The Chern Character of the Symmetric Space $E_6$

Dedicated to Professor Seiya Sasao on his 60th birthday

By

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Abstract

The purpose of this paper is to describe the Chern character homomorphism of the compact symmetric space $E_6$.

§1. Introduction

Let $E_6$ be the compact, 1-connected, exceptional Lie group of rank 6 and $PSp(4) = Sp(4)/\{\pm I_4\}$ the coset space of the symplectic group of degree 4 by its center (where $I_n$ denotes the unit matrix of degree $n$). There is an involutive automorphism $\rho : E_6 \to E_6$ whose fixed point set $E_6^{\rho}$ is $PSp(4)$ (see [9]). So the coset space $E_6/PSp(4)$ is a compact, 1-connected, irreducible symmetric space which is denoted by $E_6$ in É. Cartan's notation. The cohomology and $K$-theory of $E_6 = E_6/PSp(4)$ are known (see [3], [4] and [6]). In this paper we describe the Chern character homomorphism of $E_6$.

According to Ishitoya [3], the cohomology of $E_6$ is as follows. Its rational cohomology ring is given by

$$H^*(E_6; \mathbb{Q}) = \mathbb{Q}[e_8]/(e_8^2) \otimes \Lambda_\mathbb{Q}(e_9, e_{17}),$$

where $e_i \in H^*(E_6; \mathbb{Q})$. Note that $\dim E_6 = 42$. The integral cohomology of $E_6$ has only 2-torsion, and

$$H^*(E_6; \mathbb{Z})/\text{Tors.}H^*(E_6; \mathbb{Z})$$

$$= \mathbb{Z}[1, e_8, e_9, e_{16}, e_{17}, e_9 e_{16}, e_9 e_9, e_{16} e_9, e_{17}, e_9 e_{16} e_9 e_{17}].$$

where the relations $4e_8^2 = e_9', 2e_9 e_9 = e_{17}', 2e_9 e_{17} = e_{25}'$ and $4e_8 e_9 e_{17} = e_{34}'$ hold.

According to Minami [6], the $K$-theory of $E_6$ is as follows. The complex representation ring of $E_6$ is given by

$$R(E_6) = \mathbb{Z}[\phi_1, \phi_2, \lambda^2 \phi_1, \lambda^3 \phi_1, \lambda^2 \phi_2, \phi_6].$$

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where \( \lambda^i \) denotes the \( k \)-th exterior power operation, \( \dim \varphi_1 = \dim \varphi_6 = 27 \), \( \dim \varphi_2 = 78 \) and the relation \( \lambda^i \varphi_1 = \lambda^j \varphi_6 \) holds. Let \( \chi_i = [(H^4)_c] \in R(\text{Sp}(4)) \). Then

\[
R(\text{Sp}(4)) = \mathbb{Z}[\chi_1, \lambda^2 \chi_1, \lambda^3 \chi_1, \lambda^4 \chi_1],
\]

where \( \dim \lambda^4 \chi_1 = \binom{8}{k} \). As a subring of \( R(\text{Sp}(4)) \),

\[
R(\text{PSp}(4)) = \mathbb{Z}[\lambda^2 \chi_1, \lambda^2 \chi_1, \chi_1, (\lambda^2 \chi_1)^2, \chi_1^2 \lambda^2 \chi_1].
\]

The element \( \chi_1^2 - 64 \) belongs to the augmentation ideal \( I(\text{PSp}(4)) \). Let \( \alpha(\tilde{\chi}_1^2) \in \tilde{K}^0(\text{EI}) \) denote the image of \( \chi_1^2 - 64 \) under the composite

\[
R(\text{PSp}(4)) \xrightarrow{\alpha} K^0(\text{BPSp}(4)) \xrightarrow{\beta} K^0(\text{EI}),
\]

where \( \alpha \) is the \( \lambda \)-ring homomorphism of [1], and \( j_0 : EI \to \text{BPSp}(4) \) is the map induced from the inclusion \( i_6 : \text{PSp}(4) \to E_6 \). Let \( (I(E_6)) \) be the ideal in \( R(\text{PSp}(4)) \) generated by the image of \( i_6 : I(E_6) \to I(\text{PSp}(4)) \). Then the above composite factors to give

\[
R(\text{PSp}(4))/(I(E_6)) \to K^0(\text{EI}),
\]

where by [6, II, Theorem 5.3],

\[
R(\text{PSp}(4))/(I(E_6)) = \mathbb{Z}[\chi_1^2]/((\chi_1^2 - 64)^3).
\]

