The Converse of Minlos’ Theorem

Dedicated to Professor Tsuyoshi Ando on the occasion of his sixtieth birthday

By

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Abstract

Let $\mathcal{M}$ be the class of barrelled locally convex Hausdorff space $E$ such that $E_b'$ satisfies the property $B$ in the sense of Pietsch. It is shown that if $E \in \mathcal{M}$ and if each continuous cylinder set measure on $E'$ is $\sigma(E',E)$-Radon, then $E$ is nuclear. There exists an example of non-nuclear Fréchet space $E$ such that each continuous Gaussian cylinder set measure on $E'$ is $\sigma(E',E)$-Radon. Let $q$ be $2 \leq q < \infty$. Suppose that $E \in \mathcal{M}$ and $E$ is a projective limit of Banach space $\{E_\alpha\}$ such that the dual $E_\alpha'$ is of cotype $q$ for every $\alpha$. Suppose also that each continuous Gaussian cylinder set measure on $E'$ is $\sigma(E',E)$-Radon. Then $E$ is nuclear.

§1. Introduction

Let $E$ be a nuclear locally convex Hausdorff space, then each continuous cylinder set measure on $E'$ is $\sigma(E',E)$-Radon (Minlos’ theorem, see Badrikian [2], Gelfand and Vilenkin [4], Minlos [11], Umemura [20] and Yamasaki [21]). We consider the converse problem. Let $E$ be a locally convex Hausdorff space. If each continuous cylinder set measure on $E'$ is $\sigma(E',E)$-Radon, then is $E$ nuclear? The partial answers are known as follows.

(1) If $E$ is a $\sigma$-Hilbert space or a Fréchet space, then the answer is affirmative (see Badrikian [2], Gelfand and Vilenkin [4], Minlos, [11], Mushtari [12], Umemura [20] and Yamasaki [21]).

(2) If $E$ is barrelled and if $E$ is a projective limit of $L^0$-embeddable Banach spaces, then the answer is affirmative (see Millington [10], Mushtari [12], Okazaki and Takahashi [14]).
In this paper, we shall extend the case (1) for more general locally convex spaces. We introduce a class $\mathcal{M}$ in Section 4. $\mathcal{M}$ is the class of all barrelled locally convex Hausdorff space $E$ such that the strong dual $E'_b$ satisfies the property $B$ in the sense of Pietsch (Pietsch [15] 1.5.5). The class $\mathcal{M}$ contain $LF$-spaces, barrelled $DF$-spaces and inductive limits of them. We prove the next theorem.

**Theorem.** Let $E \in \mathcal{M}$. If each continuous discrete 1-stable cylinder set measure on $E'$ is $\sigma(E', E)$-Radon, then $E$ is nuclear.

For the Gaussian cylinder set measures, the following result is well-known.

(3) Let $E$ be a $\sigma$-Hilbert space. If each continuous Gaussian cylinder set measure on $E'$ is $\sigma(E', E)$-Radon, then $E$ is nuclear (see Gelfand and Vilenkin [4], Minlos [11], Umemura [20] and Yamasaki [21]).

In general, we can not conclude that $E$ is nuclear even if each continuous Gaussian cylinder set measure on $E'$ is $\sigma(E', E)$-Radon. We give a counter example. In this case, we prove the next result.

**Theorem.** Let $2 \leq q < \infty$ be fixed and $E \in \mathcal{M}$. Suppose that $E$ is a projective limit of Banach spaces $\{E_{\alpha}\}$ such that the dual $E'_{\alpha}$ is of cotype $q$ for every $\alpha$. Suppose also that each continuous Gaussian cylinder set measure on $E'$ is $\sigma(E', E)$-Radon. Then $E$ is nuclear.

§2. Preliminaries

Let $E$ be a locally convex Hausdorff space and $E'$ be the topological dual of $E$. Denote by $E'_b$ (resp. $E'_s$) the dual with the strong dual topology $\beta(E', E)$ (resp. weak * topology $\sigma(E', E)$). The strong bidual of $E$ is denoted by $(E'_b)''$. Let $\mu$ be a cylinder set measure on $E'$. Then we say that $\mu$ is a continuous cylinder set measure if the characteristic functional

$$\mu^\wedge(x) = \int_{E'} e^{i(x, \omega)} d\mu(\omega), \quad x \in E,$$

is continuous on $E$.

