Equivariant $K$-Theory and Maps between Representation Spheres

Dedicated to Professor Yasutoshi Nomura on his 60th birthday

By

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§1. Introduction and Statement of Results

The equivariant $K$-theory has been successfully employed in the study of equivariant maps by Marzantowicz [5], Liulevicious [7] and Bartsch [3]. In the present paper, using the equivariant $K$-theory, we will obtain a necessary condition for the existence of $G$-maps $SU \to SW$, where $SU$ and $SW$ are the unit spheres of unitary representations $U$ and $W$, respectively, of a compact Lie group $G$.

From Atiyah [1], [2] or Segal [8] we can see that the equivariant $K$-ring $K_G(SU)$ of $SU$ is isomorphic to $R(G)/\langle \lambda, U \rangle$, the complex representation ring $R(G)$ divided by the ideal $\langle \lambda, U \rangle$ of $U$ in $K_G(pt) = R(G)$. If there exists a $G$-map $\eta : SU \to SW$, then we obtain a ring homomorphism $\eta : R(G)/\langle \lambda, W \rangle \to R(G)/\langle \lambda, U \rangle$ which coincides with the homomorphism induced from the identity on $R(G)$. This implies that the condition $\lambda, W \in (\lambda, U)$ is necessary for the existence of $G$-maps $SU \to SW$. If $G$ is abelian, we will reduce this condition to more explicit form.

Let $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$ be the circle group of complex numbers with absolute value 1, and $Z_n$ the cyclic group of order $n$ considered as a subgroup of $S^1$. For any integer $i$ let $S^1$ and $Z_n$ act on $V_i = \mathbb{C}$ via $(z, v) \mapsto z^i v$ for $z \in S^1$ (or $Z_n$) and $v \in V_i$. A compact abelian group $G$ decomposes into a cartesian product

$$G = T^k \times Z_{n_1} \times \cdots \times Z_{n_k},$$

where $T^k = S^1 \times \cdots \times S^1$, the cartesian product of $k$ copies of $S^1$. Letting $\gamma$ be a sequence $(a_1, \ldots, a_k, b_1, \ldots, b_l)$ of integers, denote by $V_\gamma$ the tensor product

$$V_{a_1} \otimes \cdots \otimes V_{a_k} \otimes V_{b_1} \otimes \cdots \otimes V_{b_l},$$


1991 Mathematics Subject Classifications: 55N15, 57S99

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which can be considered as a representation of $G$ in a natural way. Let $\Gamma$ be the set of sequences

$$\gamma = (a_1, \ldots, a_k, b_1, \ldots, b_l)$$

with $a_1, \ldots, a_k \in \mathbb{Z}$ and $0 \leq b_j \leq n_j - 1$ for $1 \leq j \leq l$. The set $\{V_\gamma \mid \gamma \in \Gamma\}$ gives a complete set of irreducible unitary representations of $G$, and so any unitary representation $U$ of $G$ decomposes into a direct sum

$$U = \bigoplus_{\gamma \in \Gamma} V_\gamma^{u(\gamma)},$$

where $u(\gamma)$ is a nonnegative integer and $V_\gamma^{u(\gamma)}$ denotes the direct sum of $u(\gamma)$ copies of $V_\gamma$. We can easily see that the fixed point set $U^G$ of $U$ is $\{0\}$ if and only if $u(\gamma) = 0$ for $\gamma = (0, \ldots, 0)$. Let

$$|\gamma| = |a_i| + \cdots + |a_k| + b_1 + \cdots + b_l$$

for any $\gamma = (a_1, \ldots, a_k, b_1, \ldots, b_l) \in \Gamma$.

We are now in a position to state our main theorem.

**Theorem 1.1.** Let $U$ and $W$ be unitary representations of a compact abelian group $G$, and decompose into

$$U = \bigoplus_{\gamma \in \Gamma} V_\gamma^{u(\gamma)} \quad \text{and} \quad W = \bigoplus_{\gamma \in \Gamma} V_\gamma^{u(\gamma)}.$$

Assume that there exists a $G$-map $S_U \rightarrow S_W$. Then

1. if $\dim U = \dim W$, then there is an integer $m$ such that

$$\prod_{\gamma \in \Gamma} |\gamma|^{u(\gamma)} \equiv m \prod_{\gamma \in \Gamma} |\gamma|^{u(\gamma)} \mod d,$$

where $d$ is the greatest common divisor of $n_1, \ldots, n_l$, (if $l = 0$, then assume $d = 0$),

2. if $\dim U > \dim W$, then

$$\prod_{\gamma \in \Gamma} |\gamma|^{u(\gamma)} \equiv 0 \mod d.$$

From this theorem we obtain the following two corollaries.

