Second Order Perturbation Bounds

By

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Abstract

With a view to studying perturbation bounds, the class of functions \( f \) for which
\[
\| d f(A) \| = \| f^{(1)}(A) \| \quad \text{and} \quad \| d^2 f(A) \| = \| f^{(2)}(A) \|,
\]
where \( d^k f(A) \) (respectively \( f^{(k)}(A) \)) denotes the \( k \)-th Frechet (ordinary) derivative, \( k = 1, 2 \), has been investigated.

§1. Introduction

Let \( \mathcal{S} \) be the real space of self adjoint operators defined on a separable Hilbert space \( \mathcal{H} \) and \( \mathcal{S}_+ \) be the subset of \( \mathcal{S} \) consisting of positive operators. If \( I \) is an open interval in \( \mathbb{R} \), let \( \mathcal{S}_I \) be the set of elements of \( \mathcal{S} \) with spectrum in \( I \). Observe that \( \mathcal{S}_I \) is an open convex subset of \( \mathcal{S} \).

Let \( f \) be a real valued measurable function defined on \( I \). If \( A \) in \( \mathcal{S}_I \) has spectral decomposition \( A = \int \lambda d E_\lambda \), where \( E_\lambda \) is the left continuous spectral resolution corresponding to \( A \) [6], then denote by \( f(A) \) an operator in \( \mathcal{S} \) defined as \( f(A) = \int f(\lambda) d E_\lambda \). Thus every real measurable function \( f \) defined on \( I \) induces an operator mapping \( f : \mathcal{S}_I \rightarrow \mathcal{S} \). In this note we are interested in the mappings of positive operators, so we restrict ourselves to the interval \( (0, \infty) \) and all functions are from \( (0, \infty) \) to itself.

Let \( X \) and \( Y \) be Banach spaces and \( \Omega \) be an open subset of \( X \). Then a mapping \( f \) from \( \Omega \) into \( Y \) is Frechet differentiable at \( x \in \Omega \) if there exists a map \( df(x) \in \mathcal{L}(X, Y) \), the space of bounded linear maps from \( X \) to \( Y \), such that
\[
f(x + y) - f(x) - df(x)(y) = O(\| y \|).
\]
This map is called the Frechet derivative of $f$ at $x$. As an element of $\mathcal{L}(X,Y)$ this inherits the norm $\|df(x)\| = \sup \{\|df(x)(y)\| : \|y\| = 1\}$. If $f$ is differentiable for all $x \in \Omega$, we have a map $x \mapsto df(x)$ from $\Omega$ into $\mathcal{L}(X,Y)$. The derivative of this map, if it exists, is called the second derivative of $f$ at $x$ and is denoted by $d^2f(x)$. Clearly $d^2f(x)$ is an element of $\mathcal{L}(X,\mathcal{L}(X,Y))$ which can be identified naturally with $\mathcal{L}_2(X,Y)$, the space of bounded bilinear maps from $X \times X$ into $Y$ equipped with norm $\|\phi\| = \inf \{x : \|\phi(x,y)\| \leq x\|y\|\}$. In fact, $d^2f(x)$ is a symmetric bilinear map. The higher order derivatives can now be defined by induction. Reader may refer to [7] for systematic presentation of Frechet differential calculus and for notations we use.

Let $f^{(k)}$ denote the ordinary $k$-th derivative of $f$ when it is viewed as a function on $(0, \infty)$. Define that $f \in \mathcal{D}_k$ if for any positive operator $A$, $\|d^k f(A)\| = \|f^{(k)}(A)\|$. Note that if $I$ is the identity operator, then $\|d^k f(A)(I, \cdots, I)\| = \|f^{(k)}(A)\|$. Thus $\|d^k f(A)\| \leq \|f^{(k)}(A)\|$ for every positive operator $A$. Further it has been shown in [5] that if $f$ is a real valued $k$ times continuously differentiable function defined on $I$, then the induced operator mapping is $k$ times Frechet differentiable, i.e., for every $A \in \mathcal{I}$, $d^k f(A)$ exists.