The cylinder set measure $\mu$ on $E'$ is called a continuous discrete $p$-stable cylinder set measure on $E'$ if the characteristic functional $\mu^\wedge(x)$ is given by

$$\mu^\wedge(x) = \exp(-\|T(x)\|_p^p), \quad x \in E,$$

where $T : E \to \ell_p$ is a continuous linear operator and $0 < p \leq 2$. In the sequel, we consider only the cases $p = 1$ and 2. In the case where $p = 2$, $\mu$ is called a continuous Gaussian cylinder set measure. See Linde [7].
Let $F$, $G$ be normed spaces and $0 < q, r < \infty$. A linear operator $S: F' \to G'$ is called $(q,r)$-summing if for every $\{a_n\} \subset F'$ with $\sum_{n=1}^{\infty} \|S(a_n)\|_G < \infty$ for every $x \in F$, it holds that $\sum_{n=1}^{\infty} \|S(a_n)x\|_G < \infty$. A linear operator $T: F \to G$ is called $(q,r)$-summing if for every $\{x_n\} \subset F$ with $\sum_{n=1}^{\infty} \|x_n\|_F < \infty$ for every $a \in F'$, it holds that $\sum_{n=1}^{\infty} \|T(x_n)a\|_G < \infty$. In the case where $r = q$, $S$ and $T$ are called $r$-summing, see Pietsch [15], Schwartz [18] and Tomczak-Jaegermann [19].

Let $G$ be a Banach space and $2 \leq q < \infty$. $G$ is called of cotype $q$ if there exists $K > 0$ such that for every $n$ and every $z_1, z_2, \ldots, z_n \in G$, it holds that

$$\left( \sum_{n=1}^{\infty} \|z_n\|_G^q \right)^{1/q} \leq K \int_{\Omega} \left( \sum_{n=1}^{\infty} g_n(\omega)z_n \right)_G d\mathcal{P}(\omega),$$

where $\{g_n\}$ is a sequence of independent identically distributed Gaussian random variables on a probability space $(\Omega, \mathcal{P})$ with the characteristic functional $e^{-|t|^2}$, see Linde [7], Maurey and Pisier [9], Tomczak-Jaegermann [19].

Let $E$ be a locally convex Hausdorff space. For a closed absolutely convex neighborhood $U$ of $0$, we set $N(U) = \{x \in E: p_U(x) = 0\}$ where $p_U(x) = \inf\{t > 0: x \in tU\}$. Denote by $x(U)$ the equivalence class corresponding to $x \in E$ in the quotient space $E(U) = E/N(U)$. $E(U)$ is a normed space with norm $p[x(U)] = p_U(x)$ for $x \in E$.

For a closed absolutely convex bounded subset $A$ of $E$, we set $E(A) = \{x \in E: x \in tA \text{ for some } t > 0\}$. $E(A)$ is a linear subspace of $E$. We put the norm on $E(A)$ by $p_A(x) = \inf\{t > 0: x \in tA\}$ for $x \in E(A)$.

For a neighborhood $U$ of $0$ in $E$, the polar $U^* = \{a \in E^*: \langle a, x \rangle \leq 1 \text{ for every } x \in U\}$ is weakly compact absolutely convex subset of $E^*$. The normed space $E'(U^*)$ is a Banach space and $E(U)' = E'(U^*)$ by the duality $\langle x(U), a \rangle = \langle x, a \rangle$.

For two zero neighborhoods $U, V$ with $V \subset U$, we define a canonical mapping $E(V,U): E(V) \to E(U)$ by associating $x(U)$ with $x(V)$.

For two closed absolutely convex bounded subsets $A$ and $B$ with $A \subset B$, it holds that $E(A) \subset E(B)$ and the canonical mapping $E(A,B): E(A) \to E(B)$ is defined by $E(A,B)(x) = x$ for $x \in E(A)$.

A locally convex Hausdorff space $E$ is called nuclear if it contains a fundamental system $U_E(E)$ of zero neighborhoods which has the following equivalent properties (see Pietsch [15], 4.1.2):

1. For each $U \in U_E(E)$ there exists $V \in U_E(E)$ with $V \subset U$ such that the canonical mapping $E(V,U): E(V) \to E(U)$ is 2-summing.
2. For each $U \in U_E(E)$ there exists $V \in U_E(E)$ with $V \subset U$ such that the canonical mapping $E'(V^*, U^*): E'(V^*) \to E'(U^*)$ is 2-summing.
A locally convex Hausdorff space $E$ is called dual nuclear if the strong dual $E'$ is nuclear. For other basic notions of locally convex spaces, we refer to Schaefer [17].

§3. Summability and Dual Nuclearity

Let $E$ be a locally convex Hausdorff space and $1 \leq p < \infty$. A sequence $(x_n) \subset E$ is called weakly $p$-summable if for every neighborhood $U$ of 0, it holds that

$$\varepsilon^p_U((x_n)) = \sup \{(\sum_{n=1}^{\infty} |(x_n, a)|^p)^{1/p} : a \in U^*\} < \infty$$

Denote by $l^p[E]$ the linear space of all weakly $p$-summable sequences. The topology of $l^p[E]$ given by the seminorms $\varepsilon^p_U$, $U \in U_F(E)$, is called the $\varepsilon$-topology where $U_F(E)$ is a fundamental system of zero neighborhoods of $E$.

