Norm Additivity Conditions for Normal Linear Functionals on von Neumann Algebras

By

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§1. Introduction

Let \( M \) be a von Neumann algebra and let \( \varphi \) and \( \psi \) be bounded linear functionals on \( M \). We then have the norm inequality \( \|\varphi + \psi\| \leq \|\varphi\| + \|\psi\| \). On the other hand, it is well-known that if \( \varphi \) and \( \psi \) are positive, then \( \|\varphi + \psi\| = \|\varphi\| + \|\psi\| \). In general, however, such an equality does not necessarily holds if both \( \varphi \) and \( \psi \) are not positive. The purpose of this paper is to investigate when the norm equality \( \|\varphi + \psi\| = \|\varphi\| + \|\psi\| \) holds for given normal linear functionals \( \varphi \) and \( \psi \). Then the fact to play an essential role is the following:

Let \( M \) be a von Neumann algebra and let \( \varphi \) be a normal linear functional on \( M \). Then we have

\[
\varphi(\cdot) = |\varphi|(v' \cdot), \quad |\varphi|(\cdot) = \varphi(v' \cdot), \quad \text{and} \quad \|\varphi\| = \|\varphi\|
\]

for all partial isometries \( v' \) in \( M \) satisfying that \( \varphi(v') = |\varphi| \), where \( |\varphi| \) denotes the absolute value of \( \varphi \) (c.f. [1, Lemma 2.3]).

In connection with this fact, we may expect that the set of those elements \( x \), in the unit ball of \( M \), with \( \varphi(x) = |\varphi| \) has nice information on norms and absolute values of normal linear functionals on \( M \). In fact, by employing such a set, we will give necessary and sufficient conditions for a pair of normal linear functionals \( \varphi \) and \( \psi \) to satisfy that \( \|\varphi + \psi\| = \|\varphi\| + \|\psi\| \) or that \( |\varphi + \psi| = |\varphi| + |\psi| \) (Theorem 2.1).

§2. Results

Let \( M \) be a von Neumann algebra and let \( \varphi \) be a normal linear functional on \( M \). By the polar decomposition of \( \varphi \), we mean the following expression:

\[
\varphi(\cdot) = |\varphi|(v' \cdot) \quad \text{and} \quad |\varphi|(\cdot) = \varphi(v' \cdot)
\]
for some partial isometry \( v \) in \( M \) and a uniquely positive linear functional \( |\varphi| \) on \( M \) which satisfies that
\[
|\varphi(x)|^2 \leq |\varphi|(|\varphi(x)x|)
\]
for all \( x \) in \( M \) (cf. [2, 3.6.7], [3, 1.14.4], or [4, III. 4.2]). Note that this condition \((*)\) ensures the uniqueness of \( |\varphi| \) (cf. [2, 3.6.7] or [4, III. 4.6]). More precisely, if a positive linear functional \( \psi \) satisfies that
\[
|\psi(x)|^2 \leq |\varphi|(|\varphi(x)x|),
\]
then \( \psi = |\varphi| \). In general, there is some freedom for the choice of \( v \), as is seen from [1, Lemma 2.3]. However, if \( vv' \) is exactly equal to the support projection \( s(|\varphi|) \) of \( |\varphi| \), i.e., the smallest of all projections \( p \) such that \( |\varphi|(p-) = |\varphi|(\cdot) \), then \( v \) is uniquely determined (cf. [3, 1.14.4], [4, III. 4.6]).

Now we set
\[
M_{\varphi} = \{ x \in M \mid ||x|| \leq 1, \varphi(x) = ||\varphi|| \},
\]
which is a non-empty and weakly compact face of the unit ball of \( M \). Hence \( M_{\varphi} \) contains a partial isometry (cf. [2, 1.4.7], [3, 1.6.5], [4, 1.10.2]).

Positive linear functionals \( \varphi \) and \( \psi \) on a \( C^* \)-algebra always satisfy that
\[
|\varphi + \psi| = |\varphi| + |\psi|
\]
and \( \varphi + \psi \) is positive. These are generalized as follows.