Bhatia and Sinha [4] studied the class of $\mathcal{D}_1$ functions and showed that a large class of functions are in $\mathcal{D}_1$. However, the characterisation of class $\mathcal{D}_1$ has eluded the authors. Bhatia [3] showed that operator monotone functions are in $\mathcal{D}_n$. He points out that the problem of characterising the class $\mathcal{D}_n$ is intricate.

In this paper, we extend the techniques of [4] to study the class of functions which are in $\mathcal{D}_1 \cap \mathcal{D}_2$ and show that the function $f(t) = t^\alpha$, $t > 0$ is in $\mathcal{D}_1 \cap \mathcal{D}_2$ if $p \geq 4$ or if $-\infty < p < 1$. Moreover, $f(t) = t^\alpha$, $1 < p < \sqrt{2}$, is in $\mathcal{D}_2$ but not in $\mathcal{D}_1$ and for $2 < p < \sqrt{2} + 1$, $f$ is in $\mathcal{D}_1$ but not in $\mathcal{D}_2$.

Perturbation bounds for functions of positive Hilbert space operators are of immense interest to numerical analysts and operator theorists. Physicists too have evinced keen interest in the problem especially when $f(t)$ is either $t^{1/2}$ or $|t|$ or $t^k$, where $k$ is a positive integer. In view of Taylor expansion which takes the form

$$f(A + H) = f(A) + df(A)(H) + \frac{1}{2} d^2f(A)(H,H) + \cdots,$$

it is evident that the estimates of $\|df(A)\|$ and $\|d^2f(A)\|$ would lead to second order perturbation bounds for the function $f$. Indeed, for $f \in \mathcal{D}_1 \cap \mathcal{D}_2$, one has
\[ \|f(A + H) - f(A)\| \leq \|f^{(1)}(A)\| \|H\| + \frac{1}{2} \|f^{(2)}(A)\| \|H\|^2 + O(\|H\|^3), \]

where \( A \) and \( H \) are positive operators. In particular, if \( f(t) = t^p \), then for \( p \geq 4 \), we have

\[ \|B^p - A^p\| \leq p\|A\|^{p-1}\|B - A\| + \frac{1}{2} p(p-1)\|A\|^{p-2}\|B - A\|^2 + O(\|B - A\|^3) \]

and if \( -\infty < p < 1 \), then

\[ \|B^p - A^p\| \leq \|pA^{-1}(1-p)B - A\| + \frac{1}{2} \|p(1-p)A^{-1}\|^{2-p}\|B - A\|^2 + O(\|B - A\|^3). \]

Following are members of \( \mathcal{D}_1 \cap \mathcal{D}_2 \) ([2], [4])

(i) The functions \( f(t) = t^n \), \( n = 1, 2, \ldots \).

(ii) The functions \( f(t) = \sum_{n=0}^{\infty} a_n t^n \), \( a_n \geq 0 \) for all \( n \). In particular the exponential function is in this class.

(iii) Operator monotone functions [3].

\section{The Main Results}

It has been remarked that the exponential function is in \( \mathcal{D}_1 \cap \mathcal{D}_2 \). In what follows, we compute \( d\exp(A) \) and \( d^2\exp(A) \) and show that \( \|d\exp(A)\| = \|\exp(A)\| = \|d^2\exp(A)\| \) ([1], [8]). Indeed,

\[ d\exp(A)(B) = \int_0^1 \exp(sA)B \exp((1-s)A)ds \]

and

\[ \|d\exp(A)\| = \|\exp(A)\| = \|\exp^{(1)}(A)\|. \]

Since

\[ d^2\exp(A)(B_1, B_2) = d(d\exp(A)(B_1))(B_2) \]

\[ = \lim_{t \to 0} \frac{1}{t} (d\exp(A + tB_2)(B_1) - d\exp(A)(B_1)) \]
Consequently, it follows that

$$
\|d^2 \exp(A)(B_1, B_2)\| \leq \int_0^1 \int_0^1 [x \| B_2 \| \| B_1 \| \| \exp(A) \| + (1 - x) \| B_1 \| \| B_2 \| \| \exp(A) \|] d\beta dx
$$

Consequently,

$$
\|d^2 \exp(A)\| \leq \|\exp(A)\| = \|\exp^{(2)}(A)\|.
$$

**Theorem 2.1.** Let $f$ be a function on $(0, \infty)$ which can be written as

$$
f(t) = \int_0^\infty e^{-\lambda t} d\mu(\lambda),
$$

where $\mu$ is a positive measure on $(0, \infty)$. Then $f \in D_1 \cap D_2$.