A sequence $(x_n) \subset E$ is called absolutely $p$-summable if for every neighborhood $U$ of 0, it holds that

$$\pi^p_U((x_n)) = (\sum_{n=1}^{\infty} p_U(x_n)^p)^{1/p} < \infty.$$ 

Denote by $l^p[E]$ the linear space of all absolutely $p$-summable sequences. The topology of $l^p[E]$ given by the seminorms $\pi^p_U$, $U \in U_F(E)$, is called the $\pi$-topology, where $U_F(E)$ is a fundamental system of zero neighborhoods of $E$. It holds that $(l^p[E], \pi^p) \subset (l^p[E], \varepsilon^p)$, where the inclusion is a continuous injection.

A sequence $(x_n) \subset E$ is called totally $p$-summable if there exists a closed absolutely convex bounded subset $B$ such that $\sum_{n=1}^{\infty} p_B(x_n)^p < \infty$. Denote by $l^p\langle E \rangle$ the linear space of all totally $p$-summable sequences. It is clear that $l^p\langle E \rangle \subset l^p[E]$.

It is called that $E$ has property $B$ if for each bounded subset $\mathcal{B} \subset l^1[E]$ there exists a bounded set $B \subset E$ such that $\sum_{n=1}^{\infty} p_B(x_n) \leq 1$ for every $(x_n) \in \mathcal{B}$, see Pietsch [15], 1.5.5. If $E$ has property $B$, then it holds that $l^1[E] \subset l^1\langle E \rangle$.

The nuclearity of the strong dual $E'_b$ is characterized by the above summabilities as follows.

**Lemma 1 (Pietsch [15] Theorem 4.2.11).** If $E$ has property $B$ and $l^1[E] = l^1\langle E \rangle$, then $E'_b$ is nuclear.

It is known that the metrizable space or the dual metrizable space has property $B$ (Pietsch [15] Theorem 1.5.8). We prove that the property $B$ is retained by the projective or inductive limit operation.
Proposition 1. (1) If each $E_n$ has property $B$, then the projective limit $\lim E_n$ has property $B$.

(2) Let $E = \lim E_n$ be the strict inductive limit. Suppose that each $E_n$ has property $B$ and every bounded set $B$ of $E$ is contained and bounded in $E_k$ for some $k$ ($k$ depends on $B$). Then $E$ has property $B$.

Proof. (1) Let $\mathcal{B}$ be bounded in $l^1\{E\}$. Let $\pi_\alpha: E \to E_n$ be the canonical mapping. Then $\pi_\alpha(\mathcal{B}) = \{(\pi_\alpha x)_n : (x_n) \in \mathcal{B}\}$ is bounded in $l^1\{E_n\}$ for every $n$. By the property $B$ of $E_n$, there exists a bounded set $B_n$ in $E_n$ such that $\sup \{\sum_{n=1}^m p_{B_n}(\pi_\alpha x) : (x_n) \in \mathcal{B}\} \leq 1$ for every $n$. We set $B = \{x \in E : \sum_{n=1}^m 2^{-n} p_{B_n} \cdot (\pi_n x) \leq 1\}$. Then $B$ is bounded in $E$ and it holds that $p_B(x) = \sum_{n=1}^m 2^{-n} p_{B_n}(\pi_n x)$. So we have $\sum_{n=1}^m p_B(x_n) \leq \sum_{n=1}^m 2^{-n} < \infty$ for every $(x_n) \in \mathcal{B}$.

(2) Let $\mathcal{B} \subset l^1\{E\}$ be bounded. Then the subset $C = \{x_i^j : i = 1, 2, \ldots, (x_j) \in \mathcal{B}\}$ is bounded in $E$. There exists $k$ so that $C \subset E_k$ and $C$ is bounded in $E_k$. Since $E$ induces the topology on $E_k$, $\mathcal{B}$ is contained in $l^1\{E_k\}$ and bounded in $l^1\{E_k\}$. Hence there exists a bounded subset $B$ in $E_k$ such that $\sum_{n=1}^m p_B(x_n) \leq 1$ for every $(x_n) \in \mathcal{B}$. This proves (2).

We investigate the property $B$ of the strong dual $E'_b$.

Lemma 2. Let $E = \lim E_n$ be the inductive limit of locally convex spaces. If $E$ is barrelled and if each $(E_n)'_b$ has property $B$, then $E'_b$ has property $B$.

