k \in \mathbb{Z}} \{ y^* \in T^* Y : \text{det} (\rho_k(P)) (y^*) = 0 \quad \forall P \in \mathcal{J} \}.
$$

**Proof.** For each $k \in \mathbb{Z}$ let us denote by $L_k$ the following $\mathcal{D}_Y$-module:

$$
L_k = \mathcal{D}_Y \otimes_{\lambda_* \mathcal{D}_M(0)} \lambda_* \mathcal{D}_M(k) = \bigoplus_{|\alpha| = k} \mathcal{D}_Y \mathcal{D}_M^\alpha \quad \text{if } k \geq 0
$$

$$
\bigoplus_{|\alpha| = -k} \mathcal{D}_Y \mathcal{D}_M^\alpha \quad \text{if } k \leq 0
$$

Then we have a commutative diagram of ring homomorphisms

$$
\lambda_* \mathcal{D}_M(0) \longrightarrow \text{End}_{\mathcal{D}_Y} (L_k).
$$

We denote by $N_k$ the $\mathcal{D}_Y$-module $L_k \rho_k(\mathcal{J})$. With this notation, we see that, to prove the proposition, it is enough to prove that, for all $k \in \mathbb{Z}$,

$$
\text{Car} \left( \frac{L_k}{N_k} \right) \supseteq \bigcap_{Q \in \mathcal{J}} \text{Car} \left( \frac{L_k}{L_k Q} \right).
$$

Let $e_1, \ldots, e_s$ be a basis of the free $\mathcal{D}_Y$-module $L_k$ and we write $L_k = \bigoplus_i \mathcal{D}_Y e_i$. We denote by $\mathcal{A}_k$ the ring of $\mathcal{D}_Y$-endomorphisms $\text{End}_{\mathcal{D}_Y} (L_k)$. Then $L_k$ is a
left-right \((\mathcal{D}_Y, \mathcal{A}_k)\)-bimodule and the functor

\[
\text{Mod}(\mathcal{D}_Y) \longrightarrow \text{Mod}(\mathcal{A}_k)
\]

\[
\mathcal{M} \longmapsto \text{Hom}_{\mathcal{B}_r}(\mathcal{L}_k, \mathcal{M})
\]

is an equivalence of categories. The correspondence

\[
\text{Mod}(\mathcal{A}_k) \longrightarrow \text{Mod}(\mathcal{D}_Y)
\]

\[
\mathcal{D} \longmapsto \mathcal{L}_k \otimes_{\mathcal{A}_k} \mathcal{D}
\]

is a left adjoint functor. This gives a correspondence

\[
\mathcal{L}_k \longleftrightarrow \mathcal{A}_k
\]

\[
\mathcal{N} (\text{submodule}) \longleftrightarrow \mathcal{J} (\text{ideal})
\]

between submodules of \(\mathcal{L}_k\) and ideals of \(\mathcal{A}_k\). Thus one has a bijective correspondence

\[
\frac{\mathcal{L}_k}{\mathcal{L}_k \mathcal{J}} = \mathcal{L}_k \otimes_{\mathcal{A}_k} \left( \frac{\mathcal{A}_k}{\mathcal{J}} \right) \longleftrightarrow \frac{\mathcal{A}_k}{\mathcal{J}}.
\]

For each \((u_1, \ldots, u_s) \in \mathcal{L}_k^{o}\) let \(\phi(u_1, \ldots, u_s) : \mathcal{L}_k \longrightarrow \mathcal{L}_k\) be the homomorphism of free \(\mathcal{D}_Y\)-modules defined by \(\phi(u_1, \ldots, u_s)(e_i) = u_i\). Then \(\mathcal{L}_k^{o} \longrightarrow \mathcal{A}_k, (u_1, \ldots, u_s) \mapsto \phi(u_1, \ldots, u_s)\) is an isomorphism. Now \(\mathcal{J} \simeq \mathcal{N}^{o}\). Hence we have

\[
\bigcap_{p \in \mathcal{D}_Y \setminus \text{Car}(\frac{\mathcal{L}_k}{\mathcal{J}})} \text{Car}(\frac{\mathcal{L}_k}{\mathcal{L}_k \mathcal{J}}) = \bigcap_{(u_1, \ldots, u_s) \in \mathcal{L}_k^{o}} \text{Car}(\left. \frac{\mathcal{L}_k}{\sum_{i=1}^{s} \mathcal{D}_Y u_i} \right|_{\sum_{i=1}^{s} \mathcal{D}_Y u_i})
\]