Proof of this Theorem follows closely on the lines of (Theorem 2.1,[4]) and the definition of the second order Frechet derivative.

**Remark.** For $d\mu(\lambda) = (\lambda^{p-1} / \Gamma(p)) d\lambda$, $p > 0$, it is well known that

$$
t^{-p} = \int_0^\infty e^{-\lambda t} d\mu(\lambda)
$$

(Laplace transform of $\lambda^{p-1}$). Now it follows from Theorem 2.1 that $t^{-p} \in D_1 \cap D_2$ for $p > 0$.

**Theorem 2.2.** Let $f$ be a function on $(0, \infty)$ which can be written as

$$
f(t) = t^k \int_0^\infty e^{-\lambda t} d\mu(\lambda),
$$
where $\mu$ is a positive measure on $(0, \infty)$. Then $f \in D_1 \cap D_2$.

**Proof.** Let $g(t) = \int_0^\infty e^{-\lambda t} d\mu(\lambda)$. As in the proof of the Theorem 2.1 we can write

$$dg(A)(B) = \int_0^\infty \int_0^\lambda A^{-1} e^{-tA^{-1}} Be^{-(\lambda - s)A^{-1}} A^{-1} dsd\mu(\lambda)$$

and

$$d^2g(A)(B_1, B_2) = \int_0^\infty \int_0^\lambda \int_0^\lambda A^{-1} e^{-\beta A^{-1}} A^{-1} B_2 A^{-1} e^{-(\lambda - \beta)A^{-1}} B_1 e^{-(\lambda - s)A^{-1}} A^{-1} A^{-1} d\beta d\lambda d\mu(\lambda)$$

$$+ \int_0^\infty \int_0^\lambda \int_0^\lambda A^{-1} e^{-\alpha A^{-1}} B_1 e^{-\beta A^{-1}} A^{-1} B_2 A^{-1} e^{-(\lambda - s)A^{-1}} A^{-1} A^{-1} d\beta d\lambda d\mu(\lambda)$$

$$- \int_0^\infty \int_0^\lambda A^{-1} e^{-\alpha A^{-1}} B_1 e^{-(\lambda - s)A^{-1}} A^{-1} B_2 A^{-1} d\lambda d\mu(\lambda)$$

$$- \int_0^\infty \int_0^\lambda A^{-1} B_2 A^{-1} e^{-\alpha A^{-1}} B_1 e^{-(\lambda - s)A^{-1}} A^{-1} A^{-1} d\lambda d\mu(\lambda).$$

Since $f(A) = A^2 g(A) A^2$, we have by the rule for differentiating a product

$$df(A)(B) = (AB + BA)g(A)A^2 + A^2 g(A)(AB + BA) + A^2 dg(A)(B)A^2$$

and

$$d^2f(A)(B_1, B_2) = (AB_1 + B_1A)g(A)(AB_2 + B_2A) + (AB_2 + B_2A)g(A)(AB_1 + B_1A)$$

$$+ (B_1B_2 + B_2B_1)g(A)A^2 + A^2 g(A)(B_1B_2 + B_2B_1)$$

$$+ (AB_1 + B_1A)dg(A)(B_2)A^2 + A^2 dg(A)(B_1)(AB_2 + B_2A).$$
+ (A_2 + A_2 A) d(A)(B_1) A^2 + A^2 d(A)(B_2)(A B_1 + B_1 A) \\
+ A^2 d^2 g(A)(B_1, B_2) A^2.