Proof. Let $\mathcal{B} \subset l^1\{E'_b\}$ be bounded, that is, $\sup \{\sum_{n=1}^m p_{B_n}(a) : (a) \in \mathcal{B}\} < \infty$ for every bounded subset $B$ in $E$. Let $\pi_\alpha: E' \to E'_b$ be the canonical mapping. For every $n$, $\{(\pi_\alpha a) : (a) \in \mathcal{B}\}$ is bounded in $l^1\{(E_n)'_b\}$ since each bounded set in $E_n$ is also bounded in $E$. For every $n$, take a closed absolutely convex bounded set $K_n \subset (E_n)'_b$ such that $\sum_{n=1}^m p_{K_n}(\pi_\alpha a) \leq 1$ for every $(a) \in \mathcal{B}$. We set $K = \{a \in E' : \sum_{n=1}^m 2^{-n} p_{K_n}(a) \leq 1\}$. Then $K$ is bounded in $E'_b$ since $\pi_\alpha(K)$ is bounded in $(E_n)'_b$ for every $n$ and $E$ is barrelled (in fact, $K^*$ absorbs each point in $E$). We have $p_K(a) = \sum_{n=1}^m 2^{-n} p_{K_n}(a)$ for every $a \in E'(K)$. For each $(a) \in \mathcal{B}$, we obtain $\sum_{n=1}^m p_K(a) = \sum_{n=1}^m 2^{-n} (\sum_{n=1}^m p_{K_n}(\pi_\alpha a)) \leq \sum_{n=1}^m 2^{-n} < \infty$. Thus $E'_b$ has property $B$.

Proposition 2. Let $E$ be either

(1) metrizable,
(2) dual metrizable,
(3) LF-space,
(4) dual LF-space, or
(5) \( E = \lim_{\to} E_n \) and \( E \) is barrelled, where \( E_n \) is one of (1), (2), (3) and (4) above. Then \( E'_b \) has property B.

The next Lemma shall be used in Section 4, Theorem 2.

**Lemma 3.** Let \( q \) be \( 1 \leq q < \infty \). If \( E \) has property B, then for each bounded subset \( \mathscr{B} \subset l^q \{ \mathcal{E} \} \) there exists a bounded set \( B \subset E \) such that \( \sum_{n=1}^{\infty} p_B (x_n)^q \leq 1 \) for every \( (x_n) \in \mathscr{B} \).

*Proof.* Let \( s \) be \( 1/q + 1/s = 1 \). Then the family \( \mathscr{A} = \{ (t, x); (x, t) \in \mathscr{B} \text{ and } \| (t, x) \| \leq 1 \} \) is bounded in \( l^q \{ \mathcal{E} \} \) since it holds that for every zero neighborhood \( U \)
\[
\sum_{n=1}^{\infty} p_U (t, x_n) \leq \left( \sum_{n=1}^{\infty} \| t \|^q \right)^{1/q} \leq \left( \sum_{n=1}^{\infty} \| t \|^q \right)^{1/q} \text{ and since } \mathscr{B} \text{ bounded in } l^q \{ \mathcal{E} \}.
\]
By property B, there exists a bounded set \( B \) of \( E \) such that for every \( (t, x) \in \mathscr{A} \) it holds \( \sum_{n=1}^{\infty} p_B (t, x_n) \leq \sum_{n=1}^{\infty} \| t \|_q p_B (x_n) \leq 1 \). Thus for every \( (u, x) \in \mathscr{B} \) with \( \| u \|_q \leq 1 \), we have \( | \sum_{i} u_i p_B (x_i) | \leq \sum_{i} p_B (| u_i | x_i) \leq 1 \). By the duality of \( l^q \) and \( l^q \), it follows that \( \sum_{n=1}^{\infty} p_B (x_n)^q \leq 1 \) for every \( (x_n) \in \mathscr{B} \), which shows the assertion.

§4. Converse of Minlos’ Theorem

**Lemma 4.** Let \( F, G \) be Banach spaces, \( \psi : G \rightarrow F \) be a continuous linear mapping and \( \psi' : F' \rightarrow G' \) be the adjoint of \( \psi \). Let \( (a_i) \subset F' \) be \( \sum_{n=1}^{\infty} \| (x, a) \| < \infty \) for every \( x \in F \) and \( \mu \) be a continuous discrete 1-stable cylinder set measure on \( F' \) with \( \mu (x) = e^{-\sum_{n=1}^{\infty} \| (x, a_n) \|} \). Suppose that the image \( \psi (\mu) \) is \( \sigma (G', G) \)-Radon on \( G' \). Then it holds that \( \sum_{n=1}^{\infty} \| \psi '(a_n) \|_G < \infty \).

*Proof.* We follow Linde [7], Cor. 6.5.2 and Maurey [8], Prop.2b). For every \( N \) let \( \lambda_N, \tau_N \) be the cylinder set measures on \( G' \) with
\[
\begin{align*}
\lambda_N (z) &= e^{-\sum_{n=1}^{N} \| (z, \psi '(a_n)) \|} \\
\tau_N (z) &= e^{-\sum_{n=N+1}^{\infty} \| (z, \psi '(a_n)) \|}, \quad z \in G.
\end{align*}
\]
Then we have \( \lambda_N \ast \tau_N = \psi (\mu) \) as cylinder set measures, where \( \ast \) denotes the convolution. Since \( \psi (\mu) \) is \( \sigma (G', G) \)-Radon, \( \lambda_N \) and \( \tau_N \) are also \( \sigma (G', G) \)-Radon, see Okazaki [13], Lemma 1.

