On the Group of $S^1$-equivariant Homeomorphisms of the 3-Sphere

By

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Abstract

Let $G$ be a locally compact abelian group and let $\xi$ be a principal $G$-bundle:

$$G \longrightarrow E \longrightarrow B,$$

where $E$ is path connected and $B$ is a locally finite CW complex. We recall that $G$ acts freely on the right of $E$. Then we denote by $\text{Top}(B)$ the topological group of homeomorphisms of $B$ and by $\text{Top}_G(E)$ the group of $G$-equivariant homeomorphisms of $E$. Furthermore, let $\alpha$ be a map of $B$ into the classifying space $BG$ whose homotopy class $[\alpha]$ classifies the principal bundle $\xi$. Then we have

**Corollary 4.** Let $G$ be a locally compact abelian group, then we have the following Serre fibration:

$$\map(B, G) \longrightarrow \text{Top}_G(E) \longrightarrow \text{Top}_G(B),$$

where $\text{Top}_G(B)$ is the subspace of $\text{Top}(B)$ consisting of homeomorphisms $f : B \to B$ such that $f^*([\alpha]) = [\alpha]$.

As a special case, let $S^1 \to S^3 \to S^2$ be the Hopf principal bundle. By using Corollary 4 we have

**Theorem 5.** There exists a weak homotopy equivalence

$$\text{Top}_G(S^3) \simeq \text{Spin}^c(3).$$
§1. The $G$-equivariant Homeomorphisms

In this paper, all function spaces are supposed to have compact open topology.

Let $G$ be a locally compact topological group and let $\xi$ be a principal $G$-bundle denoted by

$$\xi : G \longrightarrow E \overset{p}{\longrightarrow} B ,$$

where $E$ is path connected, $B$ is a locally finite $CW$ complex and $G$ acts freely on the right of $E$. If we denote by $\mathcal{G}(\xi)$ the space of self bundle maps of $\xi$ and denote by $\text{map}(B, B)$ the space of self maps of $B$, by the bundle map theory (see [2] and [3]) we have the following Serre fibration:

$$\xi_0 \longrightarrow \mathcal{G}(\xi) \overset{\phi}{\longrightarrow} \text{map}(B, B) .$$

Here $\xi_0$ is weakly homotopy equivalent to the loop space $\Omega(\text{map}(B, BG; \alpha))$ of $\text{map}(B, BG; \alpha)$ ([1], [2]), which is the path connected component of $\text{map}(B, BG)$ containing the classifying map $\alpha : B \longrightarrow BG$ for the principal bundle $\xi$. We know that $\xi_0$ can be identified with the space $\text{map}(B, G)$ if $G$ is abelian ([3]).

Let $\text{Top}(B)$ denote the group of homeomorphisms of $B$ and $\text{Top}_G(E)$ denote the group of $G$-equivariant homeomorphisms of $E$. Then $\Phi$ may not be surjective but we have the following

**Lemma 1.** $\Phi^{-1}(\text{Top}(B)) = \text{Top}_G(E)$.

**Proof.** First we shall show that each map $f$ of $\Phi^{-1}(\text{Top}(B))$ is injective. Put $\Phi(f) = \tilde{f}$. For any distinct points $x_1, x_2$ of $E$ with $p(x_1) \neq p(x_2)$, we obviously see $\tilde{f}(x_1) \neq \tilde{f}(x_2)$. For distinct points $x_1, x_2$ with $p(x_1) = p(x_2)$ there exists an element $a$ of $G$ such that

$$x_2 = x_1 \cdot a \quad (a \neq e) ,$$

where $e$ is the identity element of $G$. This implies $\tilde{f}(x_2) = \tilde{f}(x_1) \cdot a$. Since the action of $G$ is free we have $\tilde{f}(x_1) \neq \tilde{f}(x_2)$.

Surjectivity of $\tilde{f}$ also can be proved easily, and continuity of $\tilde{f}^{-1}$ follows from the fact that a bijective bundle map $\tilde{f}$ is a homeomorphism if its induced map $f$ is a homeomorphism.
Let $\alpha$ be a map of $B$ into the classifying space $BG$ whose homotopy class $[\alpha]$ classifies the principal bundle $\xi$. And let $\text{Top}^{[\alpha]}(B)$ denote the subspace of $\text{Top}(B)$ consisting of homeomorphisms $f : B \to B$ which satisfy

$$f^*([\alpha]) = [\alpha].$$

Immediately we have the following

**Lemma 2.** The $\Phi$ image of $\text{Top}_G(E)$ is just $\text{Top}^{[\alpha]}(B)$.

Consequently we have

**Theorem 3.** With notation above, there exists the following Serre fibration:

$$\xi \to \text{Top}_G(E) \xrightarrow{\Phi} \text{Top}^{[\alpha]}(B).$$

**Corollary 4.** Let $G$ be a locally compact abelian group, then we have the following Serre fibration:

$$\text{map}(B, G) \to \text{Top}_G(E) \xrightarrow{\Phi} \text{Top}^{[\alpha]}(B).$$