Hence

\[ \| df(A) \| \lesssim 4 \| A \| \int_{0}^{\infty} \| e^{-\lambda A^{-1}} \| d\mu(\lambda) + \| A \| \int_{0}^{\infty} \lambda \| e^{-\lambda A^{-1}} \| d\mu(\lambda). \]

Observe that

\[ f^{(1)}(t) = 4t^3 g(t) + \int_{0}^{\infty} \lambda t^2 e^{-\lambda t} d\mu(\lambda) \]

is an increasing function of \( t \). Consequently, \( \| f^{(1)}(A) \| \geq \| df(A) \| \). Hence \( f \in D_1 \).

Now

\[ B_2 A d(A)(B_1) A^2 = B_2 \int_{0}^{\infty} \int_{0}^{\lambda} e^{-\lambda A^{-1}} B_1 e^{-(\lambda - \beta) A^{-1}} A d\beta d\mu(\lambda), \]

implies that

\[ \| B_2 A d(A)(B_1) A^2 \| \leq \| B_2 \| \| B_1 \| \int_{0}^{\infty} \lambda \| A \| \| e^{-\lambda A^{-1}} \| d\mu(\lambda). \]

Also

\[ A^2 d^2 g(A)(B_1, B_2) A^2 + A B_2 d(A)(B_1) A^2 + A^2 d(A)(B_1) B_2 A \]
\[ = \int_{0}^{\infty} \int_{0}^{\lambda} \int_{0}^{\lambda - \beta} A e^{-\beta A^{-1}} A^{-1} B_2 A^{-1} e^{-(\alpha - \beta) A^{-1}} B_1 e^{-(\lambda - z) A^{-1}} A d\beta d\alpha d\mu(\lambda) \]
\[ + \int_{0}^{\infty} \int_{0}^{\lambda} \int_{0}^{\lambda - \beta} A e^{-\alpha A^{-1}} B_1 e^{-\beta A^{-1}} A^{-1} B_2 A^{-1} e^{-(\lambda - z) A^{-1}} A d\beta d\alpha d\mu(\lambda) \]

and
Thus we have

\[ \|d^2f(A)\| \leq 12\|A\|^2 \int_0^\infty \|e^{-\lambda A^{-1}}\|d\mu(\lambda) + 6\|A\| \int_0^\infty \lambda \|e^{-\lambda A^{-1}}\|d\mu(\lambda) \]

\[ + \int_0^\infty \lambda^2 \|e^{-\lambda A^{-1}}\|d\mu(\lambda). \]

Now \( f^{(2)}(t) = 12t^2g(t) + 6t \int_0^\infty \lambda e^{-\lambda t}d\mu(\lambda) + \int_0^\infty \lambda^2 e^{-\lambda t}d\mu(\lambda) \) is an increasing function of \( t \). Hence \( \|f^{(2)}(A)\| \geq \|d^2f(A)\| \) and this implies that \( f \in \mathcal{D}_2 \). This completes the proof.

Remarks. (i) The function \( f(t) = t^p, \ p \geq 4 \), is in \( \mathcal{D}_1 \cap \mathcal{D}_2 \). This follows from the fact that for \( p > 0 \),

\[ t^p = \frac{1}{\Gamma(p)} \int_0^\infty \lambda^{p-1}e^{-\lambda t}d\lambda \]

and from Theorem 2.2.

(ii) The function \( f(t) = t^p, \ 0 < p < 1 \), being operator monotone is in \( \mathcal{D}_1 \cap \mathcal{D}_2 \). Our next proposition shows that \( f(t) = t^{p+1} \in \mathcal{D}_2 \).

**Proposition 2.3.** If \( 0 < p < 1 \) and \( f(t) = t^{1+p}, \ t \in (0, \infty) \), then \( f \in \mathcal{D}_2 \).
Proof. For \( t > 0 \) and \( 0 < p < 1 \), we have  
\[
|t^{p-1}| = \int_0^\infty \frac{d\mu(\lambda)}{\lambda + t},
\]
where \( d\mu(\lambda) = (\sin \frac{p\pi}{n})\lambda^{-1}d\lambda \). Then we have  
\[
df(A)(B) = \int_0^\infty (B - C_\lambda BC_\lambda)d\mu(\lambda),
\]
where \( C_\lambda = \lambda(\lambda + A)^{-1} \) (See Proposition 2.5 [4]). Now it follows that  
\[
d^2f(A)(B_1, B_2)
= \int_0^\infty \lambda^2[(\lambda + A)^{-1}B_2(\lambda + A)^{-1}B_1(\lambda + A)^{-1}
+ (\lambda + A)^{-1}B_1(\lambda + A)^{-1}B_2(\lambda + A)^{-1}]d\mu(\lambda).
\]
This implies that  
\[
\|d^2f(A)\| \leq \int_0^\infty 2\lambda^2(\lambda + A)^{-1}d\mu(\lambda).
\]