The homomorphism \( \rho : R(E_6) \to R(E_6) \) satisfies

\[
\rho^i(\varphi_1) = \varphi_6, \rho^i(\varphi_2) = \varphi_2, \rho^i(\lambda^i \varphi_1) = \lambda^i \varphi_6 \text{ and } \rho(\lambda^i \varphi_1) = \lambda^i \varphi_1.
\]

Let \( U \) be the infinite unitary group and \( t_o : U(n) \to U \) the canonical injection. Since \( E_6^o = \text{PSp}(4) \), if \( \rho(\lambda) = \mu \) and \( \dim \lambda = n \), there is a map \( f_\lambda : EI \to U(n) \) defined by

\[
f_\lambda(xPSp(4)) = \lambda(x)\mu(x)^{-1} \text{ for } xPSp(4) \in EI.
\]

Denote by \( \beta(\lambda - \mu) \) the homotopy class of the composite \( t_o f_\lambda \). Thus we have \( \beta(\varphi_1 - \varphi_6), \beta(\lambda^2 \varphi_1 - \lambda^2 \varphi_6) \in [EI, U] = \tilde{K}^{-1}(EI) \). Elements \( \beta(\varphi_1), \beta(\lambda^2 \varphi_1) \in \tilde{K}^{-1}(E_6) \) are defined in a similar manner. By [6, I, Proposition 7.3], the \( K \)-theory of \( EI \) is torsion-free and

\[
K^0(EI) = K^0(pt) \otimes \mathbb{Z}[\alpha(\tilde{\chi}_1^2)]/(\alpha(\tilde{\chi}_1^2)^3) \otimes A_2(\beta(\varphi_1 - \varphi_6), \beta(\lambda^2 \varphi_1 - \lambda^2 \varphi_6)).
\]

With the above notation, our main result is

**Theorem 1.** The Chern character \( ch : \tilde{K}(EI) \to \tilde{H}(EI; \mathbb{Q}) \) is given by
According to [9], there is an involutive outer automorphism \( \tau : E_6 \to E_6 \) whose fixed point set \( E_6^\tau \) is the compact exceptional group \( F_4 \) of rank 4, and there is an inner automorphism \( \gamma \) of \( E_6 \) such that \( \rho = \gamma \tau = \tau \gamma \) (where our notation is different from [9]). Since \( \rho \tau = \tau \rho \), it follows that

\[
(E_6^\gamma)^\rho = (E_6^\rho)^\gamma = E_6^\rho \cap E_6^\gamma.
\]

It is known to be \( S^3 \cdot \text{Sp}(3) = (S^3 \times \text{Sp}(3))/\mathbb{Z}_2 \), where \( \mathbb{Z}_2 = \{(1,1),(-1,-1)\} \). We denote it by \( D \). So \( D = F_4 \cap \text{PSp}(4) \). Let \( C = T^3 \cdot \text{Sp}(3) \) be as in [4]. Then \( C \subset D \).

If \( T' \) is a maximal torus of \( \text{Sp}(4) \), then \( T' = T'/{\pm I_4} \) is a maximal torus of \( C \), \( D \), \( F_4 \) and \( \text{PSp}(4) \). Choose a maximal torus \( T \) of \( E_6 \) so that \( T' \subset T \). Thus we have an inclusion \( i_e : C \to D \) and a diagram of inclusions

\[
\begin{array}{ccc}
D & \xrightarrow{i_e} & F_4 \\
\downarrow & & \downarrow \ i_6 \\
\text{PSp}(4) & \xrightarrow{i_0} & E_6.
\end{array}
\]

We also have inclusions \( i_2 = i_2 i_e : C \to F_4 \) and \( i_{10} = i_6 i_4 = i_6 i_2 : D \to E_6 \) etc.

For details of the following argument, see [4, §2]. \( F_4 \) has a system of simple roots \( \{\alpha_i | i = 1,2,3,4\} \). The corresponding fundamental weights \( \omega_i \) are given by

\[
\begin{align*}
\omega_1 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \\
\omega_2 &= 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4, \\
\omega_3 &= 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4, \\
\omega_4 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4.
\end{align*}
\]

\( D \) has a system of simple roots \( \{\beta_i | i = 0,1,2,3\} \). The corresponding fundamental weights \( \phi_i \) are given by

\[
\begin{align*}
\phi_0 &= \frac{1}{2} \beta_0, \\
\phi_1 &= \beta_1 + \beta_2 + \frac{1}{2} \beta_3, \\
\phi_2 &= \beta_1 + 2\beta_2 + \beta_3, \\
\phi_3 &= \beta_1 + 2\beta_2 + \frac{3}{2} \beta_3.
\end{align*}
\]
$\textit{PSp}(4)$ has a system of simple roots $\{\gamma_i | i = 1, 2, 3, 4\}$. The corresponding fundamental weights $\chi_i$ are given by

$$
\begin{align*}
\chi_1 &= \gamma_1 + \gamma_2 + \gamma_3 + \frac{1}{2} \gamma_4, \\
\chi_2 &= \gamma_1 + 2\gamma_2 + 2\gamma_3 + \gamma_4, \\
\chi_3 &= \gamma_1 + 2\gamma_2 + 3\gamma_3 + \frac{3}{2} \gamma_4, \\
\chi_4 &= \gamma_1 + 2\gamma_2 + 3\gamma_3 + \frac{3}{2} \gamma_4.
\end{align*}
$$