**Corollary 1.2** (cf. Liulevicious [7], Bartsch [4], Marzantowicz [6]). Let $U$ and $W$ be representations of $G = T^k$ with $W^G = \{0\}$. If there exists a $G$-map $S_U \rightarrow S_W$, then $\dim U < \dim W$.

**Corollary 1.3** (Liulevicious [7], Marzantowicz [6]). Let $U$ and $W$ be representations of $G = \mathbb{Z}_n$ with $n$ any. If $G$ acts freely on $S_W$ and if there exists a $G$-map $S_U \rightarrow S_W$, then $\dim U < \dim W$. 
Remark 1.4. If \( U \) is an orthogonal representation of \( G = T^k \) or \( Z_n \) with \( n \) odd and if \( U^G = \{0\} \), then \( U \) can be considered a unitary representation. In general, if \( U \) is orthogonal then \( U \oplus U \) becomes unitary. Since the join of two \( G \)-maps \( SU \to SW \) gives a \( G \)-map
\[
S(U \oplus U) = SU \ast SU \to SW \ast SW = S(W \oplus W),
\]
Corollaries 1.2 and 1.3 follow for orthogonal representations \( U \) and \( W \).

Remark 1.5. We should refer to a recent paper [6] of Marzantowicz. Using the Borel cohomology theory, he also studies equivariant maps between representation spheres, and obtains a necessary condition for the existence of such maps. A detailed study is done for the case of \( G = T^k \) or \( Z_p \times \cdots \times Z_p \). It is also shown that his condition is sufficient in some case.

§2. A Necessary Condition in Terms of the Euler Classes

Let \( U \) be a unitary representation of a compact Lie group \( G \). The sequence
\[
\cdots \to K^*_c(DU, SU) \to K^*_c(DU) \to K^*_c(SU) \to K^{n+1}_c(DU, SU) \to \cdots
\]
is the long exact sequence of the equivariant \( K \)-theory \( K^*_c \) for the pair \( (DU, SU) \) of the unit disk \( DU \) and the unit sphere \( SU \) of \( U \). Segal [8; Proposition 3.2] or Atiyah [2] gives the Thom isomorphism
\[
\varphi : K_c(pt) \to K_c(U) = K_c(DU, SU)
\]
such that \( \varphi \varphi(\xi) = \xi \cdot \lambda_{-i}U \) for \( \xi \in K_c(pt) \), where \( \varphi : K_c(U) \to K_c(pt) \) is the homomorphism induced from the inclusion map \( \varphi : \{pt\} \to U \),
\[
\lambda_{-i}U = \sum_i (-1)^i \wedge U \in K^*_c(pt),
\]
and \( \wedge U \) is the \( i \)-th exterior algebra of \( U \). Since \( K^*_c(DU, SU) = K^*_c(U) \equiv K^*_c(pt) = 0 \) and \( K^*_c(pt) \equiv R(G) \), the sequence (2.1) yields the exact sequence
\[
R(G) \to R(G) \to K_c(SU) \to 0,
\]
where the first homomorphism is given by multiplication by \( \lambda_{-i}U \). This argument is done in the same manner as in Atiyah [1; Lemma 2.7.4, Corollary 2.7.5] where \( G \) is finite abelian.

From the exact sequence (2.2) we obtain

**Proposition 2.3.** \( K_c(SU) \equiv R(G)/\langle \lambda_{-i}U \rangle \).

Let \( \eta : SU \to SW \) be a \( G \)-map for representations \( U \) and \( W \) of \( G \). Since the sequence (2.1) is functorial, we see that the composite
\[ R(G)/(\lambda_\gamma U) \cong K_G(SW) \cong K_G(SU) \cong R(G)/(\lambda_\gamma U) \]

coincides with the homomorphism induced from the identity on \( R(G) \). This implies the following.

**Proposition 2.4.** If there exists a \( G \)-map \( SU \to SW \), then \( \lambda_\gamma U \in (\lambda_\gamma U) \) in \( R(G) \).

§3. Calculation of \( K_G(SU) \)

In this section we will calculate the ring \( K_G(SU) \) for the case where \( G \) is abelian.

We first recall the following facts about the complex representation rings of \( G \):

1. \( R(S^1) \cong \mathbb{Z}[x, x^{-1}]/(1-xx^{-1}) \), in which the representation \( V_i \) corresponds to \( x^i \) if \( i \geq 0 \) and to \( (x^{-1})^{-i} \) if \( i \leq 0 \).
2. \( R(\mathbb{Z}_n) \cong \mathbb{Z}[x]/(1-x^n) \), in which \( V_i \) corresponds to \( x^i \).
3. \( R(G_1 \times G_2) \cong R(G_1) \otimes R(G_2) \).