If \( g(t) = 1/(\lambda + t) \), then for \( t > 0 \), it is an increasing function of \( t \). Thus  
\[
\|(\lambda + A)^{-1}\| = 1/(\lambda + \alpha), \text{ where } \alpha = \inf \{\langle Ax, x \rangle: \|x\| = 1\}.
\]
Consequently,  
\[
\|d^2f(A)\| \leq \int_0^\infty 2\lambda^2(\lambda + \alpha)^{-1}d\mu(\lambda) = p(p + 1)\alpha^{p-1}
\]
(See Step III, Example 4, Section 3).

Now \( f^{(2)}(t) = p(p + 1)t^{p-1} \) is a decreasing function of \( t \), it follows that  
\[
\|f^{(2)}(A)\| = f^{(2)}(\lambda) \geq \|d^2f(A)\|. \text{ Hence } f \in D_2.
\]

Our next result is useful in generating examples of functions which are in \( D_1 \cap D_2 \).

**Proposition 2.4.** Let \( f \) and \( g \) be functions defined on \((0, \infty)\) to itself. Assume that \( f \) and \( g \) are three times differentiable and all derivatives up to third order are positive. If \( f \) and \( g \) are in \( D_1 \cap D_2 \), then \( f + g, fg \) and \( g \circ f \) (the composite function) are in \( D_1 \cap D_2 \).

**Proof.** That \( f + g \) is in \( D_1 \cap D_2 \) is clear.
Differentiating $fg$, we have

$$d(fg)(A)(B) = df(A)(B)g(A) + f(A)dg(A)(B)$$

and

$$d^2(fg)(A)(B, B_2) = d^2f(A)(B_1, B_2)g(A) + f(A)d^2g(A)(B_1, B_2)$$

$$+ df(A)(B_1)dg(A)(B_2) + df(A)(B_2)dg(A)(B_1).$$

This gives that

$$\|d(fg)(A)\| \leq \|df(A)\| \|g(A)\| + \|f(A)\| \|dg(A)\|$$

and

$$\|d^2(fg)(A)\| \leq \|d^2f(A)\| \|g(A)\| + \|f(A)\| \|d^2g(A)\| + 2\|df(A)\| \|dg(A)\|.$$ 

Now by the hypothesis $f$ and $g$ are in $D_1 \cap D_2$ and $f$, $g$, $f^{(1)}$, $g^{(1)}$, $f^{(2)}$ and $g^{(2)}$ are increasing functions. If $s = \sup\{\langle Ax, x \rangle : \|x\| = 1\}$, then

$$\|d(fg)(A)\| \leq (fg)^{(1)}(s) = \|(fg)^{(1)}(A)\|$$

and

$$\|d^2(fg)(A)\| \leq (fg)^{(2)}(s) = \|(fg)^{(2)}(A)\|.$$ 

Hence $fg \in D_1 \cap D_2$.

That $g \circ f \in D_1 \cap D_2$ also follows similarly, noticing that for $h = g \circ f$,

$$dh(A)(B) = dg(f(A)).df(A)(B)$$

and

$$d^2h(A)(B_1, B_2) = d^2g(f(A))(df(A)(B_1), df(A)(B_2))$$

$$+ dg(f(A))(d^2f(A)(B_1, B_2)).$$

§3. Examples

1. If $p$ and $q$ are polynomials with positive coefficients, then $p(t)e^{qt}$, $t \in (0, \infty)$ is in $D_1 \cap D_2$ (Proposition 2.4).