$E_6$ has a system of simple roots $\{\delta_j | j = 1, 2, 3, 4, 5, 6\}$. The corresponding fundamental weights $\varphi_j$ are given by

$$
\begin{align*}
\varphi_1 &= \frac{4}{3} \delta_1 + \delta_2 + \frac{5}{3} \delta_3 + 2\delta_4 + \frac{4}{3} \delta_5 + \frac{2}{3} \delta_6, \\
\varphi_2 &= \delta_1 + 2\delta_2 + \frac{2}{3} \delta_3 + 3\delta_4 + \frac{2}{3} \delta_5 + \frac{1}{3} \delta_6, \\
\varphi_3 &= \frac{5}{3} \delta_1 + 2\delta_2 + \frac{10}{3} \delta_3 + 4\delta_4 + \frac{8}{3} \delta_5 + \frac{4}{3} \delta_6, \\
\varphi_4 &= 2\delta_1 + 3\delta_2 + \frac{8}{3} \delta_3 + 6\delta_4 + \frac{10}{3} \delta_5 + \frac{5}{3} \delta_6, \\
\varphi_5 &= \frac{4}{3} \delta_1 + 2\delta_2 + \frac{4}{3} \delta_3 + 2\delta_4 + \frac{5}{3} \delta_5 + \frac{4}{3} \delta_6, \\
\varphi_6 &= \frac{2}{3} \delta_1 + \delta_2 + \frac{4}{3} \delta_3 + 2\delta_4 + \frac{5}{3} \delta_5 + \frac{4}{3} \delta_6.
\end{align*}
$$

These $\{\alpha_i\}, \{\beta_i\}, \{\gamma_i\}$ can be regarded as bases for $H^2(BT';\mathbb{Q})$ and $\{\delta_j\}$ a basis for $H^2(BT;\mathbb{Q})$. We may suppose that $i_2(T') \subset T'$, $i_4(T') \subset T'$, $i_5(T') \subset T$ and $i_6(T') \subset T$.

The theory of Lie algebras for symmetric spaces [5] tells us the following facts. The dominant root with respect to the root system of $F_4$ is $\tilde{\alpha} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. As to $B_{i_2} : BT' \to BT'$, we have

$$
\begin{align*}
B_{i_2}(\alpha_2) &= \beta_3, & B_{i_2}(\alpha_3) &= \beta_2, \\
B_{i_2}(\alpha_4) &= \beta_1, & B_{i_2}(\tilde{\alpha}) &= \beta_0,
\end{align*}
$$

and so

$$
B_{i_2}(\alpha_1) = -\frac{1}{2} \beta_0 - \beta_1 - 2\beta_2 - \frac{3}{2} \beta_3.
$$

The dominant root with respect to the root system of $PSp(4)$ is $\tilde{\gamma} = 2\gamma_1 + 2\gamma_2 + 2\gamma_3 + \gamma_4$. As to $B_{i_4} : BT' \to BT'$, we have

$$
\begin{align*}
B_{i_4}(\gamma_2) &= \beta_3, & B_{i_4}(\gamma_3) &= \beta_2, \\
B_{i_4}(\gamma_4) &= \beta_1, & B_{i_4}(\tilde{\gamma}) &= \beta_0,
\end{align*}
$$

and so

$$
B_{i_4}(\gamma_1) = -\frac{1}{2} \beta_0 - \beta_1 - \beta_2 - \frac{1}{2} \beta_3.
$$

As to $B_{i_5} : BT' \to BT$, we have
\[ Bi_6(\delta_1) = Bi_6(\delta_6) = \alpha_4, \quad Bi_6(\delta_3) = \alpha_1, \]
\[ Bi_6(\delta_3) = Bi_6(\delta_6) = \alpha_3, \quad Bi_6(\delta_4) = \alpha_2. \]