From these facts we obtain

**Proposition 3.1.** If \( G = T^k \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l} \) is a compact abelian group, then

\[ R(G) \cong \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_k^{-1}, y_1, \ldots, y_l]/(X, Y), \]

where

\[ X = \{1 - x_i x_i^{-1} \mid 1 \leq i \leq k\}, \]
\[ Y = \{1 - y_j y_j \mid 1 \leq j \leq l\}, \]

and \((X, Y)\) is the ideal generated by \( X \cup Y \). The isomorphism sends the representation \( V_i \) to the monomial \( x_1^{n_1} \cdots x_k^{n_k} y_1^{n_1} \cdots y_l^{n_l} \) if \( \gamma = (a_1, \ldots, a_k, b_1, \ldots, b_l) \).

Since \( \lambda_\gamma \) is multiplicative, i.e., \( \lambda_\gamma(U_1 \otimes U_2) = \lambda_\gamma U_1 \cdot \lambda_\gamma U_2 \), Propositions 2.3 and 3.1 give the following.

**Proposition 3.2.** Let \( U = \bigoplus_{\gamma \in \gamma} V_{\nu(\gamma)} \) be a unitary representation of \( G = T^k \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l} \). Then

\[ K_G(SU) \cong \mathbb{Z}[x_1, x_1^{-1}, \ldots, x_k^{-1}, y_1, \ldots, y_l]/(X, Y, z_U), \]

where \( z_U = \prod_{\gamma} (1-(xy)^{\nu(\gamma)})(xy)^{\gamma} = x_1^{n_1} \cdots x_k^{n_k} y_1^{n_1} \cdots y_l^{n_l} \) if \( \gamma = (a_1, \ldots, a_k, b_1, \ldots, b_l) \), and \((X, Y, z_U)\) is the ideal generated by \( X \cup Y \cup \{z_U\} \).
§4. Proof of Theorem 1.1

In this section we will prove Theorem 1.1. Let \( G = T^k \times Z_{n_1} \times \cdots \times Z_{n_t} \) be a compact abelian group, and
\[
U = \bigoplus_{\gamma \in \Gamma} V^{\gamma}_U, \quad W = \bigoplus_{\gamma \in \Gamma} V^{\gamma}_W
\]
its unitary representations. Assume that there exists a \( G \)-map \( \eta : SU \rightarrow SW \). If \( W^G \neq \{0\} \), then the theorem is trivially valid. So we assume \( W^G = \{0\} \).

For the representation
\[
V_\gamma = V_{a_1} \otimes \cdots \otimes V_{a_k} \otimes V_{b_1} \otimes \cdots \otimes V_{b_l},
\]
let
\[
\overline{V}_\gamma = V_{\mu_1} \otimes \cdots \otimes V_{\mu_{k'}} \otimes V_{\nu_1} \otimes \cdots \otimes V_{\nu_{l'}},
\]
and
\[
\overline{U} = \bigoplus_{\gamma \in \Gamma} \overline{V}^{\gamma}_\gamma, \quad \overline{W} = \bigoplus_{\gamma \in \Gamma} \overline{V}^{\gamma}_\gamma.
\]
Since \( V_u \equiv V_{\mu_1} \) as real representations, we see \( U \equiv \overline{U} \) and \( W \equiv \overline{W} \). Therefore \( \eta : SU \rightarrow SW \) induces a \( G \)-map \( \overline{\eta} : SU \rightarrow SW \), and then \( \overline{\eta} \) induces a ring homomorphism \( \overline{\eta} : K_G(SW) \rightarrow K_G(SU) \). From Proposition 3.2 we obtain a ring homomorphism
\[
\overline{\eta} : Z[x, x^{-1}, \ldots, x, x^{-1}, y_1, \ldots, y_l]/(X, Y, \overline{\zeta}_U) \rightarrow Z[x, x^{-1}, \ldots, x, x^{-1}, y_1, \ldots, y_l]/(X, Y, \overline{\zeta}_U),
\]
where \( X \) and \( Y \) are as given in Proposition 3.1,
\[
\overline{\zeta}_U = \prod_{\gamma \in \Gamma} (1 - \overline{x} \overline{y}^{\gamma})^{w(\gamma)}, \quad \overline{\zeta}_W = \prod_{\gamma \in \Gamma} (1 - \overline{x} \overline{y}^{\gamma})^{w(\gamma)},
\]
and
\[
\overline{x} \overline{y}^{\gamma} = x^{\mu_1} \cdots x^{\mu_{k'}} y_1^{b_1} \cdots y_{l'}^{b_{l'}}.
\]
As in Proposition 2.4, we see \( \overline{\zeta}_W \in (X, Y, \overline{\zeta}_U) \). Then there are polynomials \( f_j \) (1 \( \leq j \leq l + 1 \)) in \( Z[x, x^{-1}, \ldots, x, x^{-1}, y_1, \ldots, y_l] \) such that
\[
\overline{\zeta}_W = \sum_{j=1}^{l} f_j \cdot (1 - y_j^{n_j}) + f_{l+1} \cdot \overline{\zeta}_U
\]
in \( Z[x, x^{-1}, \ldots, x, x^{-1}, y_1, \ldots, y_l]/(X) \). Multiplying (4.1) by \( x_1^{m_1} \cdots x_k^{m_k} \) for sufficiently large \( m_1, \ldots, m_k > 0 \), we obtain
\[
x_1^{m_1} \cdots x_k^{m_k} \overline{\zeta}_W = \sum_{j=1}^{l} \overline{f}_j \cdot (1 - y_j^{n_j}) + \overline{f}_{l+1} \cdot \overline{\zeta}_U
\]
in \( \mathbb{Z}[x_1,\ldots,x_t,y_1,\ldots,y_l] \), where \( \widetilde{f}_i (1 \leq j \leq l+1) \) are polynomials in \( \mathbb{Z}[x_1,\ldots,x_t,y_1,\ldots,y_l] \). Substituting \( x \) for all of \( x_1,\ldots,x_t,y_1,\ldots,y_l \) in (4.2), we obtain