For \( 0 < q < 1 \) it holds that
\[
\int_{G'} \|a\|_{G'}^q \, d\lambda_N(a) = \int_{G'} \|a\|_{G'}^q \, d\lambda_N(a) \, d\tau_N(b)
\]
\[
\leq 2^{-q} \int_{G'} \int_{G'} (\|a + b\|_{G'}^q + \|a - b\|_{G'}^q) \, d\lambda_N(a) \, d\tau_N(b)
\]
\[
\leq 2^{1-q} \int_{G'} \|a\|_{G'}^q \, d\psi'(\mu)(a),
\]
since \(\|2a\|_{G'}^q \leq \|a + b\|_{G'}^q + \|a - b\|_{G'}^q\) and \(\tau_N\) is symmetric, see Hoffmann-Jørgensen [4], Theorem 2.6.

Let \(\{f_n(\omega)\}\) be a sequence of independent identically distributed symmetric 1-stable random variables on a probability space \((\Omega, P)\) with the characteristic functional \(e^{-\imath t t}\). Let \(q\) be fixed such that \(0 < q < 1\). For every \(n\), we set
\[
S_N(\omega) = \sum_{n=1}^{N} \psi'(a_n) f_n(\omega).
\]

\(S_N\) is a random variable which values in a finite-dimensional subspace of \(G'\) and the distribution of \(S_N\) is \(\lambda_N\). If we set
\[
H_N(\omega) = \max_{1 \leq n \leq N} \|\psi'(a_n) f_n(\omega)\|_{G'},
\]
then by Kwapien [6], Remark 1, it follows that
\[
\int_{\Omega} H_N(\omega) \, dP(\omega) \leq 8 \int_{\Omega} S_N(\omega) \|\psi'(\mu)\|_{G'} \, dP(\omega)
\]
\[
= 8 \int_{G'} \|a\|_{G'}^q \, d\lambda_N(a).
\]

Consequently, we have
\[
\int_{\Omega} H_N(\omega) \, dP(\omega) \leq 8 \, 2^{1-q} \int_{G'} \|a\|_{G'}^q \, d\psi'(\mu)(a).
\]

Since \(\psi'(\mu)\) is a 1-stable \(\sigma(G', G')\)-Radon measure on \(G'\) and \(0 < q < 1\), we have
\[
L = \int_{G'} \|a\|_{G'}^q \, d\psi'(\mu)(a) < \infty,
\]
see de Acosta [1], Linde [7], Cor. 6.7.5. Thus we have
\[
\int_{\Omega} \max_{1 \leq n \leq N} \|\psi'(a_n) f_n(\omega)\|_{G'} \, dP(\omega) \leq 8 \, 2^{1-q} L < \infty
\]
for every \(N = 1, 2, \cdots\). Letting \(N \to \infty\), we have
\[
\int_{\Omega} \sup_n \| \psi'(a_n) f_n(\omega) \|_{G'} dP(\omega) < \infty
\]

Hence there exists \( R > 0 \) such that
\[
P(\omega : \sup_n \| \psi'(a_n) f_n(\omega) \|_{G'} \leq R) = \prod_{n=1}^{\infty} \{1 - P(\omega : \| f_n(\omega) \|_{G'} > R / \| \psi'(a_n) \|_{G'})\} > 0,
\]
where we have used the independence of \( \{f_n(\omega)\} \). This implies that
\[
\sum_{n=1}^{\infty} P(\omega : \| f_n(\omega) \|_{G'} > R / \| \psi'(a_n) \|_{G'}) < \infty.
\]

We remark that for every \( n \),
\[
\int_{\Omega} \| \psi'(a_n) f_n(\omega) \|_{G'} dP(\omega) = \| \psi'(a_n) \|_{G'}^{\alpha} \int_{\Omega} \| f_n(\omega) \|_{G'} \alpha dP(\omega) \leq 8 2^{\alpha - \alpha} L_n,
\]
that is, \( \sup_n \| \psi'(a_n) \|_{G'} < \infty \). Furthermore, it is known that
\[
P(\omega : \| f_n(\omega) \|_{G'} > t) \sim t^{-1} \quad \text{as} \quad t \to \infty,
\]
so we obtain for sufficiently large \( R \)
\[
P(\omega : \| f_n(\omega) \|_{G'} > R / \| \psi'(a_n) \|_{G'}) \sim \| \psi'(a_n) \|_{G'} / R.
\]

Hence it follows that \( \sum_{n=1}^{\infty} \| \psi'(a_n) \|_{G'} < \infty \).

**Remark 1.** If \( \psi'(\mu) \) is Radon with respect to the dual norm of \( G' \), then Lemma 4 is a direct consequence of the fact “every Banach space is of cotype 1-stable”, see Linde [7], Cor. 6.5.2 and Maurey [8], Prop. 2 b).

**Lemma 5.** Let \( E \) be a barrelled locally convex Hausdorff space. Suppose that each continuous discrete 1-stable cylinder set measure on \( E' \) is \( \sigma(E',E) \)-Radon. Then it holds that \( l^1(E'_\alpha) = l^1(\langle E'_\alpha \rangle) \).