Using these equalities and \( i_6 \mathcal{I} = i_6 \mathcal{I}_2 \), as to \( Bi_6 : BT' \to BT \), we conclude that
\[ Bi_6(\delta_1) = Bi_6(\delta_6) = \gamma_2, \]
\[ Bi_6(\delta_3) = Bi_6(\delta_4) = \gamma_3, \]
\[ Bi_6(\delta_4) = \gamma_4 \]
and so
\[ Bi_6(\delta_2) = \gamma_1 - \gamma_3 - \gamma_4. \]

From the above, it follows that
\[ Bi_2(\omega_1) = -2\phi_0, \quad Bi_4(\chi_1) = -\phi_0, \]
\[ Bi_2(\omega_2) = \phi_3 - 3\phi_0, \quad Bi_4(\chi_2) = \phi_1 - \phi_0, \]
\[ Bi_3(\omega_3) = \phi_2 - 2\phi_0, \quad Bi_4(\chi_3) = \phi_2 - \phi_0, \]
\[ Bi_4(\omega_4) = \phi_1 - \phi_0, \quad Bi_4(\chi_4) = \phi_3 - \phi_0. \]

Furthermore,
\[ Bi_6(\phi_1) = Bi_6(\phi_6) = \chi_2, \quad Bi_6(\phi_2) = 2\chi_1, \]
\[ Bi_6(\phi_3) = Bi_6(\phi_3) = \chi_1 + \chi_3, \quad Bi_6(\phi_4) = 2\chi_1 + \chi_4. \]

This result is restated in terms of representations. In fact, \( \phi_j \) of (1.1) is defined as the irreducible representation with highest weight \( \phi_j \), and \( \chi_j \) of (1.2) is just the irreducible representation with highest weight \( \chi_j \). Using (2.2), we see that \( i_6 : R(E_6) \to R(PSp(4)) \) satisfies
\[ i_6(\phi_1) = \lambda^2 \chi_1 - 1 \quad \text{and} \quad i_6(\phi_2) = \lambda^4 \chi_1 + \chi_1^2 - 2\lambda^2 \chi_1. \]

Then (1.4) follows from this, (1.1) and (1.3). (For details, see [6, II, §5].)

By [4, p. 234],
\[ H(BT'; \mathbb{Z}) = \mathbb{Z}[\omega_1, \omega_2, \omega_3, \omega_4] = \mathbb{Z}[t, y_1, y_2, y_3] \]
where \( \omega_i \in H^2(BT'; \mathbb{Z}) \) and
\[ t = \omega_1, \]
\[ y_1 = \omega_2 - \omega_3, \]
\[ y_2 = \omega_3 - \omega_4, \]
\[ y_3 = \omega_4. \]
On the other hand,

\[ H^* (B \widetilde{T}^*; \mathbb{Z}) = \mathbb{Z}[\chi_1, \chi_2, \chi_3, \chi_4] = \mathbb{Z}[t_1', t_2', \ldots, t_4'] \]

where \( \chi_i \in H^2(B \widetilde{T}^*; \mathbb{Z}) \) and

\[
\begin{align*}
t_1' &= \chi_1, \\
t_i' &= -\chi_{i-1} + \chi_i, \quad \text{for } i = 2, 3, 4.
\end{align*}
\]

(Note that \( \{\pm t_i' \mid i = 1, 2, 3, 4\} \) is the set of weights of \( \chi_1 \).) For \( i = 1, 2, 3, 4 \) let

\[ p_i = \sigma_i(t_1^2, t_2^2, t_3^2, t_4^2), \]

the \( i \)-th elementary symmetric function in the indicated variables. As is well known, the map \( B \widetilde{T}^* \to BSp(4) \) coming from the inclusion \( \widetilde{T}^* \to Sp(4) \) induces the following isomorphism

\[ H^* (BSp(4); \mathbb{Z}) \cong H^* (B \widetilde{T}^*; \mathbb{Z})^w_{w(Sp(4))} = \mathbb{Z}[p_1, p_2, p_3, p_4] \]

where the middle notation stands for the subalgebra of \( H (B \widetilde{T}^*; \mathbb{Z}) \) consisting of invariants under the action of the Weyl group \( W(Sp(4)) \). We may identify \( H^2(BT'; \mathbb{Q}) \) with \( H^2(B \widetilde{T}'; \mathbb{Q}) \). Since \( W(PSp(4)) = W(Sp(4)) \), we have

\[ H^* (BT'; \mathbb{Q})^w_{w(\tilde{P}Sp(4))} = \mathbb{Q}[p_1, p_2, p_3, p_4]. \]