\[
(4.3) \quad x^m \prod_{\gamma \in \Gamma} (1-x^{\gamma})^{u(\gamma)} = \sum_{j=1}^l g_j(x)(1-x^n) + g_{l+1}(x) \prod_{\gamma \in \Gamma} (1-x^{\gamma})^{u(\gamma)},
\]

where \( m = m_t + \cdots + m_{x_t} \), \( g_j(x) \in \mathbb{Z}[x] (1 \leq j \leq l+1) \) and \( |\gamma| = |a_i| + \cdots + |a_i| + b_1 + \cdots + b_l \) if \( \gamma = (a_1,\ldots,a_t,b_1,\ldots,b_l) \). If \( \dim U \geq \dim W \), we can divide the both sides of (4.3) by \( (1-x)^{\Sigma u(\gamma)} \), and obtain

\[
(4.4) \quad x^m \prod_{\gamma \in \Gamma} (1+x+\cdots+x^{\gamma-1})^{u(\gamma)} = h(x) + g_{l+1}(x)(1-x)^{\Sigma u(\gamma)} \prod_{\gamma \in \Gamma} (1+x+\cdots+x^{\gamma-1})^{u(\gamma)},
\]

where \( h(x) = \sum_{j=1}^l g_j(x)(1-x^n)/(1-x)^{\Sigma u(\gamma)} \in \mathbb{Z}[x] \). Since

\[
1-x^{n+1} = (1-x)(1+x+\cdots+x^{d-1})p_j(x)
\]

for any divisor \( d_j \) of \( n_j \) and some \( p_j(x) \in \mathbb{Z}[x] \), we see

\[
\sum_{j=1}^l g_j(x)(1-x^{n_j}) = (1-x)(1+x+\cdots+x^{d-1})\sum_{j=1}^l g_j(x)p_j(x),
\]

where \( d \) is the greatest common divisor of \( n_1,\ldots,n_l \). Since \( 1-x \) and \( 1+x+\cdots+x^{d-1} \) are prime to each other, \( h(x) = (1+x+\cdots+x^{d-1})q(x) \) for some \( q(x) \in \mathbb{Z}[x] \). Therefore, substituting 1 for \( x \) in (4.4), we obtain

\[
\prod_{\gamma \in \Gamma} |\gamma|^{u(\gamma)} = d \cdot q(1) + g_{l+1}(1) \prod_{\gamma \in \Gamma} |\gamma|^{u(\gamma)}
\]

if \( \sum \gamma u(\gamma) = \sum \gamma w(\gamma) \), and

\[
\prod_{\gamma \in \Gamma} |\gamma|^{u(\gamma)} = d \cdot q(1)
\]

if \( \sum \gamma u(\gamma) > \sum \gamma w(\gamma) \). This completes the proof of Theorem 1.1.

References