Now, let $\mathbb{R}^n$ be the $n$-dimensional real projective space then we have the principal bundle:

$$Z_2 \to S^n \to \mathbb{R}^n.$$

Thus we have

**Example 1.** There is a Serre fibration:

$$Z_2 \to \text{Top}_{Z_2}(S^n) \xrightarrow{\phi} \text{Top}(\mathbb{R}^n).$$

Similarly, let $\mathbb{C}^n$ be the $n$-dimensional complex projective space then we have the principal bundle:

$$S^1 \to S^{2n+1} \to \mathbb{C}^n.$$

Since $\text{map}(\mathbb{C}^n, S^1)$ is homotopy equivalent to $S^1$, we have

**Example 2.** There is a Serre fibration:

$$S^1 \to \text{Top}_{S^1}(S^{2n+1}) \xrightarrow{\phi} \text{Top}^+(\mathbb{C}^n).$$
where \( \text{Top}^+ (\mathbb{C}P^n) \) denotes the component of identity mapping in \( \text{Top} (\mathbb{C}P^n) \).

As a special case of Example 2, we have a Serre fibration:

\[
S^1 \rightarrow \text{Top}_{\mathbb{S}} (S^3) \rightarrow \text{Top}^+ (S^2)
\]

\( \textbf{§2. Top}_{\mathbb{S}} (S^3) \) and \( \text{Top}_{so(2)} (SO (3)) \)

Let us define a \((S^3 \times S^1)\)-action on \(S^3\) as follows:

\[
\rho : (S^3 \times S^1) \times S^3 \rightarrow S^3
\]

is given by

\[
\rho((q, z), q') = qq'z \quad (q, q' \in \mathbb{H}, z \in \mathbb{C}, |q| = |q'| = |z| = 1).
\]

Then we can easily verify that this action is not effective and the kernel of the action \(\rho\) is the central subgroup of the group \(S^3 \times S^1\) which consists of two elements \((1, 1)\) and \((-1, -1)\). Therefore \(\text{Spin}^c (3) \cong (S^3 \times S^1) / \{(1, 1) \cup (-1, -1)\}\) acts effectively on \(S^3\). Also we can easily prove that each element of \(\text{Spin}^c (3)\) induces an \(S^3\)-equivariant homeomorphism of \(S^3\). Thus we have the inclusion map \(i : \text{Spin}^c (3) \rightarrow \text{Top}_{\mathbb{S}} (S^3)\). With this notation, we have

**Theorem 5.** *The inclusion map \(i : \text{Spin}^c (3) \rightarrow \text{Top}_{\mathbb{S}} (S^3)\) gives a following weak homotopy equivalence*

\[
\text{Spin}^c (3) \cong \text{Top}_{\mathbb{S}} (S^3).
\]

**Proof.** We have the following principal bundle:

\[
S^1 \rightarrow \text{Spin}^c (3) \xrightarrow{\pi} S^3 / \{1, -1\}
\]

and the map \(i\) defines the following commutative diagram

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\(^1\)The fact that \((S^3 \times S^1) / \{(1, 1) \cup (-1, -1)\}\) is actually \(\text{Spin}^c (3)\) was pointed out by many participants in the Kinosaki Symposium held in autumn of 1994. The author expresses his thanks here.
\[
S^1 \longrightarrow \text{Spin}^c(3) \xrightarrow{\pi} S^3/[1, -1] \\
\downarrow \quad \downarrow i \quad \downarrow j \\
S^1 \longrightarrow \text{Top}_{\text{st}}(S^3) \xrightarrow{} \text{Top}^+(S^2)
\]

where \([1, -1]\) is the center of \(S^3\) and \(\pi\) is the map induced by the projection of \(S^3 \times S^1\) onto \(S^3\). By Kneser's theorem ([4]), we know that \(j\) is a homotopy equivalence. Considering the exactness of homotopy sequences of our fibrations, we see that \(i\) is a weak homotopy equivalence.

Next, let \(SO(2)\) act on the right of \(SO(3)\) as usual. We proceed to study the group of \(SO(2)\)-equivariant homeomorphisms of \(SO(3)\).

Let us define a \((SO(3) \times SO(2))\)-action on \(SO(3)\) similar to the case of \(S^3\) as follows:

\[
\rho' : (SO(3) \times SO(2)) \times SO(3) \longrightarrow SO(3)
\]

is given by

\[
\rho'(\sigma, g, \tau) = \sigma \tau g \quad (\sigma, \tau \in SO(3), g \in SO(2)).
\]

Then we can easily prove that this action is effective and each element of \(SO(3) \times SO(2)\) is an \(SO(2)\)-equivariant homeomorphism of \(SO(3)\). So, we have the inclusion map \(i' : SO(3) \times SO(2) \longrightarrow \text{Top}_{SO(2)}(SO(3))\).

On the other hand, we have the \(SO(2)\)-principal bundle:

\[
SO(2) \longrightarrow SO(3) \longrightarrow S^2.
\]

For this principal bundle Corollary 4 provides the following Serre fibration:

\[
SO(2) \longrightarrow \text{Top}_{SO(2)}(SO(3)) \longrightarrow \text{Top}^+(S^2).
\]

By the same manner in the proof of Theorem 5, we get the following

**Theorem 6.** The inclusion map

\[
i' : SO(3) \times SO(2) \longrightarrow \text{Top}_{SO(2)}(SO(3))
\]

gives a weak homotopy equivalence

\[
SO(3) \times SO(2) \simeq \text{Top}_{SO(2)}(SO(3)).
\]
References