Next, by [7, p. 266],

\[ H^* (BT; \mathbb{Z}) = \mathbb{Z}[\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6] = \mathbb{Z}[t_1, \ldots, t_6, x]/(t_1 + \cdots + t_6 - 3x) \]

where \( \varphi_j \in H^2(BT; \mathbb{Z}) \) and

\[
\begin{align*}
t_6 &= \varphi_6, \\
t_5 &= \varphi_5 - \varphi_6, \\
t_4 &= \varphi_4 - \varphi_5, \\
t_3 &= \varphi_2 + \varphi_3 - \varphi_4, \\
t_2 &= \varphi_1 + \varphi_2 - \varphi_3, \\
t_1 &= -\varphi_1 + \varphi_2, \\
x &= \varphi_2.
\end{align*}
\]

If we put

\[ x_j = 2t_j - x \quad \text{for } j = 1, 2, \ldots, 6, \]

the set

\[ S = \{x_i + x_j, x - x_i, -x - x_i \mid 1 \leq i < j \leq 6\} \]

is invariant under the action of \( W(E_6) \) (see [7, §4(B) and §5(B)]). For \( n \geq 1 \) let
\[ I_n = \sum_{y \in S} y^n \in H^{2n}(BT; \mathbb{Q})^{W(E_6)} \]

By [7, Lemma 5.2],
\[ H^1(BT; \mathbb{Q})^{W(E_6)} = \mathbb{Q}[I_2, I_5, I_6, I_8, I_9, I_{12}]. \]

Consider the set
\[ S' = \{ \pm I'_i \pm I'_j \mid 1 \leq i < j \leq 4 \}. \]

Then, by (2.2),
\[ \{ Bi'_i(y) \mid y \in S\} = \{ 2y' \mid y' \in S' \}. \]

For \( n \geq 1 \) let
\[ I'_n = \sum_{v \in S'} y'^v \in H^{2n}(BT'; \mathbb{Q}). \]

By [7, §5(A)], each \( I'_n \) is written as a polynomial of the \( p_i \), and the ideal generated by \( I'_n \)'s is given by
\[ (I'_n \mid n \geq 1) = (I'_2, I'_6, I'_8, I'_2) \]
\[ = (p_1, p_2, 12p_4 + p_2^3, p_2^3). \]

Let \( (H^+(BT; \mathbb{Q})^{W(E_6)}) \) be the ideal in \( H^1(BT'; \mathbb{Q})^{W(PSp(4))} \) generated by the image of
\[ Bi'_6: H^+(BT; \mathbb{Q})^{W(E_6)} \to H^+(BT'; \mathbb{Q})^{W(PSp(4))}, \]
where \( H^+(X; \mathbb{Q}) = \sum_{q>0} H^q(X; \mathbb{Q}) \). By (2.3) and (2.4),
\[ H^+(BT'; \mathbb{Q})^{W(PSp(4))}/(H^+(BT; \mathbb{Q})^{W(E_6)}) \]
\[ = \mathbb{Q}[p_1, p_2, p_3, p_4]/(p_1, p_2, 12p_4 + p_2^3, p_2^3) \]
\[ = \mathbb{Q}[p_2]/(p_2^3). \]

§3. Some Observation in Cohomology

Let \( i:T \to E_6 \) and \( i':T' \to PSp(4) \) be the inclusions respectively. The following commutative diagram

\[
\begin{array}{ccccccc}
R(E_6) & \xrightarrow{\alpha} & K^0(BE_6) & \xrightarrow{i_h} & H^1(BE_6; \mathbb{Q}) & \xrightarrow{\delta} & H^1(BT; \mathbb{Q})^{W(E_6)} \\
i_i & \downarrow & Bi_6 & \downarrow Bi'_6 & \downarrow Bi'_6 & \downarrow Bi'_6 \\
R(PSp(4)) & \xrightarrow{\alpha} & K^0(BPSp(4)) & \xrightarrow{i_h} & H^1(BPSp(4); \mathbb{Q}) & \xrightarrow{\delta} & H^1(BT'; \mathbb{Q})^{W(PSp(4))} \\
\downarrow i'_i & \downarrow i'_i & \downarrow i'_i & \downarrow i'_i \\
K^0(EI) & \xrightarrow{i_h} & H^1(EI; \mathbb{Q})
\end{array}
\]

yields a commutative square.
Note that the vertical homomorphisms are injective. We need to describe the image of the right vertical homomorphism \( j'_6 \) in terms of the generator \( e_6 \in H^8(El; \mathbb{Z}) \).