**Proof.** Let \( (a_i) \in l^1[\langle E'_\alpha \rangle] \), that is, \( \sum_{i=1}^{\infty} \langle x, a_i \rangle < \infty \) for every \( x \in E \). Since the semi-norm \( |x| = \sum_{i=1}^{\infty} |\langle x, a_i \rangle| \) is lower semicontinuous on \( E \), \( |x| \) is continuous by the barrelledness. The continuous discrete 1-stable cylinder set measure \( \mu \) on \( E' \) with \( \mu^+(x) = \exp(-\sum_{i=1}^{\infty} |\langle x, a_i \rangle|), x \in E, \) is \( \sigma(E',E) \)-Radon. We can take a \( \sigma(E',E) \)-compact set \( K \subset E'_\alpha \) of the form \( K = U, U \in U_f(E) \), satisfying that \( \mu(K) > 0 \) and \( |\mu^+(x) - 1| < 1/2 \) for \( x \in U \) by the barrelledness and the continuity of \( \mu^+(x) \). Consider the Banach space \( E'(K) = \bigcap nK \) with the unit ball \( K \). By the 0-1 law of a stable measure, it follows that \( \mu(E'(K)) = 1 \), see Dudley and Kanter [3]. Thus \( \mu \)
is a $\sigma(E'(K),E(U))$-Radon measure. We claim that $(a_t) \subset E'(K)$. Take $\ell < \infty$ so that $|\exp(-|t|) - 1| < 1/2$ implies $|t| < \ell$. Hence for every $x \in U$, it follows that $|\langle x, a_t \rangle| < \ell$ and $a_t \in \ell U^* = \ell K$, that is, $(a_t) \subset \ell K$. By Lemma 4, we obtain
\[ \sum_{t=1}^{\infty} p_k(a_t) < \infty, \] which shows $(a_t) \in l^1(E'_b).

We introduce a class $\mathcal{M}$ of locally convex spaces as follows. $\mathcal{M}$ is the set of all barrelled locally convex Hausdorff space $E$ such that the strong dual $E'_b$ has property $B$. $\mathcal{M}$ contain LF-spaces and barrelled DF-spaces. $\mathcal{M}$ is closed under the operation taking a countable inductive limit (Proposition 2).

**Theorem 1.** Let $E \in \mathcal{M}$ and suppose that every continuous discrete 1-stable cylinder set measure on $E'$ is $\sigma(E',E)$-Radon. Then $E$ is nuclear.

**Proof.** By Lemma 5, we have $l^1[E'_b] = l^1[E'_b]$. By Lemma 1, it follows that $(E'_b)'_b$ is nuclear. Since $E$ is barrelled, the topology of $E$ is induced from $(E'_b)'_b$, which proves the Theorem.

In Theorem 1, we can not replace “1-stable” by “Gaussian” in general. We give an example later on.

**Lemma 6.** Let $2 \leq q < \infty$, $E$ be a Banach space and $G = E'_b$ be the dual Banach space. Suppose that $G$ is of cotype $q$. Let $(a_n) \subset G$ be $\sum_n |\langle x, a_n \rangle|^2 < \infty$ for every $x \in E$. If the continuous Gaussian cylinder set measure $\mu$ with $\mu^\land(x) = \exp(-\sum_n |\langle x, a_n \rangle|^2)$ is $\sigma(G,E)$-Radon, then it holds that $\sum_{n=1}^{\infty} |a_n|^q_G < \infty$.

**Proof.** For every $N$, let $\lambda_N$, $\tau_N$ be the cylinder set measures on $G$ with
\[ \lambda_N^\land(x) = \exp(-\sum_{n=1}^{N} |\langle x, a_n \rangle|^2), \]
\[ \tau_N^\land(x) = \exp(-\sum_{n=N+1}^{\infty} |\langle x, a_n \rangle|^2), x \in E. \]

Then we have $\lambda_N \ast \tau_N = \mu$. Since $G$ is of cotype $q$, there exists $K > 0$ such that
\[ \left[ \sum_{n=1}^{N} \|a_n\|^q_G \right]^{1/q} \leq K \int_Q \left[ \sum_{n=1}^{N} \|a_n g_n(\omega)\|_G \right] dP(\omega) \]
\[ = K \int_G |a|_G d\lambda_N(a) \]
for every $N$ by the manner same to Lemma 4. Since $\mu$ is Gaussian, this last integral is finite, which implies the assertion.

**Theorem 2.** Let $q$ be $2 \leq q < \infty$. Let $E$ be a locally convex Hausdorff space with a fundamental system $\{U_a\}$ of zero neighborhoods such that the dual $E(U_a)' = E'(U_a)$ is of cotype $q$. Suppose that $E \in \mathcal{A}$ and each continuous Gaussian cylinder set measure on $E'$ is $\sigma(E',E)$-Radon. Then $E$ is nuclear.