Suppose now that \(p \in \mathcal{T}^*Y \setminus \text{Car}(\frac{\mathcal{L}_k}{\mathcal{J}})\). Then there is some \(Q \in \mathcal{D}_Y\) such that \(Q e_i \in \mathcal{N}_k (1 \leq i \leq s)\) and \(\sigma(Q)(p) \neq 0\). Setting \(u_i = Q e_i (i = 1, \ldots, s)\) it follows that \(\frac{\mathcal{L}_k}{\sum_{i=1}^{s} \mathcal{D}_Y u_i}\) is isomorphic as a \(\mathcal{D}_Y\)-module to \(\bigoplus_{i=1}^{s} \frac{\mathcal{D}_Y}{Q_{\epsilon_i}}\), implying \(p \notin \text{Car}(\frac{\mathcal{L}_k}{\sum_{i=1}^{s} \mathcal{D}_Y u_i})\). \(\square\)

7. Main Theorem

Now everything is prepared for the statement and proof of the main theorem.

**Theorem 7.1.** Let \(X\) be a complex analytic manifold, \(Y\) a smooth submanifold of \(X\) and \(\mathcal{M}\) a coherent \(\mathcal{D}_X\)-module. Then

\[
\text{Car}(\mathcal{M}) \supseteq \text{Car}_Y(\mathcal{M}).
\]
Proof. Let $\theta$ be a point in $T^*\gamma \setminus \text{Car}^\gamma (\mathfrak{M})$. Given a section $u$ of $\mathfrak{M}$ let $J \subset \mathfrak{D}_X$ be the annihilator of $u$. By proposition (6.1) there exists a $P \in J$ such that $\det (\rho_k (P)) (\theta) \neq 0$ for all $k \in \mathbb{Z}$. In fact, if this was not the case, then we would have $\theta \in \text{Car}^\gamma (\mathfrak{M})$ and, since $\mathfrak{D}_X / J \twoheadrightarrow \mathfrak{M}$, $P \mapsto Pu$ is an injective morphism, one would conclude by proposition (4.1) that $\theta \in \text{Car}^\gamma (\mathfrak{M})$.

As the module $\mathfrak{M}$ is locally of finite type there is a local system of generators $(u_1, \ldots, u_s)$ of $\mathfrak{M}$ and for each $u_i$ one operator $P_i$, such that $P_i u_i = 0$ and $\det (\rho_k (P_i)) (\theta) \neq 0$ for all $k \in \mathbb{Z}$ and $1 \leq i \leq s$.

Let us denote $\mathfrak{L} = \bigoplus_{i=1}^s \mathfrak{D}_X / P_i$; and let $\mathfrak{L} \xrightarrow{\phi} \mathfrak{M}$ be the morphism that sends $u_i$ to the class of 1 modulo $\mathfrak{D}_X / P_i$. Let $N$ be $\text{Ker} (\phi)$. Then there is an exact sequence of left $\mathfrak{D}_X$-modules:

\[
0 \rightarrow N \rightarrow \mathfrak{L} \rightarrow \mathfrak{M} \rightarrow 0.
\]

Applying the functor $\mathfrak{D}_Y \otimes_{\mathfrak{D}_X} \phi^*$ to the above exact sequence we get a long exact sequence of cohomology

\[
\cdots \rightarrow N^* \rightarrow \mathfrak{L}^* \rightarrow \mathfrak{M}^* \rightarrow \cdots \rightarrow \mathfrak{M}^*_Y \rightarrow 0.
\]

As the theorem was already proved for modules of type $\mathfrak{M}^*_Y$ and there exists a $k_0 \in \mathbb{Z}$ such that $N^{k_0} = 0$, one may assume, as an induction hypothesis, that $\text{Car} (\mathfrak{M}^*_Y) \subset \text{Car}^\gamma (\mathfrak{M})$ for all coherent $\mathfrak{D}_X$-module $\mathfrak{M}$ and all $k \leq k_0$. Now, from the long exact sequence (6), it follows that

\[
\text{Car} (\mathfrak{M}^{k+1}) \subset \text{Car} (\mathfrak{L}^{k+1}) \cup \text{Car} (N^*),
\]

implying that $\theta \in \text{Car} (\mathfrak{M}^{k+1})$. Hence, by induction, we finally conclude that

\[
\theta \in \bigcup_{k \in \mathbb{N}} \text{Car} (\mathfrak{M}^{-k}) = : \text{Car} (\mathfrak{M})^*,
\]

finishing the proof of the theorem. $\square$

References