By (2.1) and definitions in §2, the following relations hold in \( H^2(BT'; \mathbb{Q}) \):

\[
t'_i = \frac{1}{2} t \quad \text{and} \quad t'_i = \frac{1}{2} t + y_{5-i} \quad \text{for} \quad i = 2, 3, 4.
\]

As in [4, p. 235], put

\[
z_i = (t - y_i)y_i \quad \text{for} \quad i = 1, 2, 3
\]

and let \( q_i = \sigma_i(z_1, z_2, z_3) \). Then, by [4, p. 236],

\[
H^*(BC; \mathbb{Z}) \cong H^*(BT'; \mathbb{Z})^{w(C)} = \mathbb{Z}[t, q_1, q_2, q_3].
\]

By definition,

\[
\sum_{i=0}^{4} p_i = \prod_{i=1}^{4} (1 + t'^2)
\]

\[
= (1 + \frac{1}{4} t'^2) \prod_{i=1}^{3} (1 + \frac{1}{4} t'^2 - t_i y_i + y_i^2)
\]

\[
= z \prod_{i=1}^{3} (z - z_i) \quad \text{where} \quad z = \frac{1}{4} t'^2 + 1
\]

\[
= z(z^3 - q_1 z^2 + q_2 z - q_3).
\]

Therefore

\[
(3.2) \quad p_1 = -q_1 + t^2 \quad \text{and} \quad p_2 = q_2 - \frac{3}{4} q_1 t^2 + \frac{3}{8} t^4 \quad \text{in} \quad H^*(BT'; \mathbb{Q}).
\]

It is known that

\[
H^*(E_6/F_4; \mathbb{Z}) = \Lambda_2(e_9, e_7)
\]

where \( e_i \in H^i(E_6/F_4; \mathbb{Z}) \). As in [3, §3],

\[
H^*(F_4/D; \mathbb{Z}) = \mathbb{Z}[\chi, \beta_{14} f_8, f_{12}]/(2 \chi, f_4 \chi, \chi^3, f_4^3 - 12 f_4 f_8 + 8 f_{12}, f_4 f_{12} - 3 f_8^2, f_8^3 - f_{12}^2)
\]

where \( \chi \in H^3(F_4/D; \mathbb{Z}) \) and \( f_i \in H^i(F_4/D; \mathbb{Z}) \). By [4, Theorem 4.4],

\[
H^*(F_4/C; \mathbb{Z}) = \mathbb{Z}[t, u, v, w]/(t^3 - 2u, u^2 - 3t^2 v + 2w, 3v^2 - t^2 w, v^3 - w^2)
\]

where \( t \in H^3(F_4/C; \mathbb{Z}) \), \( u \in H^6(F_4/C; \mathbb{Z}) \), \( v \in H^8(F_4/C; \mathbb{Z}) \), \( w \in H^{12}(F_4/C; \mathbb{Z}) \) and then \( j_3: H^*(BC; \mathbb{Z}) \to H^*(F_4/C; \mathbb{Z}) \) satisfies

\[
(3.3) \quad j_3(t) = t, \quad j_3(q_1) = t^2, \quad j_3(q_2) = 3v \quad \text{and} \quad j_3(q_3) = w.
\]
Moreover, by [3, Proposition 1],
\[ H^\ast(E_6/D;\mathbb{Z}) \cong H^\ast(F_4/D;\mathbb{Z}) \otimes H^\ast(E_6/F_4;\mathbb{Z}) \]

Consider the commutative diagram
\[
\begin{array}{cccc}
F_4/C & \xrightarrow{\pi_4^i} & F_4/D & \xrightarrow{\pi_4^e} & E_6/D & \xrightarrow{\pi_4^i} & E_6 \\
j_i & & j_i & & j_i & & j_i \\
BC & \xrightarrow{B_i} & BC
\end{array}
\]

Then
\[
\pi^i_4 j^i_8 \pi^i_4 (e_6) = \pi^i_4 (-8f_8 + f_2^2) = -8v + t^4 \in H^8(F_4/C;\mathbb{Z})
\]
(see [3, §3]). Therefore
\[
\pi^i_4 j^i_8 \pi^i_4 (p_2) = j_5 B_{i_8} (p_2)
\]
\[
= j_3(q_2 - \frac{3}{4}q_1t^2 + \frac{3}{8}t^4) \quad \text{by (3.2)}
\]
\[
= 3v - \frac{3}{8}t^4 \quad \text{by (3.3)}
\]
\[
= -\frac{3}{8}(-8v + t^4)
\]
in \( H^8(F_4/C;\mathbb{Q}) \). Thus we have

\textbf{Lemma 2.} \( j_6 : H^\ast(BT';\mathbb{Q})^{W(PSp(4))}/(H^+(BT';\mathbb{Q})^{W(E_6)}) \to H^\ast(El;\mathbb{Q}) \) of (3.1) is given by

\[
j_6(p_2) = -\frac{3}{8}e_8.
\]

\textbf{§4. Proof of Theorem 1}

There exist elements \( x_j \in H^\ast(E_6;\mathbb{Z}) \) for \( j = 3, 9, 11, 15, 17, 23 \) such that \( \langle x_3x_9x_{11}x_{15}x_{17}x_{23}, [E_6] \rangle = 1 \) up to sign, where \([E_6]\) is the fundamental homology class, and

\[ H^\ast(E_6;\mathbb{Q}) = \Lambda_0(x_3, x_9, x_{11}, x_{15}, x_{17}, x_{23}), \]
where each \( x_j \in H^\ast(E_6;\mathbb{Q}) \) is primitive.