**Proof.** Firstly, we show that $l^2[E_b'] \subset l^q[E_b'] \subset l^q[E_b]$. Let $(a_i) \in l^2[E_b']$, that is, for every zero neighborhood $W$ of $E_b'$, $\sup \{\sum_{i=1}^{\infty} \langle x, a_i \rangle^2 : x \in W'\} < \infty$. Then

$$h(x) = \left( \sum_{i=1}^{\infty} |\langle x, a_i \rangle|^2 \right)^{1/2}$$

is continuous on $E$ since $E$ is barrelled and $h(x)$ is lower semicontinuous. Hence $\exp(-h(x)^2)$ determines continuous Gaussian cylinder set measure $\mu$ on $E'$ with $\mu^\alpha(x) = \exp(-h(x)^2)$ taking $T : E \to J_2$ be $T(x) = \langle (x, a_i) \rangle$. By the assumption, $\mu$ is $\sigma(E',E)$-Radon and so there exists $\alpha$ such that $\mu(E'(U_a)) = 1$ by the 0-1 law of a Gaussian measure, see the proof of Lemma 5. Since $E'_{U_a}$ is of cotype $q$ it follows that $\sum_{i=1}^{\infty} p_{U_a}(a_i)^q < \infty$ by Lemma 6.

Secondly, we show that each bounded set $B$ in $l^2[E_b']$ is bounded also in $l^q[E_b']$. For every zero neighborhood $W$ in $E_b'$, there exists $M_w > 0$ such that

$$\sup_{a_i, x, x'} \left\{ \sum_{i=1}^{\infty} \langle x, a_i \rangle^2 : x \in W' \right\} < M_w.$$

Suppose that $B$ is not bounded in $l^q[E_b']$, that is, there is a zero neighborhood $V$ in $E_b'$ such that $\sup \{\sum_{i=1}^{\infty} \langle x, a_i \rangle^q : (a_i) \in B \} = \infty$. For every $n$, take $N_n$ and $(a_i^n) \in B$ such that $\sum_{i=1}^{N_n} \langle x, a_i^n \rangle^q > 2^{nq}$. Remark that $\sup \{\sum_{i=1}^{N_n} \langle x, a_i^n \rangle^q : x \in V' \} \leq C_v^2 < \infty$ since $B$ is bounded in $l^2[E_b']$. Then we have for the sequence $\{1 \leq i \leq N_n, n = 1, 2, \ldots\}$ and for every $x \in V'$, $\sum_{i=1}^{N_n} \sum_{i=1}^{\infty} \langle x, 2^{-n} a_i^n \rangle^2 \leq \sum_{n=1}^{\infty} 2^{-2n} C_v^2 < \infty$. On the other hand, $\sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \langle x, 2^{-n} a_i^n \rangle^q \geq \sum_{n=1}^{\infty} 2^{-nq} 2^{nq} < \infty$, which contradicts to $l^2[E_b'] \subset l^q[E_b]$.

Thirdly, we prove that for every $\alpha$, there exists $\beta$ such that $E'(U_{\alpha}, U_{\beta}) : E'(U_{\alpha}) \to E'(U_{\beta})$ is $(q, 2)$-summing. For every $\alpha$, $A = U_{\alpha}$ is a bounded set in $E_b$. We set $B = \{(a_i) \in l^2[E_b] : \|a_i\|^2 = \sup \{\sum_{i=1}^{\infty} \langle x, a_i \rangle^2 : x \in W' \}$.
Since \( A \) is bounded in \( E'_b \), \( B \) is bounded in \( E'_b \) by the second step. By Lemma 3, there exists a bounded absolutely convex closed subset \( B \) in \( E'_b \) such that \( \sum_{i=1}^\infty p_{\beta}(a_i)^q \leq 1 \) for every \((a_i) \in B\). We can assume that \( B = U^\beta_\alpha \) for some \( \beta \) with \( U^\beta_\alpha \subset U_\alpha \) since \( E \) is barrelled. So we obtain \( (\sum_n p_{U^\beta_\alpha}(a_n)^q)^{1/q} \leq \sup(\langle \sum_n (x,a_n) \rangle^{1/2}; p_{U^\beta_\alpha}(x) \leq 1) \), which shows the assertion.