2. The function $f(t) = t + t^{-1}$, $t \in (0, \infty)$ is in $D_2$ but not in $D_1$. This follows from (Example 2.31, [4]) and that
Let $f(t) = t^{p+1}, t \in (0, \infty)$ and $0 < p < \sqrt{2} - 1$. Then it is shown in (Example 2.A, [4]) that $f(t)$ is not in $\mathcal{D}_1$. Further Proposition 2.3 shows that it is in $\mathcal{D}_2$.

4. Let $0 < p < 1$. It is shown in [4] that $f(t) = t^{2+p}, t \in (0, \infty)$ is in $\mathcal{D}_1$. Now we show that for some values of $p$ it is not in $\mathcal{D}_2$. This we shall accomplish in several steps.

Step I. Writing $f(A) = \frac{1}{2}(Ah(A) + h(A)A)$, where $h(t) = t^{p+1}$ and using (Proposition 2.5 [4]) we have

\[
d^2f(A)(B) = \frac{1}{2} \left[ Bh(A) + h(A)B + dh(A)(B)A + Adh(A)(B) \right]
\]

and

\[
d^2f(A)(B_1, B_2) = \frac{1}{2} \left[ B_1 dh(A)(B_2) + B_2 dh(A)(B_1) + dh(A)(B_2)B_1 + dh(A)(B_1)B_2 \right.
\]
\[
+ Ad^2h(A)(B_1, B_2) + d^2h(A)(B_1, B_2)A],
\]

where

\[
h(A) = A \int_0^\infty (\lambda + A)^{-1} d\mu(\lambda) A, \quad dh(A)(B) = \int_0^\infty (B - C_\lambda BC_\lambda) d\mu(\lambda),
\]

\[
d^2h(A)(B_1, B_2) = \int_0^\infty \frac{1}{\lambda} [C_\lambda B_2 C_\lambda B_1 C_\lambda + C_\lambda B_1 C_\lambda B_2 C_\lambda] d\mu(\lambda),
\]

\[
d\mu(\lambda) = (\sin p\pi / \pi) \lambda^{p-1} d\lambda \quad \text{and} \quad C_\lambda = \lambda(\lambda I + A)^{-1}.
\]

Step II. Now we compute $d^2f(A)(B, I)$, where $A = P + \varepsilon Q$, $P$ and $Q$ are complementary projections and $\varepsilon$ is positive. From Step I we have

\[
(C_\lambda A) / \lambda = A(A + \lambda)^{-1} = I - C_\lambda
\]

and

\[
d^2f(A)(B, I) = \int_0^\infty (2B - C_\lambda BC_\lambda^2 - C_\lambda^2 BC_\lambda) d\mu(\lambda).
\]
Using that \( C_\lambda = \frac{\lambda}{\lambda+1}P + \frac{\lambda}{\lambda+\varepsilon}Q \), we write

\[
d^2 f(A)(B,I) =
\]

\[
\int_0^\infty \left[ 2 \left( 1 - \frac{\lambda^3}{(\lambda+1)^3} \right) + \int_0^\infty \left[ 1 - \frac{\lambda^3}{(\lambda+1)(\lambda+\varepsilon)^2} \right] d\mu(\lambda)(PBQ + QBP) + \int_0^\infty \left[ 1 - \frac{\lambda^3}{(\lambda+\varepsilon)(\lambda+1)^2} \right] d\mu(\lambda)(PBQ + QBP) + \int_0^\infty \left[ 2 \left( 1 - \frac{\lambda^3}{(\lambda+\varepsilon)^3} \right) \right] d\mu(\lambda)QBQ.
\]

Step III. We next compute the above integrals. Using the representation

\[
t^{-p-1} = \int_0^\infty \frac{d\mu(\lambda)}{\lambda+t}, \quad 0 < p < 1, \quad t > 0,
\]

it follows that

\[
\int_0^\infty \frac{d\mu(\lambda)}{(\lambda+t)^2} = (1-p)t^{p-2} \text{ and }
\]

\[
\int_0^\infty \frac{d\mu(\lambda)}{(\lambda+t)^3} = \frac{1}{2} (1-p)(2-p)t^{p-3}.
\]