\textbf{Lemma 3.} \( \pi_6 : E_6 \to El \) be the projection. Then \( \pi_6 : H^\ast(El;\mathbb{Z}) \to H^\ast(E_6;\mathbb{Z}) \) satisfies

\[
\pi_6(e_9) = 2x_9 \text{ and } \pi_6(e_{17}) = 2x_{17}.
\]

\textit{Proof.} Let \( p \) be a prime and consider the Serre spectral sequence for the mod \( p \) cohomology of the fibration
If $p \geq 5$,

$$H'(E_6; \mathbb{Z}/(p)) = \Lambda_{\mathbb{Z}/(p)}(x_5, x_9, x_{11}, x_{13}, x_{17}, x_{23}),$$

where $x_i \in H'(E_6; \mathbb{Z}/(p))$, and if $p = 3$,

$$H'(E_6; \mathbb{Z}/(3)) = \mathbb{Z}/(3)[x_8]/(x_8^3) \otimes \Lambda_{\mathbb{Z}/(3)}(x_3, x_7, x_9, x_{11}, x_{15}, x_{17}),$$

where $x_i \in H'(E_6; \mathbb{Z}/(3))$. If $p \geq 3$,

$$H'(BPSp(4); \mathbb{Z}/(p)) = \mathbb{Z}/(p)[y_4, y_8, y_{12}, y_{16}],$$

where $y_i \in H'(BPSp(4); \mathbb{Z}/(p))$. If $p \geq 3$,

$$H'(E_6; \mathbb{Z}/(p)) = \mathbb{Z}/(p)[e_9]/(e_9^3) \otimes \Lambda_{\mathbb{Z}/(p)}(e_9, e_{17}),$$

where $e_i \in H'(E_6; \mathbb{Z}/(p))$. By a routine spectral sequence argument, we see that if $p \geq 3$, $\pi_0 : H(E_6; \mathbb{Z}/(p)) \to H'(E_6; \mathbb{Z}/(p))$ satisfies

$$\pi_0(e_9) = x_9 \text{ and } \pi_0(e_{17}) = x_{17}.$$ 

Let $\pi'_4 : E_6/D \to EI$ be the projection. In view of [3, §5], $\pi'_4 : H'(EI; \mathbb{Z}) \to H'(E_6/D; \mathbb{Z})$ satisfies

$$\pi'_4(e_9) = 2e_9 \text{ and } \pi'_4(e_{17}) = 2e_{17}.$$ 

Consider the Serre spectral sequence for the mod 2 cohomology of the fibration

$$E_6 \xrightarrow{\pi_0} E_6/D \xrightarrow{\pi_0} BD.$$ 

If we denote by $\Delta_{\mathbb{Z}/(2)}$ a graded ring over $\mathbb{Z}/(2)$ with a simple system of generators,

$$H(E_6; \mathbb{Z}/(2)) = \Delta_{\mathbb{Z}/(2)}(x_3, x_5, x_6, x_9, x_{11}, x_{15}, x_{23}),$$

where $x_i \in H'(E_6; \mathbb{Z}/(2))$. By [4, Corollary 4.8],

$$H(BD; \mathbb{Z}/(2)) = \mathbb{Z}/(2)[u_2, u_3, u_4, u_5, u_6, u_{12}],$$

where $u_i \in H'(BD; \mathbb{Z}/(2))$. By [3, §3],

$$H(E_6/D; \mathbb{Z}/(2)) = \mathbb{Z}/(2)[e_2, e_3, e_4, e_8, e_{12}]/(e_2^3 + e_3^2, e_3^2, e_8^2 + e_2^2, e_{12}^2 + e_2^2, e_{12}^2) \otimes \Lambda_{\mathbb{Z}/(2)}(e_9, e_{17}),$$

where $e_i \in H'(E_6/D; \mathbb{Z}/(2))$. Similarly we see that $\pi_{10} : H(E_6/D; \mathbb{Z}/(2)) \to H'(E_6; \mathbb{Z}/(2))$ satisfies

$$\pi_{10}(e_9) = x_9 \text{ and } \pi_{10}(e_{17}) = x_{17}.$$ 

Since $\pi_6 = \pi'_4 \pi_{10}$, the lemma follows.
Let us compute the image of $\chi^2_1$ under the composite

$$R(PSp(4)) \overset{i'}{\rightarrow} R(T') \overset{\alpha}{\rightarrow} K^0(BT') \overset{ch}{\rightarrow} H^\ast(BT';\mathbb{Q}).$$