Lastly, we show that for every \( \alpha \) there exists \( \beta \) such that \( E'(U^\alpha_\alpha, U^\beta_\alpha): E'(U^\alpha_\alpha) \rightarrow E'(U^\beta_\alpha) \) is 2-summing. Let \( \alpha \) be arbitrarily fixed. By the third step, there exists \( \alpha_i \) such that the canonical injection \( E'(U^\alpha_\alpha, U^\alpha_{\alpha_i}) \) is \((q,2)\)-summing. Similarly we can find \( \alpha_2 \) such that \( E'(U^\alpha_{\alpha_i}, U^\alpha_{\alpha_2}) \) is \((q,2)\)-summing. Repeatedly, we can find \( \alpha_1, \alpha_2, \ldots, \alpha_k \) such that \( E'(U^\alpha_{\alpha_{\alpha_1}}, U^\alpha_{\alpha_{\alpha_2}}) \) is \((q,2)\)-summing for every \( i \). Let \( k \) be \( k > q/2 \). Then the \( k \)-composition \( E'(U^\alpha_{\alpha_{\alpha_1}}, U^\alpha_{\alpha_{\alpha_2}}) = E'(U^\alpha_{\alpha_{\alpha_1}}, U^\alpha_{\alpha_{\alpha_2}}) \circ \cdots \circ E'(U^\alpha_{\alpha_{\alpha_k}}, U^\alpha_{\alpha_{\alpha_k+1}}) \) is 2-summing by Tomczak-Jaegermann [19], Theorem 22.5, since each \( E'(U^\alpha_{\alpha_{\alpha_k}}) \) is of cotype \( q \). This completes the proof.

**Remark 2.** In general, \( E \) is not necessarily nuclear even if each continuous cylinder set measure on \( E' \) is \( \sigma(E', E) \)-Radon. For example, let \( \tau_r \) be the Sazonov topology on the infinite-dimensional Hilbert space \( H \) and consider \( E = (H, \tau_r) \), see Sazonov [16]. Then \( E \) is not nuclear but each continuous cylinder set measure on \( E' \) is \( \sigma(E', E) \)-Radon, see Yamasaki [21], §20.

**Counterexample.** Let \( E \) be a Fréchet space. Suppose that each continuous Gaussian cylinder set measure on \( E' \) is \( \sigma \)-additive. Then can we conclude that \( E \) is nuclear? The answer is in general negative. We give a counterexample. The following result is well-known, see Schwartz [16].

**Lemma 7.** Let \( G, F \) be Banach spaces and \( \psi: G \rightarrow F \) be a continuous linear operator. Let \( \psi' \) be the adjoint of \( \psi \) and \( 0 < r < \infty \). Suppose that \( \psi' \) is \( r \)-summing. Then for every Gaussian cylinder set measure \( \mu \) on \( F' \), the image \( \psi' (\mu) \) is \( \sigma(G', G) \)-Radon.

**Example.** Let \( D_j = (n^{-(l/2)^{r_j}})_{l=1}^\infty : \ell_1 \rightarrow \ell_1 \) be the diagonal operator given by
\[
D_j \left((x_n)\right) = (n^{-(l/2)^{r_j}} x_n) = (x_n) \in \ell_1.
\]
Let \( E \) be the projective limit of \( \{\ell_1, D_j\}_{j=1}^\infty \). Explicitly, \( E \) is given by
\[
E = \left\{(x_n) \in \mathbb{R}^\infty : \sum_{n=1}^\infty n^{-(l/2)^{r_j}} |x_n| < \infty \text{ for each } j \right\}.
\]
Let \( E_j = \{(x_n) : |(x_n)_j| = \sum_n n^{-l(2)^{r_j}} |x_n| < \infty \} \) with seminorm \( | \cdot |_j \). Then we have \( E = \cap_j E_j \) and
Then the dual $E'$ is the inductive limit of $\{\ell\_\nu, D_j\}$, where $D_j: \ell\_\nu \to \ell\_\nu$ be $D_j((x\_n)) = (n^{-1/2} + k \cdot x\_n)$. For every $k$, the composition $D_k \circ D_{k-1} \cdots \circ D_1: \ell\_\nu \to \ell\_\nu$ is the diagonal operator $(n^{-1/2+k/2})$, which is not 2-summing for every $k$. Remark that the diagonal operator $A = (a\_n): \ell\_\nu \to \ell\_\nu$ (or into $\ell\_\nu$ is $r$-summing if and only if $(a\_n) \in \ell\_r$. Thus $E$ is not nuclear. We remark that each $D_j$ is 2\textsuperscript{r+2}-summing since

$$\sum_{n=1}^{\infty} (n^{-1/2+k})^{2/r^2} = \sum_{n=1}^{\infty} n^{-2} < \infty.$$ 

By Lemma 7, for each continuous Gaussian cylinder set measure on $E'$ is $\sigma$-additive on $E'\_{r+1}$ since the natural injection $t\_{r+1}: E' \to E'_{r+1}$ is 2\textsuperscript{r+2}-summing. Hence each continuous Gaussian cylinder set measure on $E'$ is also $\sigma(E', E)$-Radon.

**Remark 3.** In the above example, $D_j$ is in fact defined on $\ell\_2^{r+1}$ into $\ell\_2^{r+2}$ which is also 2\textsuperscript{r+2}-summing. And the composition $D_k \circ D_{k-1} \cdots \circ D_1: \ell\_4 \to \ell\_2^{r+2}$ is not 2-summing. This shows that, in Theorem 2, we can not relax the condition “$E'(U\_a)$ is of cotyple $q$” by “$E'(U\_a)$ is of finite cotype”. In Theorem 2, $q$ must be uniform for every $E\_a$.

**References**