Elementary calculations show that:

\[
\int_0^\infty \left[ 1 - \frac{\lambda^3}{(\lambda+1)^3} \right] d\mu(\lambda) = \frac{(p+1)(p+2)}{2}
\]

\[
\int_0^\infty \left[ 1 - \frac{\lambda^3}{(\lambda+1)(\lambda+\varepsilon)^2} \right] d\mu(\lambda) = \frac{1 - \varepsilon^{p+1}((p+1)(1-\varepsilon) + 1)}{1-\varepsilon^2}
\]

\[
\int_0^\infty \left[ 1 - \frac{\lambda^3}{(\lambda+\varepsilon)(\lambda+1)^2} \right] d\mu(\lambda) = \frac{\varepsilon^{p+2} + (p+2)(1-\varepsilon) - 1}{1-\varepsilon^2}
\]

\[
\int_0^\infty \left[ 1 - \frac{\lambda^3}{(\lambda+\varepsilon)^3} \right] d\mu(\lambda) = \frac{\varepsilon^p(p+1)(p+2)}{2}
\]

Step IV. On the space \( H = C^2 \), consider the following matrices:
\[ A = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}, \quad 0 < \varepsilon < 1, \quad B = \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}, \quad b > 0. \] Then we have that
\[ d^2 f(A)(B,I) = \begin{bmatrix} \alpha & \beta \\ \beta & \chi \end{bmatrix}, \quad \text{where} \quad \alpha = (p+1)(p+2), \quad \chi = -\varepsilon^p(p+1)(p+2) \quad \text{and} \]
\[ \beta = \frac{\varepsilon^{p+1} + (p+2)(1-\varepsilon) - \varepsilon^{p+1}(p+1)(1-\varepsilon)+1}{(1-\varepsilon)^2}. \]

If we call this matrix as \( X(\varepsilon) \), then
\[ X(0) = \begin{bmatrix} (p+1)(p+2) & (p+2)b \\ (p+2)b & 0 \end{bmatrix}. \]

It follows that
\[ \| X(0) \| = (p+2)(p+1) + \sqrt{(p+1)^2 + 4b^2}. \]

Also
\[ \| f^{(2)}(A) \| \| B \| \| I \| = (p+1)(p+2) \| A \|^{p+1}(1+b^2)^{1/2}. \]

After some algebraic manipulation, we have that
\[ \| X(0) \| > \| f^{(2)}(A) \| \| B \| \| I \| \quad \text{if and only if} \quad p^2 + 2p < (1+b^2)^{-1/2}. \]

When \( b \) is near zero, the right hand side of the above inequality is near 1. So, in this case, the inequality is true if \( 0 < p < \sqrt{2} - 1 \). Thus for \( 0 < p < \sqrt{2} - 1 \), \( t^{2+p} \) does not belong to \( \mathcal{D}_2 \).

5. Let
\[ f(\lambda) = \begin{cases} 0 & 0 < \lambda < \alpha \\ e^{-\beta(\lambda-\alpha)} & \lambda > \alpha \end{cases} \]
where \( \alpha, \beta > 0 \). Then
\[ \int_0^\alpha e^{-\gamma} f(\lambda) d\lambda = (t+\beta)^{-1} e^{-\alpha t}. \]

Using Theorem 1, Section 2, we get \( (t+\beta)^{-1} e^{-\alpha t} \in \mathcal{D}_1 \cap \mathcal{D}_2 \).
§4. Concluding Remark

Let \( f(t) = t^p, \ t > 0 \). Then \( f(t) \in \mathcal{D}_1 \cap \mathcal{D}_2 \) if \( p \geq 4 \) or \( -\infty < p < 1 \). If \( 1 < p < \sqrt{2} \), then \( f \) is in \( \mathcal{D}_2 \) but not in \( \mathcal{D}_1 \) and if \( 2 < p < \sqrt{2} + 1 \), then \( f \) is in \( \mathcal{D}_1 \) but not in \( \mathcal{D}_2 \). Here our results may be compared with those obtained by Bhatia and Sinha [4]. It would be interesting to know the status of the functions involved for remaining values of \( p \).

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References