Since $\chi_1$ has weights $\pm t_i', i = 1, 2, 3, 4$ (see §2), $\chi^2_1$ has weights

$$\pm 2t_i', \pm t_i' \pm t_j', \pm t_i', 0, \ldots, 0$$

where $1 \leq i < j \leq 4$. For $n \geq 1$ let

$$s_n = \sum_{i=1}^{4} t_i^n \in H^{2n}(BT';\mathbb{Z}).$$

Then

$$ch\alpha i' (\chi^2_1) = \sum_{n \geq 0} \sum_{i=1}^{4} \frac{(2t_i')^n + (-2t_i')^n}{n!} + \frac{2I'_n}{n!} + 8.$$

Denoting by $ch^q$ the $2q$-dimensional component, we have

$$ch^{2n}\alpha i' (\chi^2_1) = \frac{2}{(2n)!} (I'_{2n} + 2^{2n} s_{2n}).$$

Now we use the expressions of $I'_{2n}$ and $s_{2n}$ in terms of the $p_i$ given in [7, p. 271]. For $n = 2$ we have

$$ch^2\alpha i' (\chi^2_1) = \frac{1}{12} (12p_2^2 + 16(-2p_2 + p_2^2)) \equiv -\frac{8}{3} p_2 \mod(p_1).$$

By this, (1.4) and (2.5), the upper horizontal homomorphism $ch\alpha$ of (3.1) is given by

$$ch\alpha (\chi^2_1) = 64 - \frac{8}{3} p_2.$$

Combining this, Lemma 2 and (3.1), we find that the coefficient of $e_8$ in $ch(\alpha(\chi^2_1))$ is 1. Similarly for $n = 4$ we have

$$ch^4\alpha i' (\chi^2_1) = \frac{1}{20160} (80(12p_4^2 + p_4^2) + 256(-4p_4 + 2p_4^2)) \mod(p_1, p_3)$$

$$\equiv \frac{1}{1260} (-4p_4 + 37p_4^2) \mod(p_1, p_3)$$

$$\equiv \frac{4}{135} p_2^2 \mod(p_1, p_3, 12p_4 + p_4^2).$$

Combining this, Lemma 2 and (3.1), we find that the coefficient of $e_8^2$ in $ch(\alpha(\chi^2_1))$ is 1/240. Since $4e'_6 = e_8^2$, that of $e'_6$ in $ch(\alpha(\chi^2_1))$ is 1/60. Thus we obtain the first equality of Theorem 1.

Let us define a map $\xi_\rho : EI \rightarrow E_6$ by
By the definition of $\beta(\lambda - \mu)$ and (1.5), it is easy to see that
\[ \xi_p(\beta(\varphi_i)) = \beta(\varphi_i - \varphi_6) \] and
\[ \xi_p(\beta(\lambda^2 \varphi_i)) = \beta(\lambda^2 \varphi_i - \lambda^2 \varphi_6). \]

By [2, §2], if $x \in H^j(E_6; \mathbb{Q})$ is primitive, then
\[ \pi_6 \xi_p(x) = x - \rho(x) \]
and
\[ \rho(x_j) = \begin{cases} x_j & \text{for } j = 3, 11, 15, 23 \\ -x_j & \text{for } j = 9, 17 \end{cases} \]
in $H^j(E_6; \mathbb{Z})$ (see also [8, (1.5)]). Therefore, by Lemma 3,
\[ \xi_p(x_j) = \begin{cases} 0 & \text{for } j = 3, 11, 15, 23 \\ e_j & \text{for } j = 9, 17 \end{cases} \]
in $H^j(E_6; \mathbb{Z})$. We quote from [8, Theorem 1] that
\[ \text{ch}(\beta(\varphi_i)) = 6x_3 + \frac{1}{2} x_9 + \frac{1}{20} x_{11} + \frac{1}{16} x_{15} + \frac{1}{480} x_{17} + \frac{1}{443520} x_{23}, \]
\[ \text{ch}(\beta(\lambda^2 \varphi_i)) = 150x_3 + \frac{11}{2} x_9 - \frac{1}{4} x_{11} - \frac{101}{168} x_{15} - \frac{229}{480} x_{17} - \frac{2021}{443520} x_{23}. \]
Then, by applying $\xi_p$ to these equalities, the second and third equalities of Theorem 1 follow. This completes the proof.

References