**Theorem 2.1.** Let \( M \) be a von Neumann algebra and let \( \varphi \) and \( \psi \) be normal linear functionals on \( M \). Then the following conditions are equivalent:

1. \( M_{\varphi} \cap M_{\psi} \) is not empty.
2. \( |\varphi + \psi| = |\varphi| + |\psi| \).
3. \( \varphi + \psi = |\varphi| + |\psi| \).
4. \( M_{\varphi} \cap M_{\psi} = M_{\varphi + \psi} \).

**Proof.** We first show that \( M_{\varphi} \cap M_{\psi} \subset M_{\varphi + \psi} \). Without loss of generality, we can assume that \( M_{\varphi} \cap M_{\psi} \) is not empty. For any \( x \) in \( M_{\varphi} \cap M_{\psi} \), we have
\[
|\varphi| + |\psi| = \varphi(x) + \psi(x) = (\varphi + \psi)(x) \leq \varphi + \psi.
\]
Since this mean that \((\varphi + \psi)(x) = |\varphi + \psi|\), we see that \( x \in M_{\varphi + \psi} \).

(1) \(\implies\) (2). Let \( v \) be a partial isometry in \( M_{\varphi} \cap M_{\psi} \). Since \( v \in M_{\varphi + \psi} \), it follows from [1, Lemma 2.3] that
\[
|\varphi + \psi(\cdot) = (\varphi + \psi)(v \cdot) = \varphi(v \cdot) + \psi(v \cdot) = |\varphi(\cdot)| + |\psi(\cdot)|.
\]

(2) \(\implies\) (3). Since \( |\varphi + \psi|, |\varphi| \) and \( |\psi| \) are positive linear functionals, we have
\[
|\varphi + \psi| = |\varphi + \psi| = |\varphi + \psi(1) = (|\varphi| + |\psi|)(1)
\]
\[
= |\varphi(1) + |\psi(1) = ||\varphi|| + ||\psi|| = ||\varphi|| + ||\psi||.
\]

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1 As the definition of a polar decomposition, we adopt the (right) polar decomposition mentioned in [2, 3.6.7] and our absolute value \( |\varphi| \) means \( |\varphi| \) in the sense of the (left) polar decomposition mentioned in [3, 1.14.4] and [4, III. §4].
(3) ⇒ (4). We have only to show that $M_{\varphi + \psi} \subset M_{\varphi} \cap M_{\psi}$. Take any $x$ from $M_{\varphi + \psi}$. We then have

$$\varphi(x) + \psi(x) = (\varphi + \psi)(x) = \|\varphi + \psi\| = \|\varphi\| + \|\psi\|.$$  

Now denote by $\Re z$ the real part of a complex number $z$ and by $\Im z$ the imaginary part of $z$, respectively. Since $\|\varphi\| + \|\psi\|$ is a real number, it follows from the above equality that

$$\Re \varphi(x) + \Re \psi(x) = \Re(\varphi(x) + \psi(x)) = \|\varphi\| + \|\psi\|.$$  

Here remark that

$$\|\omega\| \geq |\omega(x)| \geq |\Re \omega(x)| \quad (**)$$  

for every bounded linear functional $\omega$ on $M$. We thus have

$$0 \leq \|\varphi\| - \Re \varphi(x) = \Re \psi(x) - \|\psi\| \leq 0.$$  

Hence we conclude that $\|\varphi\| = \Re \varphi(x)$ and $\|\psi\| = \Re \psi(x)$. This and the inequality (**) show that $\Im \varphi(x) = 0$, i.e., $\|\varphi\| = \varphi(x)$. Similarly we obtain that $\|\psi\| = \psi(x)$. We thus see that $x \in M_{\varphi} \cap M_{\psi}$.

(4) ⇒ (1). Since $M_{\varphi + \psi}$ is not empty, this implication is clear. Q.E.D.

Here recall that positive linear functionals $\varphi$ and $\psi$ on a $C^*$-algebra are said to be orthogonal if $\|\varphi - \psi\| = \|\varphi\| + \|\psi\|$.

**Corollary 2.2.** Let $M$ be a von Neumann algebra and let $\varphi$ and $\psi$ be positive normal linear functionals on $M$. Then the following conditions are equivalent:

1. $\varphi$ and $\psi$ are orthogonal.
2. $M_{\varphi} \cap M_{\psi}$ is not empty.

**References**


